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Feng Dai
Yuan Xu

Approximation Theory and Harmonic Analysis on Spheres and Balls

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Feng Dai • Yuan Xu

Approximation Theory and Harmonic Analysis on Spheres and Balls

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Feng Dai
Department of Mathematical
and Statistical Sciences
University of Alberta
Edmonton, AB, Canada

Yuan Xu
Department of Mathematics
University of Oregon
Eugene, OR, USA

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Preface

This book is written as an introduction to analysis on the sphere and on the ball, and it provides a cohesive account of recent developments in approximation theory and harmonic analysis on these domains. Analysis on the unit sphere appears as part of Fourier analysis, in the study of homogeneous spaces, and in several fields in applied mathematics, from numerical analysis to geoscience, and it has seen increased activity in recent years. Its materials, however, are mostly scattered in papers and sections of books that cover more general topics. Our goals are twofold. The first is to provide a self-contained background for readers who are interested in analysis on the sphere. The second is to give a complete treatment of some recent advances in approximation theory and harmonic analysis on the sphere developed in the last fifteen years or so, of which both authors are among the earnest participants, and several chapters of the book are based on materials from their own research.

The book is loosely divided into four parts. The first part deals with analysis on the sphere with respect to the surface measure $\mathrm{d}\sigma$, the only rotation-invariant measure on the sphere. We give a self-contained exposition on spherical harmonics, written with analysis in mind, in the first chapter, and present classical results of harmonic analysis on the sphere, including convolution structure, Cesàro summability of orthogonal expansions, the Littlewood–Paley theory, and the multiplier theorem due to Bonami–Clerc in the next two chapters. Approximation on the sphere is discussed in the fourth section, where a recent characterization of best approximation by polynomials on the sphere is given in terms of a modulus of smoothness and its equivalent K-functional. An introduction to cubature formulas, which are necessary for discretizing integrals to obtain discrete processes of approximation, is given in the sixth chapter. A recent proof of a conjecture on spherical design, synonym of equal-weight cubature formulas, by Bondarenko, Radchenko, and Viazovska, is included, for which the necessary ingredient of the Marcinkiewicz–Zygmund inequality is established in the fifth chapter, where the inequality and several others are established for the doubling weight on the sphere.

The second part discusses analysis in weighted spaces on the sphere. The background of this part is a far-reaching extension of spherical harmonics due to C. Dunkl, in which the role of the orthogonal group is replaced by a finite reflection group, the measure $\mathrm{d}\sigma$ is replaced by $h_\kappa^2(x)\mathrm{d}\sigma$, where h_κ is a weight function invariant under a reflection group with κ being a parameter, and spherical harmonics are replaced by h-spherical harmonics associated with the Dunkl operators, a family of commuting differential–difference operators that replace partial derivatives. The study of h-spherical harmonic expansions started about fifteen years ago. Many deeper results in analysis were established only in the case of the group $\mathbb{Z}_2^d$, for which h_κ is given by $h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$. In order to avoid heavy algebraic preparations, we give a self-contained exposition of Dunkl's theory in the case of $\mathbb{Z}_2^d$, which is composed to highlight its parallel to the theory of spherical harmonics. Most results on ordinary spherical harmonic expansions can be extended to h-spherical harmonic expansions, including finer L^p results on projection operators and the Cesàro means, maximal functions, and multiplier theorem, as well as a characterization of best approximation that was developed by many authors. We give complete proofs of these results, which are more challenging than proofs for classical results for $\mathrm{d}\sigma$, and in fact, in some cases, simplify those proofs for classical results when the parameters κ are set to zero.

The third part deals with analysis on the unit ball and on the simplex. There are close relations between analysis on spheres and that on balls of different dimensions, which enables us to utilize the results in the part two to develop a parallel theory for approximation theory and harmonic analysis on the unit ball. There is also a connection between analysis on the ball and that on the simplex, which carries much, but not all, of analysis on the ball over to the simplex. These results are composed in parallel to the development on the sphere.

The fourth part consists of one chapter, the last chapter of the book, which discusses five topics related to the main theme of the book: highly localized polynomial frames, distribution of nodes of positive cubature, positive and strictly positive definite functions, asymptotics of minimal discrete energy, and computerized tomography.

Analysis on the sphere has seen increased activity in the past two decades. There are other related topics that we decided not to include, for example scattered data interpolation, applications of spherical radial basis functions (zonal functions), and numerical or computational analysis on the sphere. These topics are more closely related to the applied and computational branches of approximation theory. Our choices, dictated by our own strengths and limitations, are those topics that are closely related to the main theme—approximation theory *and* harmonic analysis—of this book.

We keep the references in the text to a mininum and leave references and historical remarks to the last section of each chapter, entitled "Notes and Further Results," where we also point out further results related to the materials in the chapter. Some common notation and terminology are given in the preamble at the front of the book, and there are two fairly detailed indexes: a subject index and a symbol index.

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Edmonton, Canada — Feng Dai
Eugene, OR — Yuan Xu

To My Teacher
Professor Kunyang Wang
F.D.

To Litian
With Appreciation
Y.X.

Contents

Preamble

Basics. Let $\mathbb{R}^d$ denote d-dimensional Euclidean space. For $x \in \mathbb{R}^d$, we write $x = (x_1, \dots, x_d)$. The inner product of $x, y \in \mathbb{R}^d$ is denoted by $\langle x, y \rangle := \sum_{i=1}^d x_i y_i$, and the norm of x is denoted by $\|x\| := \sqrt{\langle x, x \rangle}$.

The unit sphere $\mathbb{S}^{d-1}$ and the unit ball $\mathbb{B}^d$ of $\mathbb{R}^d$ are defined by

$$\mathbb{S}^{d-1} := \{x : \|x\| = 1\} \quad \text{and} \quad \mathbb{B}^d := \{x : \|x\| \le 1\}.$$

Distance on the sphere. The distance on the sphere is the geodesic distance, or the distance between x and y on the largest circle on $\mathbb{S}^{d-1}$ that passes through x and y on the sphere:

$$\mathtt{d}(x, y) := \arccos \langle x, y \rangle, \qquad x, y \in \mathbb{S}^{d-1}.$$

Distance on the ball. The distance on the ball is the projection of the geodesic distance on $\mathbb{S}^d$ onto $\mathbb{B}^d$:

$$\mathtt{d}_{\mathbb{B}}(x, y) := \arccos \left(\langle x, y \rangle + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \right), \quad x, y \in \mathbb{B}^d.$$

Multi-index notation. Let $\mathbb{N}_0$ denote the set of nonnegative integers. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, a monomial x^α is a product $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$, which has degree $|\alpha| := \alpha_1 + \cdots + \alpha_d$. For $a \in \mathbb{R}$ and $n \in \mathbb{N}_0$, the Pochhammer symbol $(a)_n$ is defined by

$$(a)_n := a(a+1) \cdots (a+n-1).$$

If a is not a negative integer, then $(a)_n = \Gamma(a+n)/\Gamma(a)$.

Polynomial spaces. The space of polynomials of degree n in d variables is denoted by Π_n^d. The space of homogeneous polynomials of degree n in d variables is denoted by $\mathscr{P}_n^d$. The restriction of Π_n^d to $\mathbb{S}^{d-1}$ is the space of spherical polynomials, denoted by $\Pi_n(\mathbb{S}^{d-1})$.

L^p spaces. For a weight function w defined on a domain Ω, we define $L^p(w)$ as the space of functions on Ω with finite $\|f\|_{p,w}$ norm, where

$$\|f\|_{p,w} := \left(\int_\Omega |f(x)|^p w(x)\,\mathrm{d}x \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

and we retain this notation for $0 < p < 1$ even it is no longer a norm. For $p = \infty$, we consider the space of continuous functions with the uniform norm

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

Constants. In various inequalities and estimates in this book, we will use $c, c_1, c_2, \ldots$ to denote positive constants, possibly different at every occurrence. The notation $A \sim B$ means that $c_1 A \le B \le c_2 A$.

Chapter 1
Spherical Harmonics

In this chapter we introduce spherical harmonics and study their properties. Most of the material of this chapter, except the last section, is classical. We strive for a succinct account of the theory of spherical harmonics. After a standard treatment of the space of spherical harmonics and orthogonal bases in the first section, the orthogonal projection operator and reproducing kernels, also known as zonal harmonics, are developed in greater detail in the second section, because of their central role in harmonic analysis and approximation theory. As an application of the addition formula, it is shown in the third section that there exist bases of spherical harmonics consisting of entirely zonal harmonics. The Laplace–Beltrami operator is discussed in the fourth section, where an elementary and self-contained approach is adopted. Spherical coordinates and an explicit orthonormal basis of spherical harmonics in these coordinates are presented the fifth section. These formulas in two and three variables are collected in the sixth section for easy reference, since they are most often used in applications. The connection to group representation is treated briefly in the seventh section. The last section deals with derivatives and integrals on the sphere. With the introduction of angular derivatives that are first-order differential operators acting on the large circles of intersections of the sphere and the coordinate planes, it is shown that the Laplace–Beltrami operator can be decomposed into second-order angular derivatives. These derivative operators will play an important role in approximation theory on the sphere. They are used to derive several integral formulas on the sphere.

1.1 Space of Spherical Harmonics and Orthogonal Bases

We begin by introducing some notation that will be used throughout this book. For $x \in \mathbb{R}^d$, we write $x = (x_1, \ldots, x_d)$. The inner product of $x, y \in \mathbb{R}^d$ is denoted by $\langle x, y \rangle := \sum_{i=1}^d x_i y_i$, and the norm of x is denoted by $\|x\| := \sqrt{\langle x, x \rangle}$. Let $\mathbb{N}_0$ denote the set of nonnegative integers. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, a monomial x^α is a product $x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$, which has degree $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

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A homogeneous polynomial P of degree n is a linear combination of monomials of degree n, that is, $P(x) = \sum_{|\alpha|=n} c_\alpha x^\alpha$, where c_α are either real or complex numbers. A polynomial of (total) degree at most n is of the form $P(x) = \sum_{|\alpha|\le n} c_\alpha x^\alpha$. Let $\mathscr{P}_n^d$ denote the space of real homogeneous polynomials of degree n, and let Π_n^d denote the space of real polynomials of degree at most n. Counting the cardinalities of $\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\}$ and $\{\alpha \in \mathbb{N}_0^d : |\alpha| \le n\}$ shows that

$$\dim \mathscr{P}_n^d = \binom{n+d-1}{n} \quad \text{and} \quad \dim \Pi_n^d = \binom{n+d}{n}.$$

Let ∂_i denote the partial derivative in the ith variable and Δ the Laplacian operator

$$\Delta := \partial_1^2 + \cdots + \partial_d^2.$$

Definition 1.1.1. For $n = 0,1,2,\ldots$, let $\mathscr{H}_n^d$ be the linear space of real harmonic polynomials, homogeneous of degree n, on $\mathbb{R}^d$, that is,

$$\mathscr{H}_n^d := \left\{ P \in \mathscr{P}_n^d : \Delta P = 0 \right\}.$$

Spherical harmonics are the restrictions of elements in $\mathscr{H}_n^d$ to the unit sphere. If $Y \in \mathscr{H}_n^d$, then $Y(x) = \|x\|^n Y(x')$, where $x = \|x\| x'$ and $x' \in \mathbb{S}^{d-1}$. Strictly speaking, one should make a distinction between $\mathscr{H}_n^d$ and its restriction to the sphere. We will, however, also call $\mathscr{H}_n^d$ the space of spherical harmonics. When it is necessary to emphasize the restriction to the sphere, we shall use the notation $\mathscr{H}_n^d|_{\mathbb{S}^{d-1}}$. In the same vein, we shall define $\mathscr{P}_n(\mathbb{S}^{d-1}) := \mathscr{P}_n^d|_{\mathbb{S}^{d-1}}$ and $\Pi_n(\mathbb{S}^{d-1}) := \Pi_n^d|_{\mathbb{S}^{d-1}}$.

Spherical harmonics of different degrees are orthogonal with respect to

$$\langle f,g\rangle_{\mathbb{S}^{d-1}} := \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(x)g(x)\,\mathrm{d}\sigma(x), \tag{1.1.1}$$

where $\mathrm{d}\sigma$ is the surface area measure and ω_d denotes the surface area of $\mathbb{S}^{d-1}$,

$$\omega_d := \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{1.1.2}$$

Theorem 1.1.2. *If $Y_n \in \mathscr{H}_n^d$, $Y_m \in \mathscr{H}_m^d$, and $n \ne m$, then $\langle Y_n, Y_m\rangle_{\mathbb{S}^{d-1}} = 0$.*

Proof. Let $\frac{\partial}{\partial r}$ denote the normal derivative. Since Y_n is homogeneous, $Y_n(x) = r^n Y_n(x')$, where $x = rx'$ and $x' \in \mathbb{S}^{d-1}$, so that $\frac{\partial Y_n}{\partial r}(x') = nY_n(x')$ for $x' \in \mathbb{S}^{d-1}$ and $n \ge 0$. By Green's identity,

$$\begin{aligned}(n-m)\int_{\mathbb{S}^{d-1}} Y_n Y_m \mathrm{d}\sigma &= \int_{\mathbb{S}^{d-1}} \left(Y_m \frac{\partial Y_n}{\partial r} - Y_n \frac{\partial Y_m}{\partial r} \right) \mathrm{d}\sigma \\ &= \int_{\mathbb{B}^d} (Y_m \Delta Y_n - Y_n \Delta Y_m)\,\mathrm{d}x = 0,\end{aligned}$$

since $\Delta Y_n = 0$ and $\Delta Y_m = 0$. □

Theorem 1.1.3. *For $n = 0,1,2,\ldots$, there is a decomposition of $\mathscr{P}_n^d$,*

$$\mathscr{P}_n^d = \bigoplus_{0\le j\le n/2} \|x\|^{2j}\mathscr{H}_{n-2j}^d. \tag{1.1.3}$$

In other words, for each $P \in \mathscr{P}_n^d$, there is a unique decomposition

$$P(x) = \sum_{0\le j\le n/2} \|x\|^{2j} P_{n-2j}(x) \quad \text{with} \quad P_{n-2j} \in \mathscr{H}_{n-2j}^d. \tag{1.1.4}$$

Proof. The proof uses induction. Evidently $\mathscr{P}_0^d = \mathscr{H}_0^d$ and $\mathscr{P}_1^d = \mathscr{H}_1^d$. Since $\Delta \mathscr{P}_n^d \subset \mathscr{P}_{n-2}^d$, $\dim \mathscr{H}_n^d \ge \dim \mathscr{P}_n^d - \dim \mathscr{P}_{n-2}^d$. Suppose the statement holds for $m = 0,1,\ldots,n-1$. Then $\|x\|^2\mathscr{P}_{n-2}^d$ is a subspace of $\mathscr{P}_n^d$, and it is isomorphic to $\mathscr{P}_{n-2}^d$. By the induction hypothesis, $\|x\|^2\mathscr{P}_{n-2}^d = \bigoplus_{0\le j\le n/2-1} \|x\|^{2j+2}\mathscr{H}_{n-2-2j}^d$. Hence, by the previous theorem, $\mathscr{H}_n^d$ is orthogonal to $\|x\|^2\mathscr{P}_{n-2}^d$, so that $\dim \mathscr{H}_n^d + \dim \mathscr{P}_{n-2}^d \le \dim \mathscr{P}_n^d$. Consequently, $\mathscr{P}_n^d = \mathscr{H}_n^d \oplus \|x\|^2\mathscr{P}_{n-2}^d$. □

Corollary 1.1.4. *For $n = 0,1,2,\ldots$,*

$$\dim \mathscr{H}_n^d = \dim \mathscr{P}_n^d - \dim \mathscr{P}_{n-2}^d = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}, \tag{1.1.5}$$

where it is agreed that $\dim \mathscr{P}_{n-2}^d = 0$ *for $n = 0,1$.*

Corollary 1.1.5. *For $n \in \mathbb{N}$, $\Pi_n(\mathbb{S}^{d-1}) = \mathscr{P}_n(\mathbb{S}^{d-1}) \oplus \mathscr{P}_{n-1}(\mathbb{S}^{d-1})$ and*

$$\dim \Pi_n(\mathbb{S}^{d-1}) = \dim \mathscr{P}_n^d + \dim \mathscr{P}_{n-1}^d = \binom{n+d-1}{n} + \binom{n+d-2}{n-1}. \tag{1.1.6}$$

Proof. By Theorem 1.1.3, $\Pi_n(\mathbb{S}^{d-1})$ can be written as a direct sum of $\mathscr{H}_k^d$ for $0 \le k \le n$, which gives the stated decomposition by Eq. (1.1.3). Moreover,

$$\dim \Pi_n(\mathbb{S}^{d-1}) = \sum_{k=0}^{n} \dim \mathscr{H}_k^d = \sum_{k=0}^{n} (\dim \mathscr{P}_k^d - \dim \mathscr{P}_{k-2}^d)$$

by Eq. (1.1.5), which simplifies to Eq. (1.1.6). □

The orthogonality and homogeneity define spherical harmonics.

Proposition 1.1.6. *If P is a homogeneous polynomial of degree n and P is orthogonal to all polynomials of degree less than n with respect to $\langle\cdot,\cdot\rangle_{\mathbb{S}^{d-1}}$, then $P \in \mathscr{H}_n^d$.*

Proof. Since $P \in \mathscr{P}_n^d$, P can be expressed as in Eq. (1.1.4). The orthogonality then shows that $P = P_n \in \mathscr{H}_n^d$. □

Let $O(d)$ denote the orthogonal group, the group of $d \times d$ orthogonal matrices, and let $SO(d) = \{g \in O(d) : \det g = 1\}$ be the special orthogonal group. A rotation in $\mathbb{R}^d$ is determined by an element in $SO(d)$.

Theorem 1.1.7. *The space $\mathscr{H}_n^d$ is invariant under the action $f(x) \mapsto f(Qx)$, $Q \in O(d)$. Moreover, if $\{Y_\alpha\}$ is an orthonormal basis of $\mathscr{H}_n^d$, then so is $\{Y_\alpha(Q\{\cdot\})\}$.*

Proof. Since Δ is invariant under the orthogonal group $O(d)$ (writing $\Delta = \nabla \cdot \nabla$ and changing variables), if $Y \in \mathscr{H}_n^d$ and $Q \in O(d)$, then $Y(Qx) \in \mathscr{H}_n^d$. That $\{Y_\alpha(Qx)\}$ is an orthonormal basis of $\mathscr{H}_n^d$ whenever $\{Y_\alpha(x)\}$ is follows from

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} Y_\alpha(Qx)Y_\beta(Qx)\mathrm{d}\sigma(x) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} Y_\alpha(x)Y_\beta(x)\mathrm{d}\sigma(x) = \delta_{\alpha,\beta},$$

which holds under a change of variables, since $\mathrm{d}\sigma$ is invariant under $O(d)$. ◻

Besides $\langle f,g\rangle_{\mathbb{S}^{d-1}}$, another useful inner product can be defined on $\mathscr{P}_n^d$ through the action of differentiation. For $\alpha \in \mathbb{N}_0^d$, let $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$. Let $(a)_n := a(a+1)\cdots(a+n-1)$ be the Pochhammer symbol.

Theorem 1.1.8. *For $p,q \in \mathscr{P}_n^d$, define a bilinear form*

$$\langle p,q\rangle_\partial := p(\partial)q, \tag{1.1.7}$$

where $p(\partial)$ is the differential operator defined by replacing x^α in $p(x)$ by ∂^α. Then

1. *$\langle p,q\rangle_\partial$ is an inner product on $\mathscr{P}_n^d$;*
2. *the reproducing kernel of this inner product is $k_n(x,y) := \langle x,y\rangle^n/n!$; that is,*

$$\langle k_n(x,\cdot),p\rangle_\partial = p(x), \qquad \forall p \in \mathscr{P}_n^d;$$

3. *for $p \in \mathscr{P}_n^d$ and $q \in \mathscr{H}_n^d$,*

$$\langle p,q\rangle_\partial = 2^n \left(\frac{d}{2}\right)_n \langle p,q\rangle_{\mathbb{S}^{d-1}}.$$

Proof. Let $p,q \in \mathscr{P}_n^d$ be given by $p(x) = \sum_{|\alpha|=n} a_\alpha x^\alpha$ and $q(x) = \sum_{|\alpha|=n} b_\alpha x^\alpha$, where $a_\alpha, b_\alpha \in \mathbb{R}$. Then,

$$\langle p,q\rangle_\partial = \sum_{|\alpha|=n} a_\alpha \partial^\alpha \sum_{|\beta|=n} b_\beta x^\beta = \sum_{|\alpha|=n} \alpha! a_\alpha b_\alpha, \tag{1.1.8}$$

which implies, in particular, that $\langle p,p\rangle_\partial > 0$ for $p \neq 0$. It follows then that $\langle\cdot,\cdot\rangle_\partial$ is an inner product on $\mathscr{P}_n^d$. By the multinomial formula, for $q_\alpha(x) = x^\alpha$, $|\alpha| = n$,

$$\langle k_n(x,\cdot),q_\alpha\rangle_\partial = \frac{1}{n!}\sum_{|\beta|=n}\binom{n}{\beta}x^\beta\frac{\partial^\beta}{\partial y^\beta}y^\alpha = q_\alpha(x),$$

which shows that $k_n(x,y)$ is the reproducing kernel with respect to $\langle\cdot,\cdot\rangle_\partial$.

We now prove item (3). Integrating by parts shows that

$$\int_{\mathbb{R}^d} \partial_i f(x) g(x) e^{-\|x\|^2/2} \mathrm{d}x = -\int_{\mathbb{R}^d} f(x)(\partial_i g(x) - x_i g(x)) e^{-\|x\|^2/2} \mathrm{d}x.$$

Since $p(\partial)q$ is a constant, using this integration by parts repeatedly shows that

$$\begin{aligned} \langle p,q\rangle_\partial &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} p(\partial)q(x) e^{-\|x\|^2/2} \mathrm{d}x \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} q(x)(p(x)+s(x)) e^{-\|x\|^2/2} \mathrm{d}x, \end{aligned}$$

where $s \in \Pi_{n-1}^d$. Since $q \in \mathscr{H}_n^d$ and $p \in \mathscr{P}_n^d$, switching to a polar integral and using the orthogonality of $\mathscr{H}_n^d$, we obtain

$$\langle p,q\rangle_\partial = \frac{1}{(2\pi)^{d/2}} \int_0^\infty r^{2n+d-1} e^{-r^2/2} \mathrm{d}r \int_{\mathbb{S}^{d-1}} q(x')p(x')\mathrm{d}\sigma(x').$$

Evaluating the integral in r and simplifying by Eq. (1.1.2) concludes the proof. □

A large number of spherical harmonic polynomials can be defined explicitly through differentiation. Let us denote the standard basis of $\mathbb{R}^d$ by

$$e_1 = (1,0,\ldots,0), e_2 = (0,1,0,\ldots,0)\ldots, e_d = (0,\ldots,0,1).$$

Theorem 1.1.9. *Let $d > 2$. For $\alpha \in \mathbb{N}_0^d$, $n = |\alpha|$, define*

$$p_\alpha(x) := \frac{(-1)^n}{2^n\left(\frac{d-2}{2}\right)_n} \|x\|^{2|\alpha|+d-2} \partial^\alpha \{\|x\|^{-d+2}\}. \tag{1.1.9}$$

Then

1. *$p_\alpha \in \mathscr{H}_n^d$ and p_α is the monic spherical harmonic of the form*

$$p_\alpha(x) = x^\alpha + \|x\|^2 q_\alpha(x), \qquad q_\alpha \in \mathscr{P}_{n-2}^d. \tag{1.1.10}$$

2. *p_α satisfies the recurrence relation*

$$p_{\alpha+e_i}(x) = x_i p_\alpha(x) - \frac{1}{2n+d-2} \|x\|^2 \partial_i p_\alpha(x). \tag{1.1.11}$$

3. *$\{p_\alpha : |\alpha| = n, \alpha_d = 0 \text{ or } 1\}$ is a basis of $\mathscr{H}_n^d$.*

Proof. Taking the derivative of $p_\alpha(x)$ gives immediately the recurrence relation (1.1.11). Clearly $p_0(x) = 1$. By induction, the recurrence relation shows that p_α is a homogeneous polynomial of degree n and that it is of the form (1.1.10). We now show that p_α is a spherical harmonic. For $g \in \mathscr{P}_n^d$ and $\rho \in \mathbb{R}$, a quick computation using $\sum_{i=1}^d x_i \partial_i g(x) = ng(x)$ shows that

$$\Delta(\|x\|^\rho g) = \rho(2n+\rho+d-2)\|x\|^{\rho-2} g + \|x\|^\rho \Delta g. \tag{1.1.12}$$

In particular, setting $n=0$ and $g(x)=1$ gives $\Delta(\|x\|^{-d+2}) = 0$. Furthermore, setting $g = p_\alpha$ and $\rho = -2n-d+2$ in Eq. (1.1.12) leads to

$$\Delta p_\alpha(x) = \frac{(-1)^n}{2^n(d/2-1)_n}\|x\|^{2|\alpha|+d-2}\partial^\alpha \Delta\{\|x\|^{-d+2}\} = 0.$$

Thus, $p_\alpha \in \mathscr{H}_n^d$. Since $\|x\|^2 q(x)$ is a linear combination of the monomials x^β with $\beta_d \ge 2$, by Eq. (1.1.10) and the linear independence of $\{x^\alpha : |\alpha| = n, \alpha_d = 0 \text{ or } 1\}$, it follows that the elements in the set $\{p_\alpha : |\alpha| = n, \alpha_d = 0 \text{ or } 1\}$ are linearly independent. The cardinality of the set is

$$\dim \mathscr{P}_n^{d-1} + \dim \mathscr{P}_{n-1}^{d-1} = \binom{n+d-2}{d-2} + \binom{n+d-3}{d-2},$$

which is, by a simple identity of binomial coefficients and Eq. (1.1.5), precisely $\dim \mathscr{H}_n^d$. This completes the proof. □

The right-hand side of Eq. (1.1.9) is called Maxwell's representation of harmonic polynomials [88, 125]. The complete set of $\{p_\alpha : |\alpha| = n\}$ is necessarily linearly dependent by its cardinality. Moreover, by Eq. (1.1.9),

$$p_{\alpha+2e_1} + \cdots + p_{\alpha+2e_d} = \frac{(-1)^n}{2^n(\frac{d}{2})_n}\|x\|^{2|\alpha|+d-2}\partial^\alpha \Delta\{\|x\|^{-d+2}\} = 0,$$

which gives $\dim \mathscr{P}_{n-2}^d$ linearly dependent relations among $\{p_\alpha : |\alpha| = n\}$. The set $\{p_\alpha : |\alpha| = n\}$ evidently contains many bases of $\mathscr{H}_n^d$. The basis in item (3) of Theorem 1.1.9 is but one convenient choice. The proof of Theorem 1.1.9 relies on the fact that $\|x\|^{-d+2}$ is a harmonic function in $\mathbb{R}^d \setminus \{0\}$ for $d > 2$. In the case of $d = 2$, we need to replace this function by $\log \|x\|$. Since the case $d = 2$ corresponds to the classical Fourier series, we leave the analogue of Theorem 1.1.9 for $d = 2$ to the interested reader.

The basis $\{p_\alpha : |\alpha| = n, \alpha_d = 0 \text{ or } 1\}$ of $\mathscr{H}_n^d$ is not orthonormal. In fact, the elements of this basis are not mutually orthogonal. Orthonormal bases can be constructed by applying the Gram–Schmidt process. An explicit orthonormal basis for $\mathscr{H}_n^d$ will be given in Sect. 1.5 in terms of spherical coordinates.

1.2 Projection Operators and Zonal Harmonics

Let $L^2(\mathbb{S}^{d-1})$ denote the space of square integrable functions on $\mathbb{S}^{d-1}$. Let

$$\operatorname{proj}_n : L^2(\mathbb{S}^{d-1}) \mapsto \mathscr{H}_n^d$$

denote the orthogonal projection from $L^2(\mathbb{S}^{d-1})$ onto $\mathscr{H}_n^d$. If $P \in \mathscr{P}_n^d$, then $P = P_n + \|x\|^2 Q_n$, where $P_n \in \mathscr{H}_n^d$ and $Q_n \in \mathscr{P}_{n-2}^d$, by Eq. (1.1.4), so that $\operatorname{proj}_n P = P_n$. In particular, Eq. (1.1.11) shows that p_α defined in Eq. (1.1.9) is the orthogonal projection of the function $q_\alpha(x) = x^\alpha$; that is, $p_\alpha = \operatorname{proj}_n q_\alpha$. This leads to the following:

Lemma 1.2.1. *Let $p \in \mathscr{P}_n^d$. Then*

$$\operatorname{proj}_n p = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{4^j j! (-n+2-d/2)_j} \|x\|^{2j} \Delta^j p. \tag{1.2.1}$$

Proof. By linearity, it suffices to consider p being $q_\alpha(x) = x^\alpha$. By Theorem 1.1.9, $\operatorname{proj}_n q_\alpha(x) = p_\alpha(x)$, and the proof amounts to showing that $p_\alpha(x)$ defined in Eq. (1.1.9) can be expanded as in Eq. (1.2.1). We use induction on n. The case $n = 0$ is evident. Assume that Eq. (1.2.1) has been established for $m = 0, 1, \ldots, n$. Applying Eq. (1.2.1) to $q_\alpha(x)$, $|\alpha| = n$, it follows that

$$\partial^\alpha \left\{ \|x\|^{-d+2} \right\} = (-1)^n 2^n \left(\tfrac{d}{2} - 1\right)_n \|x\|^{-2n-d+2} \times \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{4^j j! (-n+2-d/2)_j} \|x\|^{2j} \Delta^j \{x^\alpha\}.$$

Applying ∂_i to this identity, we obtain

$$\partial_i \partial^\alpha \left\{ \|x\|^{-d+2} \right\} = (-1)^n 2^n \left(\tfrac{d}{2} - 1\right)_n (-2n-d+2) \|x\|^{-2n-d+2} \times \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \frac{1}{4^j j! (-n+1-d/2)_j} \|x\|^{2j} [x_i \Delta^j \{x^\alpha\} + 2j \Delta^{j-1} \partial_i \{x^\alpha\}].$$

The terms in the square brackets are exactly $\Delta^j \{x_i x^\alpha\}$, and the constant in front simplifies to $(-1)^{n+1} 2^{n+1} \left(\frac{d}{2} - 1\right)_{n+1}$, so that Eq. (1.2.1) holds for $p(x) = x_i x^\alpha$. This completes the induction. □

Definition 1.2.2. The reproducing kernel $Z_n(\cdot,\cdot)$ of $\mathscr{H}_n^d$ is uniquely determined by

$$\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} Z_n(x,y) p(y) \mathrm{d}\sigma(y) = p(x), \quad \forall p \in \mathscr{H}_n^d, \quad x \in \mathbb{S}^{d-1}, \tag{1.2.2}$$

and the requirement that $Z_n(x,\cdot)$ be an element of $\mathscr{H}_n^d$ for each fixed x.

That the kernel is well defined and unique follows from the Riesz representation theorem applied to the linear functional $L(Y) := Y(x)$, $Y \in \mathscr{H}_n^d$, for a fixed $x \in \mathbb{S}^{d-1}$.

Lemma 1.2.3. *In terms of an orthonormal basis $\{Y_j : 1 \le j \le \dim \mathscr{H}_n^d\}$ of $\mathscr{H}_n^d$,*

$$Z_n(x,y) = \sum_{k=1}^{\dim \mathscr{H}_n^d} Y_k(x) Y_k(y), \quad x, y \in \mathbb{S}^{d-1}, \tag{1.2.3}$$

and despite Eq. (1.2.3), Z_n is independent of the particular choice of basis of $\mathscr{H}_n^d$.

Proof. Since $Z_n(x,\cdot) \in \mathscr{H}_n^d$, it can be expressed as $Z_n(x,y) = \sum_k c_k Y_k(y)$, where the coefficients are determined by Eq. (1.2.2) as $c_k = Y_k(x)$. The uniqueness implies that Z_n is independent of the choice of basis. This can also be shown directly as follows. Let $\mathbb{Y}_n = (Y_1, \ldots, Y_N)$ with $N = \dim \mathscr{H}_n^d$, and regard it as a column vector. Then $Z_n(x,y) = [\mathbb{Y}_n(x)]^{\mathrm{tr}} \mathbb{Y}_n(y)$. If $\{Y_j' : 1 \le j \le N\}$ is another orthonormal basis of $\mathscr{H}_n^d$, then $\mathbb{Y}_n' = Q \mathbb{Y}_n$. Since the orthonormality of $\{Y_j\}$ can be expressed as the fact that $\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \mathbb{Y}_n(x) [\mathbb{Y}_n(x)]^{\mathrm{tr}} \mathrm{d}\sigma(x)$ is an identity matrix, it follows readily that Q is an orthogonal matrix. Hence $Z_n(x,y) = [\mathbb{Y}_n(x)]^{\mathrm{tr}} Q^{\mathrm{tr}} Q \mathbb{Y}_n(y) = [\mathbb{Y}_n'(x)]^{\mathrm{tr}} \mathbb{Y}_n'(y)$. □

The reproducing kernel is also the kernel for the projection operator.

Lemma 1.2.4. *The projection operator can be written as*

$$\operatorname{proj}_n f(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) Z_n(x,y) \mathrm{d}\sigma(y). \tag{1.2.4}$$

Proof. Since $\operatorname{proj}_n f \in \mathscr{H}_n^d$, it can be expanded in terms of the orthonormal basis $\{Y_j, 1 \le j \le N_n\}$, $N_n = \dim \mathscr{H}_n^d$, of $\mathscr{H}_n^d$, where the coefficients are determined by the orthonormality,

$$\operatorname{proj}_n f(x) = \sum_{j=1}^{N_n} c_j Y_j(x) \quad \text{with} \quad c_j = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) Y_j(y) \mathrm{d}\sigma(y).$$

If we pull out the integral in front of the sum, this is Eq. (1.2.4) by Eq. (1.2.3). □

Lemma 1.2.5. *The kernel $Z_n(\cdot,\cdot)$ satisfies the following properties:*

1. For every $\xi, \eta \in \mathbb{S}^{d-1}$,

$$\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} Z_n(\xi,y) Z_n(\eta,y) \mathrm{d}\sigma(y) = Z_n(\xi,\eta). \tag{1.2.5}$$

2. $Z_n(x,y)$ depends only on $\langle x,y \rangle$.

Proof. By Corollary 1.1.7, the uniqueness of $Z_n(x,y)$ shows that $Z_n(Qx,Qy) = Z_n(x,y)$ for all $Q \in O(d)$. Since for $x,y \in \mathbb{S}^{d-1}$, there exists a $Q \in SO(d)$ such that $Qx = (0,\ldots,0,1)$ and $Qy = \left(0,\ldots,0,\sqrt{1-\langle x,y\rangle^2}, \langle x,y \rangle\right)$, this shows that $Z_n(x,y)$ depends only on $\langle x,y \rangle$. □

From the second property of the lemma, $Z_n(x,y) = F_n(\langle x,y\rangle)$, which is often called a zonal harmonic, since it is harmonic and depends only on $\langle x,y\rangle$. We now derive a closed formula for F_n, which turns out to be a multiple of the Gegenbauer polynomial C_n^λ of degree n defined, for $\lambda > 0$ and $n \in \mathbb{N}_0$, by

$$C_n^\lambda(x) := \frac{(\lambda)_n 2^n}{n!} x^n {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ 1-n-\lambda \end{matrix}; \frac{1}{x^2}\right), \tag{1.2.6}$$

where ${}_2F_1$ is the hypergeometric function. The properties of the Gegenbauer polynomials are collected in Appendix B.

Theorem 1.2.6. *For $n \in \mathbb{N}_0$ and $x,y \in \mathbb{S}^{d-1}$, $d \geq 3$,*

$$Z_n(x,y) = \frac{n+\lambda}{\lambda} C_n^\lambda(\langle x,y\rangle), \quad \lambda = \frac{d-2}{2}. \tag{1.2.7}$$

Proof. Let $p \in \mathscr{H}_n^d$. By Theorem 1.1.8, $p(x) = \langle k_n(x,\cdot), p\rangle_\partial$. For fixed x, it follows from the same theorem that

$$\begin{aligned} p(x) &= \langle k_n(x,\cdot), p\rangle_\partial = \langle \operatorname{proj}_n(k_n(x,\cdot)), p\rangle_\partial \\ &= \frac{2^n (d/2)_n}{\omega_d} \int_{\mathbb{S}^{d-1}} \operatorname{proj}_n[k_n(x,\cdot)](y) p(y) \mathrm{d}\sigma(y). \end{aligned}$$

Since the kernel $Z_n(\cdot,\cdot)$ is uniquely determined by the reproducing property, this shows that $Z_n(x,y) = 2^n(\frac{d}{2})_n \operatorname{proj}_n[k_n(x,\cdot)](y)$. Since $k_n(x,\cdot)$ is a homogeneous polynomial of degree n and, taking the derivative on y, we have $\Delta^j k_n(x,y) = \|x\|^{2j} k_{n-2j}(x,y)$, as is easily seen from $\partial_i k_n(x,y) = x_i k_{n-1}(x,y)$, Lemma 1.2.1 shows, for $x,y \in \mathbb{S}^{d-1}$, that

$$Z_n(x,y) = 2^n \left(\frac{d}{2}\right)_n \operatorname{proj}_n[k_n(x,\cdot)](y) = \sum_{j=0}^{\lfloor n/2\rfloor} \frac{(\frac{d}{2})_n 2^{n-2j}}{j!(1-n-\lambda)_j} k_{n-2j}(x,y).$$

Using the fact $1/(n-2j)! = (-n)_{2j}/n! = 2^{2j}(-\frac{n}{2})_j(\frac{-n+1}{2})_j/n!$, we conclude then

$$\begin{aligned} Z_n(x,y) &= \frac{n+\lambda}{\lambda} \frac{(\lambda)_n 2^n}{n!} \sum_{j=0}^{\lfloor n/2\rfloor} \frac{(-\frac{n}{2})_j(\frac{-n+1}{2})_j}{j!(1-n-\lambda)_j} \langle x,y\rangle^{n-2j} \\ &= \frac{n+\lambda}{\lambda} \frac{(\lambda)_n 2^n}{n!} \langle x,y\rangle^n {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ 1-n-\lambda \end{matrix}; \frac{1}{\langle x,y\rangle^2}\right), \end{aligned}$$

from which the stated result follows from Eq. (1.2.6). □

Let $\{Y_i : 1 \le i \le \dim \mathscr{H}_n^d\}$ be an orthonormal basis of $\mathscr{H}_n^d$. Then Eq. (1.2.7) states that

$$\sum_{j=1}^{\dim \mathscr{H}_n^d} Y_j(x)Y_j(y) = \frac{n+\lambda}{\lambda} C_n^\lambda(\langle x,y\rangle), \qquad \lambda = \frac{d-2}{2}. \tag{1.2.8}$$

This identity is usually referred to as the addition formula of spherical harmonics, since for $d=2$, it is the addition formula of the cosine function (see Sect. 1.6).

Corollary 1.2.7. *For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{S}^{d-1}$, $d \ge 3$,*

$$|Z_n(x,y)| \le \dim \mathscr{H}_n^d \quad \textit{and} \quad Z_n(x,x) = \dim \mathscr{H}_n^d. \tag{1.2.9}$$

Proof. Set $F_n(t) := \frac{n+\lambda}{\lambda} C_n^\lambda(t)$. By Eq. (1.2.7), $Z_n(x,x) = F_n(1)$ is a constant for all $x \in \mathbb{S}^{d-1}$. Setting $x = y$ in Eq. (1.2.3) and integrating over $\mathbb{S}^{d-1}$, we obtain

$$F_n(1) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} Z_n(\langle x,x\rangle)\mathrm{d}\sigma(x) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} \sum_{k=1}^{\dim \mathscr{H}_n^d} Y_k^2(x)\mathrm{d}\sigma(x) = \dim \mathscr{H}_n^d.$$

The inequality follows from applying the Cauchy–Schwarz inequality to Eq. (1.2.3). □

Because of the relation (1.2.7), the Gegenbauer polynomials with $\lambda = \frac{d-2}{2}$ are also called ultraspherical polynomials. A number of properties of the Gegenbauer polynomials can be obtained from the zonal spherical harmonics. For example, the corollary implies that $C_n^\lambda(1) = \frac{\lambda}{n+\lambda}\dim \mathscr{H}_n^d$. Here is another example:

Corollary 1.2.8. *For $\lambda = \frac{d-2}{2}$, the Gegenbauer polynomials C_n^λ satisfy the orthogonality relation*

$$\frac{\omega_{d-1}}{\omega_d}\int_{-1}^{1} C_n^\lambda(t)C_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}}\mathrm{d}t = h_n^\lambda \delta_{m,n}, \tag{1.2.10}$$

where

$$h_n^\lambda = \frac{\lambda}{n+\lambda} C_n^\lambda(1).$$

Proof. Set again $F_n(t) = \frac{n+\lambda}{\lambda} C_n^\lambda(t)$. Since $Z_n(x,\cdot)$ and $Z_m(x,\cdot)$ are orthogonal over $\mathbb{S}^{d-1}$, and by Eq. (A.5.1), their integrals can be written as an integral of one variable, we obtain

$$\begin{aligned}\frac{\omega_{d-1}}{\omega_d}\int_{-1}^{1} F_n(t)F_m(t)(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t &= \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} F_n(\langle x,y\rangle)F_m(\langle x,y\rangle)\mathrm{d}\sigma(y)\\ &= F_n(1)\delta_{m,n} = (\dim \mathscr{H}_n^d)\delta_{m,n},\end{aligned}$$

where the second line follows from Eq. (1.2.5) and Corollary 1.2.7. □

The functions on $\mathbb{S}^{d-1}$ that depend only on $\langle x,y\rangle$ are analogues of radial functions on $\mathbb{R}^d$. For such functions, there is a Funk–Hecke formula given below.

Theorem 1.2.9. *Let f be an integrable function such that $\int_{-1}^{1}|f(t)|(1-t^2)^{(d-3)/2}\mathrm{d}t$ is finite and $d \ge 2$. Then for every $Y_n \in \mathscr{H}_n^d$,*

$$\int_{\mathbb{S}^{d-1}} f(\langle x,y\rangle)Y_n(y)\mathrm{d}\sigma(y) = \lambda_n(f)Y_n(x), \quad x \in \mathbb{S}^{d-1}, \tag{1.2.11}$$

where $\lambda_n(f)$ is a constant defined by

$$\lambda_n(f) = \omega_{d-1}\int_{-1}^{1} f(t)\frac{C_n^{\frac{d-2}{2}}(t)}{C_n^{\frac{d-2}{2}}(1)}(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t.$$

Proof. If f is a polynomial of degree m, then we can expand f in terms of the Gegenbauer polynomials

$$f(t) = \sum_{k=0}^{m} \lambda_k \frac{k+\frac{d-2}{2}}{\frac{d-2}{2}} C_k^{\frac{d-2}{2}}(t),$$

where λ_k are determined by the orthogonality of the Gegenbauer polynomials,

$$\lambda_k = \frac{c_d}{C_k^{\frac{d-2}{2}}(1)}\int_{-1}^{1} f(t)C_k^{\frac{d-2}{2}}(t)(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t,$$

and $c_d^{-1} = \int_{-1}^{1}(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t = \omega_d/\omega_{d-1}$. From Eq. (1.2.7) and the reproducing property of $Z_n(x,y)$, it follows that for $n \le m$,

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(\langle x,y\rangle)Y_n(y)\mathrm{d}\sigma(y) = \lambda_n Y_n(x), \quad x \in \mathbb{S}^{d-1}.$$

Since $\lambda_n/\omega_d = \lambda_n(f)$ by definition, we have established the Funk–Hecke formula (1.2.11) for polynomials, and hence by the Weierstrass theorem, for continuous functions, and the function satisfying the integrable condition in the statement can be approximated by a sequence of continuous functions. □

1.3 Zonal Basis of Spherical Harmonics

In view of the addition formula of spherical harmonics, one may ask whether there is a basis of spherical harmonics that consists entirely of zonal harmonics. This question is closely related to the problem of interpolation on the sphere.

Throughout this section, we fix n, set $N = \dim \mathscr{H}_n^d$, and fix $\{Y_1, \ldots, Y_N\}$ as an orthonormal basis for $\mathscr{H}_n^d$. Let $\{x_1, \ldots, x_N\}$ be a collection of points on $\mathbb{S}^{d-1}$. We let $M_1 := Y_1(x_1)$ and for $k = 2, 3, \ldots, N$, define matrices

$$M_k := \begin{bmatrix} Y_1(x_1) & \ldots & Y_1(x_k) \\ \vdots & \ldots & \vdots \\ Y_k(x_1) & \ldots & Y_k(x_k) \end{bmatrix}, \quad M_k(x) := \left[\begin{array}{c|c} M_{k-1} & \begin{matrix} Y_1(x) \\ \vdots \\ Y_{k-1}(x) \end{matrix} \\ \hline Y_k(x_1), \ldots Y_k(x_{k-1}) & Y_k(x), \end{array}\right].$$

The product of M_N and its transpose M_N^T can be summed, on applying the addition formula (1.2.8), as $M_N^T M_N = [Z_n(x_i, x_j)]_{i,j=1}^N$, which shows, in particular,

$$\det[Z_n(x_i, x_j)]_{i,j=1}^N = (\det M_N)^2 \geq 0. \tag{1.3.1}$$

This motivates the following definition.

Definition 1.3.1. A collection of points $\{x_1, \ldots, x_N\}$ in $\mathbb{S}^{d-1}$ is called a fundamental system of degree n on the sphere $\mathbb{S}^{d-1}$ if

$$\det \left[C_n^\lambda(\langle x_i, x_j \rangle) \right]_{i,j=1}^N > 0, \qquad \lambda = \frac{d-2}{2}.$$

Lemma 1.3.2. *There exists a fundamental system of degree n on the sphere.*

Proof. The existence of a fundamental system follows from the linear independence of $\{Y_1, \ldots, Y_N\}$. Indeed, we can clearly choose $x_1 \in \mathbb{S}^{d-1}$ such that $\det M_1 = Y_1(x_1) \neq 0$. Assume that $x_1, \ldots, x_k$, $1 \leq k \leq N-1$, have been chosen such that $\det M_k \neq 0$. The determinant $\det M_{k+1}(x)$ is a polynomial of x and cannot be identically zero by the linear independence of $\{Y_1, \ldots, Y_{k+1}\}$, so that there is an $x_{k+1} \in \mathbb{S}^{d-1}$ such that $\det M_{k+1} = \det M_k(x_{k+1}) \neq 0$. In this way, we end up with a collection of points $\{x_1, \ldots, x_N\}$ on $\mathbb{S}^{d-1}$ that satisfies $\det M_N \neq 0$, which implies $\det \left[C_n^\lambda(\langle x_i, x_j \rangle) \right]_{i,j=1}^N = c_N (\det M_N)^2 > 0$, where $c_N = \lambda^N / (n+\lambda)^N$. □

The proof shows, in fact, that there are infinitely many fundamental systems. Indeed, regarding $x_1, \ldots, x_N$ as variables, we see that $\det M_N$ is a $(d-1)N$-dimensional polynomial in these variables, and its zero set is an algebraic surface of $\mathbb{R}^{(d-1)N}$, which necessarily has measure zero.

Theorem 1.3.3. *If $\{x_1, \ldots, x_N\}$ is a fundamental system of points on the sphere, then $\{C_n^\lambda(\langle \cdot, x_i \rangle) : i = 1, 2, \ldots, N\}$, $\lambda = \frac{d-2}{2}$, is a basis of $\mathscr{H}_n^d|_{\mathbb{S}^{d-1}}$.*

Proof. Let $P_i(x) = \frac{n+\lambda}{\lambda} C_n^\lambda(\langle \cdot, x_i \rangle)$. The addition theorem (1.2.8) gives

$$P_i(x) = \sum_{k=1}^N Y_k(x_i) Y_k(x), \qquad i = 1, 2, \ldots, N,$$

which shows that $\{P_1,\ldots,P_N\}$ is expressed in the basis $\{Y_1,\ldots,Y_N\}$ with transition matrix given by $M_N = [Y_k(x_i)]_{k,i=1}^N$. Since $\{x_1,\ldots,x_N\}$ is fundamental, the matrix is invertible, by Eq. (1.3.1). We can then invert the system to express Y_k as a linear combination of $P_1,\ldots,P_N$, which completes the proof. □

A word of caution is in order. The polynomial $C_n^\lambda(\langle x,x_i\rangle)$ is, for $x\in\mathbb{S}^{d-1}$, a linear combination of the spherical harmonics according to the addition formula. It is not, however, a homogeneous polynomial of degree n in $x\in\mathbb{R}^d$; rather, it is the restriction of the homogeneous polynomial $\|x\|^n C_n^\lambda(\langle x/\|x\|,y\rangle)$ to the sphere. This is a situation in which the distinction between $\mathscr{H}_n^d$ and $\mathscr{H}_n^d|_{\mathbb{S}^{d-1}}$ is called for; see the discussion below Definition 1.1.1.

Fundamental sets of points are closely related to the problem of interpolation. Indeed, it can be stated as follows: for a given set of data $\{(x_j,y_j): 1\le j\le N\}$, $x_j\in\mathbb{S}^{d-1}$ and $y_j\in\mathbb{R}$, there is a unique element $Y\in\mathscr{H}_n^d$ such that $Y(x_j)=y_j$, $j=1,\ldots,N$, if and only if the points $\{x_1,\ldots,x_N\}$ form a fundamental system on the sphere. Much more interesting and challenging is the problem of choosing points in such a way that the resulting basis of zonal spherical harmonics has a relatively simple structure.

A related result is a zonal basis for the space $\mathscr{P}_n$ of homogeneous polynomials of degree n and the space Π_n^d of all polynomials of degree at most n.

Theorem 1.3.4. *There exist points $\xi_{j,n}\in\mathbb{S}^{d-1}$, $1\le j\le r_n^d := \dim\mathscr{P}_n^d$, such that*

- (*i*) *$\{\langle x,\xi_{j,n}\rangle^n : 1\le j\le r_n^d\}$ is a basis for $\mathscr{P}_n^d$ of degree n.*
- (*ii*) *For each polynomial $f\in\Pi_n^d$, there exist polynomials $p_j:[-1,1]\mapsto\mathbb{R}$ for $1\le j\le r_n^d$ such that*

$$f(x)=\sum_{j=1}^{r_n^d} p_j(\langle x,\xi_{j,n}\rangle).$$

Proof. Following the proof of the existence of a fundamental system of points, it is easy to see that there exist points $\xi_{j,n}$ such that $f(\xi_{j,n})=0$ if and only if $f=0$ for all $f\in\mathscr{P}_n^d$. For the proof of (i), we first deduce by the binomial formula that

$$f_{j,n}(x) := \langle x,\xi_{j,n}\rangle^n = \sum_{|\alpha|=n}\frac{n!}{\alpha!}\xi_{j,n}^\alpha x^\alpha, \quad 1\le j\le r_n^d.$$

Let $f\in\mathscr{P}_n^d$. Then $f(x)=\sum_{|\alpha|=n} a_\alpha x^\alpha$. By Eq. (1.1.8),

$$\langle f,f_{j,n}\rangle_\partial = n!\sum_{|\alpha|=n} a_\alpha \xi_{j,n}^\alpha = n! f(\xi_{j,n}),$$

which implies, by the choice of $\xi_{j,n}$, that $\langle f,f_{j,n}\rangle_\partial=0$, $1\le j\le r_n^d$, if and only if $f=0$. Thus $\{f_{j,n}\}^\perp=\{0\}$. Since $f_{j,n}\in\mathscr{P}_n^d$. This proves (i).

For the proof of (ii), let m be an integer, $0\le m\le n-1$, and let $f_{j,m,n}:=\langle x,\xi_{j,n}\rangle^m$. For $f\in\mathscr{P}_m^d$, it follows as above that $\langle f,f_{j,m,n}\rangle_\partial = m! f(\xi_{j,m})$, $1\le j\le r_n^d$. Hence if

$\langle f, f_{j,m,n} \rangle_\partial = 0$ for $1 \le j \le r_n^d$, then $f(\xi_{j,m}) = (fg)(\xi_{j,m}) = 0$ for $1 \le j \le r_n^d$ and all $g \in \mathscr{P}_{n-m}^d$, which implies, by the choice of $\xi_{j,n}$ and the fact that $fg \in \mathscr{P}_n^d$, that $fg = 0$ or $f = 0$. Consequently, $\mathscr{P}_m^d = \operatorname{span}\{f_{j,m,n} : 0 \le j \le r_n^d\}$ for $0 \le m \le n$. Since $\Pi_n^d = \sum_{m=0}^n \mathscr{P}_m^d$, this proves (ii). □

1.4 Laplace–Beltrami Operator

The operator in the section heading is the spherical part of the Laplace operator, which we denote by Δ_0. The operator Δ_0 plays an important role in analysis on the sphere. The usual approach to deriving this operator relies on an expression of the Laplace operator under a change of variables, which we describe first.

For $x \in \mathbb{R}^d$, let $x \mapsto u = u(x)$ be a change of variables that is a bijection, so that we can also write $x = x(u)$. Introduce the tensors

$$g_{i,j} := \sum_{k=1}^d \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} \quad \text{and} \quad g^{i,j} := \sum_{k=1}^d \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}, \quad 1 \le i, j \le d,$$

and let $g := \det(g_{i,j})_{i,j=1}^d$. Then $(g_{i,j})^{-1} = (g^{i,j})$. A general result in Riemannian geometry, or a bit of tensor analysis, shows that the Laplace operator satisfies

$$\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} = \frac{1}{\sqrt{g}} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial}{\partial u_i} \sqrt{g} g^{i,j} \frac{\partial}{\partial u_j}. \tag{1.4.1}$$

The Laplace–Beltrami operator, i.e., the spherical part of the Laplace operator, can then be derived from Eq. (1.4.1) by the change of variables $x \mapsto (r, \xi_1, \ldots, \xi_{d-1})$, where $r > 0$ and $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{S}^{d-1}$. For this approach and a derivation of Eq. (1.4.1), see [125]. We shall adopt an approach that is elementary and self-contained.

Lemma 1.4.1. *In the spherical–polar coordinates $x = r\xi$, $r > 0$, $\xi \in \mathbb{S}^{d-1}$, the Laplace operator satisfies*

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0, \tag{1.4.2}$$

where

$$\Delta_0 = \sum_{i=1}^{d-1} \frac{\partial^2}{\partial \xi_i^2} - \sum_{i=1}^{d-1}\sum_{j=1}^{d-1} \xi_i \xi_j \frac{\partial^2}{\partial \xi_i \partial \xi_j} - (d-1)\sum_{i=1}^{d-1} \xi_i \frac{\partial}{\partial \xi_i}. \tag{1.4.3}$$

Proof. Since $\xi \in \mathbb{S}^{d-1}$, we have $\xi_1^2 + \cdots + \xi_d^2 = 1$. We evaluate the Laplacian Δ under a change of variables $(x_1, \ldots, x_d) \mapsto (r, \xi_1, \ldots, \xi_{d-1})$ under $x = r\xi$, which has inverse $\xi_1 = x_1/\|x\|, \ldots, \xi_{d-1} = x_{d-1}/\|x\|, r = \|x\|$. The chain rule leads to

$$\frac{\partial}{\partial x_i} = \frac{1}{r}\frac{\partial}{\partial \xi_i} - \frac{\xi_i}{r}\sum_{j=1}^{d-1}\xi_j\frac{\partial}{\partial \xi_j} + \xi_i\frac{\partial}{\partial r}, \quad 1 \le i \le d-1,$$

$$\frac{\partial}{\partial x_d} = -\frac{x_d}{r^2}\sum_{j=1}^{d-1}\xi_j\frac{\partial}{\partial \xi_j} + \frac{x_d}{r}\frac{\partial}{\partial r}. \tag{1.4.4}$$

If we apply the product rule for the partial derivative on x_d, it follows that

$$\Delta = \sum_{i=1}^{d-1}\left(\frac{1}{r}\frac{\partial}{\partial \xi_i} - \frac{\xi_i}{r}\sum_{j=1}^{d-1}\xi_j\frac{\partial}{\partial \xi_j} + \xi_i\frac{\partial}{\partial r}\right)^2$$
$$+ \xi_d^2\left(-\frac{1}{r}\sum_{j=1}^{d-1}\xi_j\frac{\partial}{\partial \xi_j} + \frac{\partial}{\partial r}\right)^2 + (1-\xi_d^2)\left(-\frac{1}{r^2}\sum_{j=1}^{d-1}\xi_j\frac{\partial}{\partial \xi_j} + \frac{1}{r}\frac{\partial}{\partial r}\right),$$

where we have used $x_d = r\xi_d$, from which a straightforward, though tedious, computation, and simplification using $\xi_1^2 + \cdots + \xi_d^2 = 1$, establishes Eqs. (1.4.2) and (1.4.3). □

The Laplace–Beltrami operator also satisfies a recurrence relation that can be used to derive an explicit formula for Δ_0 under a given coordinate system of $\mathbb{S}^{d-1}$. We write $\Delta_{0,d}$ instead of Δ_0 when we need to emphasize the dimension.

Lemma 1.4.2. *Let $\Delta_{0,d}$ be the Laplace–Beltrami operator for $\mathbb{S}^{d-1}$. For $\xi \in \mathbb{S}^{d-1}$, write $\xi = (\sqrt{1-t^2}\eta, t)$ with $-1 \le t \le 1$ and $\eta \in \mathbb{S}^{d-2}$. Then*

$$\Delta_{0,d} = \frac{1}{(1-t^2)^{\frac{d-3}{2}}}\frac{\partial}{\partial t}\left((1-t^2)^{\frac{d-1}{2}}\frac{\partial}{\partial t}\right) + \frac{1}{1-t^2}\Delta_{0,d-1}. \tag{1.4.5}$$

Proof. We work with the expression for $\Delta_{0,d}$ in Eq. (1.4.3) and make a change of variables $(\xi_1,\ldots,\xi_{d-1}) \mapsto (\eta_1,\ldots,\eta_{d-2},t)$ defined by

$$\xi_1 = \sqrt{1-t^2}\eta_1, \quad \ldots, \quad \xi_{d-2} = \sqrt{1-t^2}\eta_{d-2}, \quad \xi_{d-1} = t,$$

where we have switched ξ_{d-1} and ξ_d for convenience. The chain rule gives

$$\frac{\partial}{\partial \xi_i} = \frac{1}{\sqrt{1-t^2}}\frac{\partial}{\partial \eta_i}, \quad 1 \le i \le d-2, \quad \text{and} \quad \frac{\partial}{\partial \xi_{d-1}} = \frac{t}{1-t^2}\sum_{j=1}^{d-2}\eta_j\frac{\partial}{\partial \eta_j} + \frac{\partial}{\partial t},$$

which can be used iteratively to compute $\Delta_{0,d}$ on writing Eq. (1.4.3) as

$$\Delta_{0,d} = (1-t^2)\frac{\partial^2}{\partial \xi_{d-1}^2} + \sum_{i=1}^{d-2}\frac{\partial^2}{\partial \xi_i^2} - 2t\sum_{i=1}^{d-2}\xi_i\frac{\partial^2}{\partial \xi_i \partial \xi_{d-1}}$$
$$- \sum_{i=1}^{d-2}\sum_{j=1}^{d-2}\xi_i\xi_j\frac{\partial^2}{\partial \xi_i \partial \xi_j} - (d-1)\sum_{i=1}^{d-1}\frac{\partial}{\partial \xi_i}.$$

A straightforward computation and another use of Eq. (1.4.3) then leads to

$$\Delta_{0,d} = (1-t^2)\frac{\partial^2}{\partial t^2} - (d-1)t\frac{\partial}{\partial t} + \frac{1}{1-t^2}\Delta_{0,d-1},$$

which is precisely Eq. (1.4.5). □

The formula (1.4.3) gives an explicit expression for Δ_0 in the local coordinates of $\mathbb{S}^{d-1}$. An explicit formula for Δ_0 in terms of spherical coordinates will be given in Sect. 1.5.

Let $\nabla = (\partial_1, \ldots, \partial_d)$. The proof of Lemma 1.4.1 also shows that

$$\nabla = \frac{1}{r}\nabla_0 + \xi\frac{\partial}{\partial r}, \qquad x = r\xi, \quad \xi \in \mathbb{S}^{d-1}, \tag{1.4.6}$$

where ∇_0 is the spherical gradient, which is the spherical part of ∇ and involves only derivatives in ξ. Its explicit expression can be read off from Eq. (1.4.4). We shall not need this expression and will be content with the following expression.

Corollary 1.4.3. *Let $f \in C^2(\mathbb{S}^{d-1})$. Define $F(y) := f(y/\|y\|)$, $y \in \mathbb{R}^d$. Then*

$$\Delta_0 f(x) = \Delta F(x) \quad \text{and} \quad \nabla_0 f(x) = \nabla F(x), \quad x \in \mathbb{S}^{d-1}. \tag{1.4.7}$$

The corollary follows immediately from Eqs. (1.4.2) and (1.4.6), since $x/\|x\|$ is independent of r. The expressions in Eq. (1.4.7) show that Δ_0 and ∇_0 are independent of the coordinates of $\mathbb{S}^{d-1}$. In fact, we could take Eq. (1.4.7) as the definition of Δ_0 and ∇_0.

The usual Laplacian Δ can be expressed in terms of the dot product of ∇, $\Delta = \nabla\cdot\nabla$, which can also be written—as is often done in physics textbooks—as $\Delta = \nabla^2$. The analogue of this identity also holds on the sphere.

Lemma 1.4.4. *The Laplace–Beltrami operator satisfies*

$$\Delta_0 = \nabla_0 \cdot \nabla_0. \tag{1.4.8}$$

Proof. An application of Eq. (1.4.6) gives immediately

$$\Delta = \nabla\cdot\nabla = \frac{1}{r^2}\nabla_0\cdot\nabla_0 + \frac{1}{r}\nabla_0\left(\xi\frac{\partial}{\partial r}\right) + \xi\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla_0\right) + \frac{\partial^2}{\partial r^2}.$$

We note that $\xi\cdot\nabla_0 f(\xi) = 0$, since $\xi \in \mathbb{S}^{d-1}$ is in the normal direction of ξ, whereas $\nabla_0 f(\xi)$, by Eq. (1.4.6), is on the tangent plane at ξ. Hence, we see that

$$\xi\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla_0\right) = -\frac{1}{r^2}\xi\cdot\nabla_0 + \frac{1}{r}\xi\cdot\nabla_0\frac{\partial}{\partial r} = 0.$$

Using Eq. (1.4.6), a quick computation gives $\nabla_0 \cdot \xi = d-1$, so that by the product rule,

$$\nabla_0 \left(\xi \frac{\partial}{\partial r} \right) = \nabla_0 \cdot \xi \frac{\partial}{\partial r} + \frac{\partial}{\partial r} \xi \cdot \nabla_0 = (d-1)\frac{\partial}{\partial r}.$$

Consequently, we conclude that

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\nabla_0 \cdot \nabla_0.$$

Comparing this with Eq. (1.4.2) completes the proof. □

Our next result shows that the spherical harmonics are eigenfunctions of the Laplace–Beltrami operator.

Theorem 1.4.5. *The spherical harmonics are eigenfunctions of Δ_0,*

$$\Delta_0 Y(\xi) = -n(n+d-2)Y(\xi), \quad \forall Y \in \mathscr{H}_n^d, \quad \xi \in \mathbb{S}^{d-1}. \tag{1.4.9}$$

Proof. Let $x = r\xi$, $\xi \in \mathbb{S}^{d-1}$. Since $Y \in \mathscr{H}_n^d$ is homogeneous, $Y(x) = r^n Y(\xi)$, and by Eq. (1.4.2),

$$0 = \Delta Y(x) = n(n-1)r^{n-2}Y(\xi) + (d-1)nr^{n-2}Y(\xi) + r^{n-2}\Delta_0 Y(\xi),$$

which is Eq. (1.4.9) upon dividing by r^{n-2}. □

The identity (1.4.9) also implies that Δ_0 is self-adjoint, which can also be proved directly and will be treated in the last section of this chapter, together with a number of other properties of the Laplace–Beltrami operator.

1.5 Spherical Harmonics in Spherical Coordinates

The polar coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$, $r \ge 0$, $0 \le \theta \le 2\pi$, give coordinates for $\mathbb{S}^1$ when $r = 1$. The high-dimensional analogue is the spherical polar coordinates defined by

$$\begin{cases} x_1 = r\sin\theta_{d-1} \ldots \sin\theta_2 \sin\theta_1, \\ x_2 = r\sin\theta_{d-1} \ldots \sin\theta_2 \cos\theta_1, \\ \cdots \\ x_{d-1} = r\sin\theta_{d-1}\cos\theta_{d-2}, \\ x_d = r\cos\theta_{d-1}, \end{cases} \tag{1.5.1}$$

where $r \ge 0$, $0 \le \theta_1 \le 2\pi$, $0 \le \theta_i \le \pi$ for $i = 2, \dots, d-1$. When $r = 1$, these are the coordinates for the unit sphere $\mathbb{S}^{d-1}$, and they are in fact defined recursively by

$$x = (\xi \sin\theta_{d-1}, \cos\theta_{d-1}) \in \mathbb{S}^{d-1}, \qquad \xi \in \mathbb{S}^{d-2}.$$

Let $\mathrm{d}\sigma = \mathrm{d}\sigma_d$ be Lebesgue measure on $\mathbb{S}^{d-1}$. Then it is easy to verify that

$$\mathrm{d}\sigma_d(x) = (\sin\theta_{d-1})^{d-2}\mathrm{d}\theta_{d-1}\mathrm{d}\sigma_{d-1}(\xi). \tag{1.5.2}$$

Since the Lebesgue measure of $\mathbb{S}^1$ is $\mathrm{d}\theta_1$, it follows by induction that

$$\mathrm{d}\sigma = \mathrm{d}\sigma_d = \prod_{j=1}^{d-2}\left(\sin\theta_{d-j}\right)^{d-j-1}\mathrm{d}\theta_{d-1}\dots\mathrm{d}\theta_2\mathrm{d}\theta_1 \tag{1.5.3}$$

in the spherical coordinates (1.5.1). Furthermore, Eq. (1.5.2) shows that

$$\int_{\mathbb{S}^{d-1}} f(x)\mathrm{d}\sigma_d(x) = \int_0^\pi \int_{\mathbb{S}^{d-2}} f(\xi\sin\theta, \cos\theta)\mathrm{d}\sigma_{d-1}(\xi)(\sin\theta)^{d-2}\mathrm{d}\theta. \tag{1.5.4}$$

The orthogonality (1.2.10) of the Gegenbauer polynomials can be written as

$$\int_0^\pi C_n^\lambda(\cos\theta)C_m^\lambda(\cos\theta)(\sin\theta)^{d-2}\mathrm{d}\theta = \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})}h_n^\lambda\delta_{m,n}, \quad \lambda = \frac{d-2}{2}. \tag{1.5.5}$$

Together with Eq. (1.5.4), this allows us to write down a basis of spherical harmonics in terms of the Gegenbauer polynomials in the spherical coordinates.

Theorem 1.5.1. *For $d > 2$ and $\alpha \in \mathbb{N}_0^d$, define*

$$Y_\alpha(x) := [h_\alpha]^{-1} r^{|\alpha|} g_\alpha(\theta_1) \prod_{j=1}^{d-2} (\sin\theta_{d-j})^{|\alpha^{j+1}|} C_{\alpha_j}^{\lambda_j}(\cos\theta_{d-j}), \tag{1.5.6}$$

where $g_\alpha(\theta_1) = \cos\alpha_{d-1}\theta_1$ for $\alpha_d = 0$, $\sin\alpha_{d-1}\theta_1$ for $\alpha_d = 1$, $|\alpha^j| = \alpha_j + \cdots + \alpha_{d-1}$, $\lambda_j = |\alpha^{j+1}| + (d-j-1)/2$, and

$$[h_\alpha]^2 := b_\alpha \prod_{j=1}^{d-2} \frac{\alpha_j!(\frac{d-j+1}{2})_{|\alpha^{j+1}|}(\alpha_j + \lambda_j)}{(2\lambda_j)_{\alpha_j}(\frac{d-j}{2})_{|\alpha^{j+1}|}\lambda_j},$$

in which $b_\alpha = 2$ if $\alpha_{d-1} + \alpha_d > 0$, while $b_\alpha = 1$ otherwise. Then $\{Y_\alpha : |\alpha| = n, \alpha_d = 0, 1\}$ is an orthonormal basis of $\mathscr{H}_n^d$; that is, $\langle Y_\alpha, Y_\beta\rangle_{\mathbb{S}^{d-1}} = \delta_{\alpha,\beta}$.

Proof. To see that Y_α is a homogeneous polynomial, we use by Eq. (1.5.1) the relation $\cos\theta_k = x_{k+1}/\sqrt{x_1^2 + \cdots + x_{k+1}^2}$ for $1 \le k \le d-1$ to rewrite Eq. (1.5.6) as

$$Y_\alpha(x) = [h_\alpha]^{-1} g(x) \prod_{j=1}^{d-2} (x_1^2 + \cdots + x_{d-j+1}^2)^{\alpha_j/2} C_{\alpha_j}^{\lambda_j} \left(\frac{x_{d-j+1}}{\sqrt{x_1^2 + \cdots + x_{d-j+1}^2}} \right),$$

where $g(x) = \rho^{\alpha_{d-1}} \cos \alpha_{d-1}\theta_1$ for $\alpha_d = 0$, $\rho^{\alpha_{d-1}} \cos \alpha_{d-1}\theta_1$ for $\alpha_d = 1$, with $\rho = \sqrt{x_1^2 + x_2^2}$. Since $x_1 = \rho \sin \theta_1$ and $x_2 = \rho \cos \theta_1$ by Eq. (1.5.1), $g(x)$ is either the real part or the imaginary part of $(x_2 + ix_1)^{\alpha_{d-1}}$, which shows that it is a homogeneous polynomial of degree α_{d-1} in x. Since $C_n^\lambda(t)$ is even when n is even, and odd when n is odd, we see that $Y_\alpha \in \mathscr{P}_n^d$. Using Eq. (1.5.4), we see that

$$\begin{aligned} \langle Y_\alpha, Y_{\alpha'} \rangle_{\mathbb{S}^{d-1}} = & \frac{h_\alpha^{-1} h_{\alpha'}^{-1}}{\omega_d} \int_0^{2\pi} g_\alpha(\theta_1) g_{\alpha'}(\theta_1) \mathrm{d}\theta_1 \\ & \times \prod_{j=1}^{d-2} \int_0^\pi C_{\alpha_j}^{\lambda_j}(\cos \theta_{d-j}) C_{\alpha_j'}^{\lambda_j}(\cos \theta_{d-j}) (\sin \theta_{d-j})^{2\lambda_j} \mathrm{d}\theta_{d-j}, \end{aligned}$$

from which the orthogonality follows from the orthogonality of the Gegenbauer polynomials (1.5.4) and that of $\cos m\theta$ and $\sin m\theta$ on $[0, 2\pi)$, and the formula for h_α follows from the normalizing constant of the Gegenbauer polynomial. □

For $d = 2$ and the polar coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$, it is easy to see that $\nabla_0 = \partial_\theta$, where $\partial_\theta = \partial/\partial\theta$. Hence by Eq. (1.4.8), the Laplace–Beltrami operator for $d = 2$ is $\Delta_0 = \partial_\theta^2$. Using Eq. (1.4.5) iteratively, we see that the Laplace–Beltrami operator Δ_0 has an explicit formula in the spherical coordinates (1.5.1),

$$\begin{aligned} \Delta_0 = & \frac{1}{\sin^{d-2}\theta_{d-1}} \frac{\partial}{\partial \theta_{d-1}} \left[\sin^{d-2}\theta_{d-1} \frac{\partial}{\partial \theta_{d-1}} \right] \\ & + \sum_{j=1}^{d-2} \frac{1}{\sin^2\theta_{d-1} \dots \sin^2\theta_{j+1} \sin^{j-1}\theta_j} \frac{\partial}{\partial \theta_j} \left[\sin^{j-1}\theta_j \frac{\partial}{\partial \theta_j} \right]. \end{aligned} \tag{1.5.7}$$

1.6 Spherical Harmonics in Two and Three Variables

Since spherical harmonics in two and three variables are used most often in applications, we state their properties in this section.

1.6.1 Spherical Harmonics in Two Variables

For $d = 2$, $\dim \mathscr{H}_n^2 = 2$. An orthogonal basis of $\mathscr{H}_n^2$ is given by the real and imaginary parts of $(x_1 + ix_2)^n$, since both are homogeneous of degree n and are harmonic as the real and imaginary parts of an analytic function. In polar coordinates $(x_1,x_2) = (r\cos\theta, r\sin\theta)$ of $\mathbb{R}^2$, this basis is given by

$$Y_n^{(1)}(x) = r^n \cos n\theta, \qquad Y_n^{(2)}(x) = r^n \sin n\theta. \tag{1.6.1}$$

Hence, restricting to the circle $\mathbb{S}^1$, the spherical harmonics are precisely the cosine and sine functions. In particular, spherical harmonic expansions on $\mathbb{S}^1$ are the classical Fourier expansions in cosine and sine functions.

As homogeneous polynomials, the basis (1.6.1) is given explicitly in terms of the Chebyshev polynomials T_n and U_n defined by

$$T_n(t) = \cos n\theta \quad \text{and} \quad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{where} \quad t = \cos\theta,$$

which are related to the Gegenbauer polynomials: $U_n(t) = C_n^1(t)$ and

$$\lim_{\lambda\to 0+} \frac{1}{\lambda} C_n^\lambda(x) = \frac{2}{n} T_n(x).$$

The basis in Eq. (1.6.1) can be rewritten then as

$$Y_n^{(1)}(x) = r^n T_n\left(\frac{x_1}{r}\right), \quad Y_n^{(2)}(x) = r^{n-1} x_2 U_{n-1}\left(\frac{x_1}{r}\right), \tag{1.6.2}$$

which shows explicitly that these are homogeneous polynomials, since $r = \sqrt{x_1^2 + x_2^2}$ and both $T_n(t)$ and $U_n(t)$ are even if n is even, odd if n is odd.

When $d = 2$, the zonal polynomial is given by $T_n(\langle x,y\rangle) = \cos n(\theta - \phi)$, and the addition formula (1.2.8) becomes, by Eq. (1.6.1), the addition formula

$$\cos n\theta \cos n\phi + \sin n\theta \sin n\phi = \cos n(\theta - \phi).$$

The expression (1.4.2) of the Laplace operator in polar coordinates becomes

$$\Delta = \frac{\mathrm{d}^2}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2};$$

in particular, the Laplace–Beltrami operator on $\mathbb{S}^1$ is simply $\Delta_0 = \mathrm{d}^2/\mathrm{d}\theta^2$.

1.6.2 Spherical Harmonics in Three Variables

The space $\mathscr{H}_n^3$ of spherical harmonics of degree n has dimension $2n+1$. For $d = 3$, the spherical polar coordinates (1.5.1) are written as

$$\begin{cases} x_1 = r\sin\theta\sin\phi, \\ x_2 = r\sin\theta\cos\phi, \\ x_3 = r\cos\theta, \end{cases} \qquad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi, \quad r > 0. \tag{1.6.3}$$

The surface area of $\mathbb{S}^2$ is 4π, and the integral over $\mathbb{S}^{d-1}$ is parameterized by

$$\int_{\mathbb{S}^2} f(x)\mathrm{d}\sigma = \int_0^{\pi}\int_0^{2\pi} f(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)\mathrm{d}\phi\,\sin\theta\mathrm{d}\theta. \tag{1.6.4}$$

The orthogonal basis (1.5.6) in spherical coordinates becomes

$$\begin{aligned} Y_{k,1}^n(\theta,\phi) &= (\sin\theta)^k C_{n-k}^{k+\frac{1}{2}}(\cos\theta)\cos k\phi, \quad 0 \le k \le n, \\ Y_{k,2}^n(\theta,\phi) &= (\sin\theta)^k C_{n-k}^{k+\frac{1}{2}}(\cos\theta)\sin k\phi, \quad 1 \le k \le n. \end{aligned} \tag{1.6.5}$$

Their $L^2(\mathbb{S}^2)$ norms can be deduced from Eq. (1.5.6). This basis is often written in terms of the associated Legendre polynomials $P_n^k(t)$ defined by

$$P_n^k(x) := (-1)^n(1-x^2)^{k/2}\frac{\mathrm{d}^k}{\mathrm{d}x^k}P_n(x) = (2k-1)!!(-1)^n(1-x^2)^{k/2}C_{n-k}^{k+1/2}(x),$$

where $P_n(t) = C_n^0(t)$ denotes the Legendre polynomial of degree n (see Appendix B for properties of P_n and P_n^k), and in terms of $\{e^{ik\phi}, e^{-ik\phi}\}$ instead of $\{\cos k\phi, \sin k\phi\}$. In this way, an orthonormal basis of $\mathscr{H}_n^3$ is given by

$$Y_{k,n}(\theta,\phi) = \left(\frac{(2n+1)(n-k)!}{(n+k)!}\right)^{1/2} P_n^k(\cos\theta)e^{ik\phi}, \quad -n \le k \le n. \tag{1.6.6}$$

The addition formula (1.2.8) then reads, assuming that x and y have spherical coordinates (θ,ϕ) and (θ',ϕ'), respectively,

$$\sum_{k=-n}^{n} Y_{k,n}(\theta,\phi)Y_{k,n}(\theta',\phi') = (2n+1)P_n(\langle x,y\rangle). \tag{1.6.7}$$

In terms of the coordinates (1.6.3), the Laplace–Beltrami operator is given by

$$\Delta_0 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{1.6.8}$$

as seen from Eq. (1.5.7).

1.7 Representation of the Rotation Group

In this section, we show that the representation of the group $SO(d)$ in spaces of harmonic polynomials is irreducible.

A representation of $SO(d)$ is a homomorphism from $SO(d)$ to the group of nonsingular continuous linear transformations of $L^2(\mathbb{S}^{d-1})$. We associate with each element $Q \in SO(d)$ an operator $T(Q)$ in the space of $L^2(\mathbb{S}^{d-1})$, defined by

$$T(Q)f(x) = f(Q^{-1}x), \qquad x \in \mathbb{S}^{d-1}. \tag{1.7.1}$$

Evidently, for each $Q \in SO(d)$, $T(Q)$ is a nonsingular linear transformation of $L^2(\mathbb{S}^{d-1})$ and T is a homomorphism,

$$T(Q_1Q_2) = T(Q_1)T(Q_2), \quad \forall Q_1, Q_2 \in SO(d).$$

Thus, T is a representation of $SO(d)$. Since $\mathrm{d}\sigma$ is invariant under rotations, $\|T(Q)f\|_2 = \|f\|_2$ in the $L^2(\mathbb{S}^{d-1})$ norm, so that $T(Q)$ is unitary.

A linear space $\mathscr{U}$ is called invariant under T if $T(Q)$ maps $\mathscr{U}$ to itself for all $Q \in SO(d)$. The null space and $L^2(\mathbb{S}^{d-1})$ itself are trivial invariant subspaces. A representation T is called irreducible if it has only trivial invariant subspaces. The space $\mathscr{H}_n^d$ is an invariant subspace of T in Eq. (1.7.1).

Let $T_{n,d}$ denote the representation of $SO(d)$ corresponding to T in the invariant subspace $\mathscr{H}_n^d$. We want to show that $T_{n,d}$ is irreducible.

Lemma 1.7.1. *A spherical harmonic $Y \in \mathscr{H}_n^d$ is invariant under all rotations in $SO(d)$ that leaves x_d fixed if and only if*

$$Y(x) = c\|x\|^n C_n^\lambda\left(\frac{x_d}{\|x\|}\right), \qquad \lambda = \frac{d-2}{2}, \tag{1.7.2}$$

where c is a constant.

Proof. If Y is invariant under rotations that fix x_d and is a homogeneous polynomial of degree n, then it can be written as

$$Y(x) = \sum_{0 \le j \le n/2} b_j x_d^{n-2j}(x_1^2 + \cdots + x_{d-1}^2)^j = \sum_{0 \le j \le n/2} c_j x_d^{n-2j}\|x\|^{2j},$$

where the second equal sign follows from expanding $(\|x\|^2 - x_d^2)^j$ and changing the order of summation. Since Y is harmonic, computing $\Delta Y(x) = 0$ shows that c_j satisfies the recurrence relation

$$4(j+1)(n-j-1)c_{j+1} + (n-2j)(n-2j-1)c_j = 0.$$

Solving the recurrence equation for c_j, we conclude that

$$Y(x) = c_0 \sum_{0 \le j \le n/2} \frac{(-\frac{n}{2})_j(\frac{1-n}{2})_j}{j!(1-n-d-2)_j} x_d^{n-2j}\|x\|^{2j}.$$

Consequently, Eq. (1.7.2) follows from the formula (B.2.5) for the Gegenbauer polynomials. Since the function Y in Eq. (1.7.2) is clearly invariant under all rotations that fix x_d and we have just shown that it is harmonic, the proof is complete. □

Theorem 1.7.2. *The representation $T_{n,d}$ of $SO(d)$ on $\mathscr{H}_n^d$ is irreducible.*

Proof. Assume that $\mathscr{U}$ is an invariant subspace of $\mathscr{H}_n^d$ and $\mathscr{U}$ is not a null space. Let $\{Y_j : 1 \le j \le M\}$, $M \le \dim \mathscr{H}_n^d$, be an orthonormal basis of $\mathscr{U}$. Following the proof of Lemma 1.2.5, there is a polynomial $F(t)$ of one variable such that $\sum_{j=1}^M Y_j(x)Y_j(y) = F(\langle x,y\rangle)$. In particular, setting $y = e_d = (0,\dots,0,1)$ shows that $F(\langle x,e_d\rangle)$ is in $\mathscr{H}_n^d$, and it is evidently invariant under rotations in $SO(d)$ that fix x_d. Hence, by Lemma 1.7.1, $F(\langle x,e_d\rangle) = c\|x\|^n C_n^\lambda(\frac{x_d}{\|x\|})$. In particular, this shows that $\|x\|^n C_n^\lambda(\frac{x_d}{\|x\|}) \in \mathscr{U}$. On the other hand, let $\mathscr{U}^\perp$ denote the orthogonal complement of $\mathscr{U}$ in $\mathscr{H}_n^d$. If $f \in \mathscr{U}^\perp$ and $g \in \mathscr{U}$, then $\langle T(Q)f,g\rangle_{\mathbb{S}^{d-1}} = \langle f, T(Q^{-1})g\rangle_{\mathbb{S}^{d-1}} = 0$, which shows that $\mathscr{U}^\perp$ is also an invariant subspace of $\mathscr{H}_n^d$. Applying the same argument as for $\mathscr{U}$ shows then $\|x\|^n C_n^\lambda(\frac{x_d}{\|x\|}) \in \mathscr{U}^\perp$, which contradicts $\mathscr{U} \cap \mathscr{U}^\perp = \{0\}$. Thus, $\mathscr{U}$ must be trivial. □

1.8 Angular Derivatives and the Laplace–Beltrami Operator

Consider the case $d = 2$ and the polar coordinates $(x_1,x_2) = (r\cos\theta, r\sin\theta)$. Let $\partial_r = \partial/\partial r$ and $\partial_\theta = \partial/\partial\theta$, while we retain ∂_1 and ∂_2 for the partial derivatives with respect to x_1 and x_2. It then follows that

$$\partial_1 = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta, \quad \partial_2 = \sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta.$$

From these relations it follows easily that the angular derivative ∂_θ can also be written as $\partial_\theta = x_1\partial_2 - x_2\partial_1$, and the operator Δ_0 is $\Delta_0 = \partial_\theta^2$. We introduce angular derivatives in higher dimensions as follows.

Definition 1.8.1. For $x \in \mathbb{R}^d$ and $1 \le i \ne j \le d$, define

$$D_{i,j} := x_i\partial_j - x_j\partial_i = \frac{\partial}{\partial\theta_{i,j}}, \tag{1.8.1}$$

where $\theta_{i,j}$ is the angle of polar coordinates in the (x_i,x_j)-plane, defined by $(x_i,x_j) = r_{i,j}(\cos\theta_{i,j}, \sin\theta_{i,j})$, $r_{i,j} \ge 0$, and $0 \le \theta_{i,j} \le 2\pi$.

By its definition with partial derivatives on $\mathbb{R}^d$, $D_{i,j}$ acts on $\mathbb{R}^d$, yet the second equality in Eq. (1.8.1) shows that it acts on the sphere $\mathbb{S}^{d-1}$. Thus, for f defined on $\mathbb{R}^d$,

$$(D_{i,j}f)(\xi) = D_{i,j}[f(\xi)], \qquad \xi \in \mathbb{S}^{d-1}, \tag{1.8.2}$$

where the right-hand side means that $D_{i,j}$ is acting on $f(\xi)$.

Since $D_{j,i} = -D_{i,j}$, the number of distinct operators $D_{i,j}$ is $\binom{d}{2}$. The operator Δ_0 can be decomposed in terms of them.

Theorem 1.8.2. *On $\mathbb{S}^{d-1}$, Δ_0 satisfies the decomposition*

$$\Delta_0 = \sum_{1\le i<j\le d} D_{i,j}^2. \tag{1.8.3}$$

Proof. Let $F(x) = f\left(\frac{x}{\|x\|}\right)$. A straightforward computation shows that

$$\sum_{1\le i<j\le d} D_{i,j}^2 F(x) = (\Delta F)(x) - \frac{1}{\|x\|^2}\sum_{i=1}^{d}\sum_{j=1}^{d} x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j} - \frac{d-1}{\|x\|}\sum_{i=1}^{d}\frac{\partial F}{\partial x_i}.$$

Consequently, restricting to $\mathbb{S}^{d-1}$ and comparing with Eqs. (1.4.3), (1.8.3) follows. □

Let $Q_{i,j,\theta}$ denote a rotation by the angle θ in the (x_i,x_j)-plane, oriented so that $(x_i,x_j) = (s\cos\theta, s\sin\theta)$. Then $T(Q_{i,j,\theta})$, defined in Eq. (1.7.1), maps f into $T(Q_{i,j,\theta})f(x) = f(Q_{i,j,-\theta}x)$. Written explicitly, for example for $(i,j) = (1,2)$, we have

$$T(Q_{1,2,\theta}f)(x) = f(x_1\cos\theta + x_2\sin\theta, -x_1\sin\theta + x_2\cos\theta, x_3,\ldots,x_d). \tag{1.8.4}$$

Then $D_{i,j}$ is the infinitesimal operator of $T(Q)$,

$$\left.\frac{\mathrm{d}T(Q_{i,j,\theta})}{\mathrm{d}\theta}\right|_{\theta=0} = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i} = D_{i,j}, \tag{1.8.5}$$

where the first equality follows from Eq. (1.8.4). The infinitesimal operator plays an important role in representation theory; see, for example, [169].

The operators $D_{i,j}$ will play an important role in approximation theory on the sphere. We state several more properties of these operators.

Lemma 1.8.3. *For $1 \le i < j \le d$, the operators $D_{i,j}$ commute with Δ_0. In particular, $D_{i,j}$ maps $\mathscr{H}_n^d$ to itself.*

Proof. By symmetry, we need to show only that $D_{1,2}$. Let $[A,B] = AB - BA$ denote the commutator of A and B. A quick computation shows that

$$[\partial_j, D_{k,l}] = \delta_{j,k}\partial_l - \delta_{j,l}\partial_k, \quad [x_i, D_{k,l}] = \delta_{i,k}x_l - \delta_{i,l}x_k,$$

from which it is easy to see that for $1 \le i < j \le d$, $1 \le k < l \le d$, we have

$$[D_{i,j}, D_{k,l}] = -\delta_{i,k}D_{j,l} + \delta_{i,l}D_{j,k} + \delta_{j,k}D_{i,l} - \delta_{j,l}D_{i,k}. \tag{1.8.6}$$

Using Eq. (1.8.6), a simple computation shows that $[D_{1,2}, D_{1,l}^2] = -(D_{1,l}D_{2,l} + D_{2,l}D_{1,l})$ and $[D_{1,2}, D_{2,l}^2] = D_{1,l}D_{2,l} + D_{2,l}D_{1,l}$, so that $[D_{1,2}, D_{1,l}^2 + D_{2,l}^2] = 0$ for $l \ge 2$. Moreover, by Eq. (1.8.6), $[D_{1,2}, D_{k,l}^2] = 0$ whenever $3 \le k < l \le d$. Summing over (k,l) for $1 \le k < l \le d$ then proves $[D_{1,2}, \Delta_0] = 0$. □

Proposition 1.8.4. *For $f, g \in C^1(\mathbb{S}^{d-1})$ and $1 \le i \ne j \le d$,*

$$\int_{\mathbb{S}^{d-1}} f(x) D_{i,j} g(x) \mathrm{d}\sigma(x) = -\int_{\mathbb{S}^{d-1}} D_{i,j} f(x) g(x) \mathrm{d}\sigma(x). \tag{1.8.7}$$

Proof. By the rotation invariance of the Lebesgue measure $\mathrm{d}\sigma$, we obtain, for every $\theta \in [-\pi, \pi]$,

$$\int_{\mathbb{S}^{d-1}} f(x) g(Q_{i,j,-\theta} x) \,\mathrm{d}\sigma(x) = \int_{\mathbb{S}^{d-1}} f(Q_{i,j,\theta} x) g(x) \,\mathrm{d}\sigma(x).$$

Differentiating both sides of this identity with respect to θ and evaluating the resulting equation at $\theta = 0$ leads to, by Eq. (1.8.5), the desired Eq. (1.8.7). □

Equation (1.8.7) allows us to define distributional derivatives $D_{i,j}^r$ on $\mathbb{S}^{d-1}$ for $r \in \mathbb{N}$ via the identity, with $g \in C^\infty(\mathbb{S}^{d-1})$,

$$\int_{\mathbb{S}^{d-1}} D_{i,j}^r f(x) g(x) \mathrm{d}\sigma(x) = (-1)^r \int_{\mathbb{S}^{d-1}} f(x) D_{i,j}^r g(x) \mathrm{d}\sigma(x). \tag{1.8.8}$$

Summing Eq. (1.8.8) with $r = 2$ over $i < j$ and applying Eq. (1.8.3), it follows immediately that Δ_0 is self-adjoint, which can also be deduced from Theorem 1.4.5.

Corollary 1.8.5. *For $f, g \in C^2(\mathbb{S}^{d-1})$,*

$$\int_{\mathbb{S}^{d-1}} f(x) \Delta_0 g(x) \mathrm{d}\sigma = \int_{\mathbb{S}^{d-1}} \Delta_0 f(x) g(x) \mathrm{d}\sigma.$$

The spherical gradient ∇_0 is a vector of first-order differential operator on the sphere, which can be written in terms of $D_{i,j}$ as follows.

Lemma 1.8.6. *For $f \in C^1(\mathbb{S}^{d-1})$ and $1 \le j \le d$, the jth component of $\nabla_0 f$ satisfies*

$$(\nabla_0)_j f(\xi) = \sum_{1 \le i \le d, i \ne j} \xi_i D_{i,j} f(\xi), \quad \xi \in \mathbb{S}^{d-1}. \tag{1.8.9}$$

Furthermore, for $f, g \in C^1(\mathbb{S}^{d-1})$, the following identity holds:

$$\nabla_0 f(\xi) \cdot \nabla_0 g(\xi) = \sum_{1 \le i < j \le d} D_{i,j} f(\xi) D_{i,j} g(\xi), \quad \xi \in \mathbb{S}^{d-1}. \tag{1.8.10}$$

Proof. Let $F(y) = f(y/\|y\|)$. From the definition and $\|\xi\| = 1$, we obtain

$$(\nabla_0)_j f(\xi) = \frac{\partial}{\partial x_j} F(\xi) = \partial_j f - \xi_j \sum_{i=1}^{d} \xi_i \partial_i f \tag{1.8.11}$$

$$= \partial_j f \sum_{i=1}^{d} \xi_i^2 - \xi_j \sum_{i=1}^{d} \xi_i \partial_i f = \sum_{i=1}^{d} \xi_i D_{i,j} f,$$

which gives Eq. (1.8.9), since $D_{i,i} f = 0$. Equation (1.8.11) means that

$$\nabla_0 f(\xi) = \nabla f(\xi) - \xi(\xi \cdot \nabla f). \tag{1.8.12}$$

Since $\xi \cdot \nabla_0 f(\xi) = 0$, using Eq. (1.8.11) and the definition of $D_{i,j}$, it follows that

$$\nabla_0 f \cdot \nabla_0 g = \nabla_0 f \cdot \nabla g = \sum_{j=1}^{d} \partial_j f \partial_j g - \sum_{j=1}^{d} \sum_{i=1}^{d} \xi_i \xi_j \partial_i f \partial_j g = \sum_{i<j} D_{i,j} f(\xi) D_{i,j} g(\xi),$$

where the second equality uses $\|\xi\| = 1$. □

As an application, we state an integration by parts formula on the sphere.

Proposition 1.8.7. *For $f, g \in C^1(\mathbb{S}^{d-1})$,*

$$\int_{\mathbb{S}^{d-1}} f(x) \nabla_0 g(x) \mathrm{d}\sigma = -\int_{\mathbb{S}^{d-1}} \left(\nabla_0 f(x) - (d-1) x f(x)\right) g(x) \mathrm{d}\sigma. \tag{1.8.13}$$

Furthermore, for $f \in C^2(\mathbb{S}^{d-1})$ and $g \in C^1(\mathbb{S}^{d-1})$,

$$\int_{\mathbb{S}^{d-1}} \nabla_0 f \cdot \nabla_0 g \mathrm{d}\sigma = -\int_{\mathbb{S}^{d-1}} \Delta_0 f(x) g(x) \mathrm{d}\sigma. \tag{1.8.14}$$

Proof. Using the expression (1.8.11) and applying Eq. (1.8.7), we obtain

$$\int_{\mathbb{S}^{d-1}} f(x) (\nabla_0)_j g(x) \mathrm{d}\sigma = \sum_{1 \le i \le d, i \ne j} \int_{\mathbb{S}^{d-1}} x_i f(x) D_{i,j} g(x) \mathrm{d}\sigma$$

$$= -\sum_{1 \le i \le d, i \ne j} \int_{\mathbb{S}^{d-1}} D_{i,j}(x_i f(x)) g(x) \mathrm{d}\sigma.$$

By the chain rule, recalling Eq. (1.8.2) if necessary, we have $D_{i,j}(x_i f(x)) = x_i D_{i,j} f - x_j f(x)$. Hence we obtain

$$\int_{\mathbb{S}^{d-1}} f(x) (\nabla_0)_j g(x) \mathrm{d}\sigma = -\int_{\mathbb{S}^{d-1}} \Bigg(\sum_{1 \le i \le d, i \ne j} x_i D_{i,j} f(x) - (d-1) f(x) \Bigg) g(x) \mathrm{d}\sigma,$$

which gives the jth component of Eq. (1.8.13). Equation (1.8.14) follows immediately from Eqs. (1.8.10), (1.8.7), and (1.8.3). □

1.9 Notes

Spherical harmonics appear in many disciplines and in many different branches of mathematics. Many books contain parts of the theory of spherical harmonics. Our treatment covers what is needed for harmonic analysis and approximation theory in this book. Below we comment on some books that we have consulted.

A classical treatise on spherical harmonics is [88], a good source for classical results. A short but nice expository work is [124], which was later expanded into [125]. The reference book [71] contains a chapter on spherical harmonics. A rich resource for spherical harmonics in Fourier analysis is [159]. Applications to and connections with group representations are studied extensively in [169]; see also [83]. For their role in the context of orthogonal polynomials of several variables, see [67] as well as [71]. The book [5] contains a chapter on spherical harmonics in light of special functions. The theory of harmonic functions is treated in [7], including material on spherical harmonics. The book [78] deals with spherical harmonics in geometric applications. Finally, the recent book [6] provides an introduction to spherical harmonics and approximation on the sphere from the perspective of applications in numerical analysis.

Aside from their role in representation theory, the operators $D_{i,j}$ do not seem to have received much attention in analysis. Most of the materials in Sect. 1.8 have not previously appeared in books. These operators play an important role in our development of approximation theory on the sphere.

Chapter 2
Convolution Operator and Spherical Harmonic Expansion

The convergence of spherical harmonic expansions is studied through projection operators and various summability methods. We start with translation and convolution operators on the sphere in the first section, which are essential for the rest of the book. In particular, the projection operators and the Poisson integrals for the Fourier expansion in spherical harmonics, discussed in the second section, are convolution operators, which are also multiplier operators. The convolution and translation operators are used to define and study the Hardy–Littlewood maximal function on the sphere in the third section. As in the classical Fourier series, spherical harmonic series do not in general converge beyond the L^2 metric. It is then necessary to consider summation methods. One family of summation methods is that of Cesàro means, which will serve as an important tool in our later chapters. In the fourth section, we define the Cesàro means (C,δ) of the spherical harmonics and establish their convergence for δ above the critical index. Further results, in greater depth, on the convergence of these means are collected in the fifth section. A family of convolution operators that combine the polynomial-preserving property of the partial sum operator and the convergence of the Cesàro means is defined in terms of a smooth cutoff function in the sixth section. These operators provide near-optimal polynomial approximation, and their convolution kernels are proven to be highly localized in the sense that they decay faster than any polynomial order away from the diagonal. Such operators and their kernels will be instrumental for approximation on the sphere.

2.1 Convolution and Translation Operators on the Sphere

The distance between two points $x,y \in \mathbb{S}^{d-1}$ is defined as the geodesic distance

$$\mathtt{d}(x,y) := \arccos\langle x,y\rangle,$$

F. Dai and Y. Xu, *Approximation Theory and Harmonic Analysis on Spheres and Balls*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-6660-4_2,

and the reproducing kernel of $\mathscr{H}_n^d$ depends only on $\langle x,y\rangle$. This suggests a definition of a convolution operator on the sphere. Let

$$w_\lambda(x) = (1-x^2)^{\lambda-1/2}, \quad \lambda > -\frac{1}{2}, \quad x \in (-1,1).$$

Definition 2.1.1. For $f \in L^1(\mathbb{S}^{d-1})$ and $g \in L^1(w_\lambda;[-1,1])$ with $\lambda = \frac{d-2}{2}$,

$$(f*g)(x) := \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)g(\langle x,y\rangle)\mathrm{d}\sigma(y). \tag{2.1.1}$$

Denote the norm of the space $L^p(w_\lambda;[-1,1])$ by $\|\cdot\|_{\lambda,p}$; for $g \in L^p(w_\lambda;[-1,1])$,

$$\|g\|_{\lambda,p} := \left(c_\lambda \int_{-1}^1 |g(x)|^p w_\lambda(x)\mathrm{d}x\right)^{1/p}, \quad 1 \le p \le \infty,$$

where c_λ is the normalization constant such that $c_\lambda \int_{-1}^1 w_\lambda(t)\mathrm{d}t = 1$, and the norm is taken as the uniform norm when $p=\infty$. The convolution on the sphere satisfies Young's inequality:

Theorem 2.1.2. *Let $p,q,r \ge 1$ and $p^{-1} = r^{-1}+q^{-1}-1$. For $f \in L^q(\mathbb{S}^{d-1})$ and $g \in L^r(w_\lambda;[-1,1])$ with $\lambda = \frac{d-2}{2}$,*

$$\|f*g\|_p \le \|f\|_q \|g\|_{\lambda,r}. \tag{2.1.2}$$

In particular, for $1 \le p \le \infty$,

$$\|f*g\|_p \le \|f\|_p \|g\|_{\lambda,1} \quad \textit{and} \quad \|f*g\|_p \le \|f\|_1 \|g\|_{\lambda,p}. \tag{2.1.3}$$

Proof. The standard proof (cf. [15, p. 6]) applies in this setting. By Minkowski's inequality,

$$\|f*g\|_q \le \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} |f(y)| \left(\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} |g(\langle x,y\rangle)|^q \mathrm{d}\sigma(y)\right)^{1/q} \mathrm{d}\sigma(x) = \|f\|_1 \|g\|_{\lambda,q},$$

on using (A.5.1). And by Hölder's inequality and (A.5.1), it follows readily that

$$\|f*g\|_\infty \le \|f\|_{q'} \|g\|_{\lambda,q}, \quad \frac{1}{q'}+\frac{1}{q} = 1.$$

Applying the Riesz–Thorin theorem to interpolate the above two inequalities with $\theta = q(1-\frac{1}{p})$ gives the stated result. □

In particular, (2.1.3) shows that $f*g$ is well defined. By (1.2.4) and (1.2.7), proj_n is a convolution operator:

$$\operatorname{proj}_n f = f * Z_n, \qquad Z_n(t) := \frac{n+\lambda}{\lambda} C_n^{\lambda}(t) \quad \text{with} \quad \lambda = \frac{d-2}{2}. \tag{2.1.4}$$

For $g \in L^1(w_\lambda; [-1,1])$, let $\hat{g}_n^\lambda$ denote the Fourier coefficient of g with respect to the Gegenbauer polynomials,

$$\hat{g}_n^\lambda := c_\lambda \int_{-1}^{1} g(t) \frac{C_n^\lambda(t)}{C_n^\lambda(1)} (1-t^2)^{\lambda - \frac{1}{2}} \mathrm{d}t.$$

Theorem 2.1.3. *For $f \in L^1(\mathbb{S}^{d-1})$ and $g \in L^1(w_\lambda; [-1,1])$ with $\lambda = \frac{d-2}{2}$,*

$$\operatorname{proj}_n (f * g) = \hat{g}_n^\lambda \operatorname{proj}_n f, \quad n = 0, 1, 2 \ldots. \tag{2.1.5}$$

Proof. By (1.2.4) and the Funk–Hecke formula in Theorem 1.2.9,

$$\begin{aligned}
\operatorname{proj}_n (f * g)(x) &= \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} (f * g)(\xi) Z_n(x, \xi) \mathrm{d}\sigma(\xi) \\
&= \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) \left(\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} g(\langle \xi, y \rangle) Z_n(x, \xi) \mathrm{d}\sigma(\xi) \right) \mathrm{d}\sigma(y) \\
&= \hat{g}_n^\lambda \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) Z_n(x, y) \mathrm{d}\sigma(y) = \hat{g}_n^\lambda \operatorname{proj}_n f(x),
\end{aligned}$$

where we have used the fact that $c_\lambda = \omega_{d-1}/\omega_d$ when $\lambda = \frac{d-2}{2}$. □

The identity (2.1.5) can be viewed as an analogue of the fact that the Fourier transform of $f * g$ is equal to the product of the Fourier transforms of f and g. It justifies our calling the right-hand side of (2.1.1) a convolution.

The translation operator $T_\theta f$ on the sphere can be interpreted in terms of the geodesic distance. It is defined as follows:

Definition 2.1.4. For $0 \le \theta \le \pi$ and $f \in L^1(\mathbb{S}^{d-1})$, define

$$T_\theta f(x) := \frac{1}{\omega_{d-1} (\sin \theta)^{d-1}} \int_{\langle x, y \rangle = \cos \theta} f(y) \mathrm{d}\ell_{x,\theta}(y), \tag{2.1.6}$$

where $\mathrm{d}\ell_{x,\theta}(y)$ denotes Lebesgue measure on the set $\{y \in \mathbb{S}^{d-1} : \langle x, y \rangle = \cos \theta\}$.

The basic properties of the translation operator are listed below:

Proposition 2.1.5. *Let $0 \le \theta \le \pi$ and $f \in L^2(\mathbb{S}^{d-1})$. Then*

1. *Let $\mathbb{S}_x^\perp := \{y \in \mathbb{S}^{d-1} : \langle x, y \rangle = 0\}$, the equator in $\mathbb{S}^{d-1}$ with respect to x; then*

$$T_\theta f(x) = \frac{1}{\omega_{d-1}} \int_{\mathbb{S}_x^\perp} f(x \cos \theta + u \sin \theta) \mathrm{d}\sigma(u). \tag{2.1.7}$$

In particular, if $f_0(x) := 1$, then $T_\theta f_0(x) = 1$.

2. *For a generic* $g : [-1,1] \mapsto \mathbb{R}$,

$$(f * g)(x) = \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) T_\theta f(x) (\sin\theta)^{d-2} \mathrm{d}\theta. \tag{2.1.8}$$

Proof. The first item follows from a change of variable $y \mapsto x\cos\theta + u\sin\theta$. For the second, we choose a coordinate system such that x becomes the north pole and set again $y = x\cos\theta + u\sin\theta$ to obtain

$$\begin{aligned}(f * g)(x) &= \frac{1}{\omega_d} \int_0^\pi g(\cos\theta) \int_{\mathbb{S}_x^\perp} f(x\cos\theta + u\sin\theta) \mathrm{d}\sigma(u) (\sin\theta)^{d-2} \mathrm{d}\theta \\ &= \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) T_\theta f(x) (\sin\theta)^{d-2} \mathrm{d}\theta,\end{aligned}$$

since $\mathbb{S}_x^\perp$ is isomorphic to the sphere $\mathbb{S}^{d-2}$. □

The next proposition gives the interaction between T_θ and orthogonal expansions:

Lemma 2.1.6. *The operator* $T_\theta f$ *maps* $\Pi_n(\mathbb{S}^{d-1})$ *onto itself; for* $f \in L^1(\mathbb{S}^{d-1})$,

$$\operatorname{proj}_n T_\theta f = \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} \operatorname{proj}_n f, \qquad \lambda = \frac{d-2}{2}. \tag{2.1.9}$$

Proof. Let $Y \in \mathscr{H}_n^d$. Denote by $\langle f, Y \rangle$ the Fourier coefficient of f with respect to Y. By Theorem 2.1.3,

$$\begin{aligned}\langle f * g, Y \rangle &= \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \operatorname{proj}_n (f * g)(x) Y(x) \mathrm{d}\sigma(x) \\ &= \langle f, Y \rangle \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} (\sin\theta)^{d-2} \mathrm{d}\theta.\end{aligned}$$

On the other hand, by (2.1.8),

$$\langle f * g, Y \rangle = \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) \langle T_\theta f, Y \rangle (\sin\theta)^{d-2} \mathrm{d}\theta.$$

Since the above holds for a generic g whenever the integrals make sense, this shows that the Fourier coefficient of $T_\theta f$ with respect to Y satisfies $\langle T_\theta f, Y \rangle = \langle f, Y \rangle C_n^\lambda(\cos\theta)/C_n^\lambda(1)$, which proves the stated formula. □

Lemma 2.1.7. *For* $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p < \infty$, *or* $f \in C(\mathbb{S}^{d-1})$ *and* $p = \infty$,

$$\|T_\theta f\|_p \le \|f\|_p \qquad \text{and} \qquad \lim_{\theta \to 0^+} \|T_\theta f - f\|_p = 0.$$

Proof. For $f \in L^1(\mathbb{S}^{d-1})$ and $\lambda = \frac{d-2}{2}$, we have

$$\|T_\theta f\|_1 \le \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} T_\theta(|f|)\,\mathrm{d}\sigma(x) = \mathrm{proj}_0(T_\theta|f|)$$
$$= \frac{C_0^\lambda(\cos\theta)}{C_0^\lambda(1)}\mathrm{proj}_0(|f|) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} |f(x)|\,\mathrm{d}\sigma(x) = \|f\|_1,$$

where we have used the positivity of T_θ in the first step, and Lemma 2.1.6 in the third step. On the other hand, it follows directly from the definition that $\|T_\theta f\|_\infty \le \|f\|_\infty$. Thus, using the Riesz–Thorin interpolation theorem, we deduce that $\|T_\theta f\|_p \le \|f\|_p$ for all $1 \le p \le \infty$. This further implies $\|T_\theta f - f\|_p \le 2\|f-P\|_p + \|T_\theta P - P\|_p$ for every polynomial P. By Lemma 2.1.6,

$$T_\theta P - P = \sum_{j=0}^{n}\left(\frac{C_j^\lambda(\cos\theta)}{C_j^\lambda(1)} - 1\right)\mathrm{proj}_j P, \qquad P \in \Pi_n(\mathbb{S}^{d-1}),$$

so that $T_\theta P - P \to 0$ as $\theta \to 0^+$, from which the convergence of $\|T_\theta f - f\|_p$ follows from the density of polynomials. □

2.2 Fourier Orthogonal Expansions

With respect to an orthonormal basis $\{Y_\alpha\}$, say (1.5.6), a function f in $L^2(\mathbb{S}^{d-1})$ can be expanded in a Fourier series

$$f(x) = \sum c_\alpha Y_\alpha(x), \quad \text{where} \quad c_\alpha = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)Y_\alpha(y)\mathrm{d}\sigma.$$

It is often more convenient to consider the orthogonal expansions in terms of the spaces $\mathcal{H}_n^d$. Collecting terms of spherical harmonics of the same degree, the Fourier series takes the form, by (1.2.4) and (1.2.3),

$$f(x) = \sum_{n=0}^{\infty}\mathrm{proj}_n f(x), \tag{2.2.1}$$

where $\mathrm{proj}_n f$ is the orthogonal projection of f onto the space $\mathcal{H}_n^d$. The formulation of (2.2.1) is independent of a particular choice of orthogonal basis. In particular, the nth partial sum of (2.2.1) is defined by

$$S_n f := \sum_{k=0}^{n}\mathrm{proj}_k f. \tag{2.2.2}$$

By (1.2.4), $S_n f$ can be written as an integral operator whose kernel enjoys a closed form in terms of Jacobi polynomials.

Proposition 2.2.1. *For $n = 0, 1, 2, \ldots$,*

$$S_n f(x) = (f * K_n)(x), \qquad x \in \mathbb{S}^{d-1}, \tag{2.2.3}$$

where the kernel K_n satisfies, with $\lambda = \frac{d-2}{2}$,

$$K_n(t) := \sum_{k=0}^{n} \frac{k+\lambda}{\lambda} C_k^{\lambda}(t) = \frac{(2\lambda+1)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda+\frac{1}{2},\lambda-\frac{1}{2})}(t). \tag{2.2.4}$$

Proof. The definition follows from the closed form of the zonal harmonics in (1.2.7). The closed form follows from the Eq. (B.1.9). □

Since the space of spherical polynomials is dense in $C(\mathbb{S}^{d-1})$ by Weierstrass's theorem and, as a consequence, dense in $L^2(\mathbb{S}^{d-1})$, the following theorem is a standard Hilbert space result for $L^2(\mathbb{S}^{d-1})$:

Theorem 2.2.2. *The family of spherical harmonics is dense in $L^2(\mathbb{S}^{d-1})$, and*

$$L^2(\mathbb{S}^{d-1}) = \sum_{n=0}^{\infty} \mathscr{H}_n^d \quad \text{i.e.} \quad f = \sum_{n=0}^{\infty} \operatorname{proj}_n f$$

in the sense that $\lim_{n\to\infty} \|f - S_n f\|_2 = 0$ *for every $f \in L^2(\mathbb{S}^{d-1})$. In particular, for $f \in L^2(\mathbb{S}^{d-1})$, Parseval's identity holds,*

$$\|f\|_2^2 = \sum_{n=0}^{\infty} \|\operatorname{proj}_n f\|_2^2.$$

Just as in the case of classical Fourier series in several variables, $S_n f$ does not in general converge either pointwise or in L^p for $p \neq 2$. The summability of Fourier series will be studied in the next chapter. Here we are content with one result.

Definition 2.2.3. For $f \in L^1(\mathbb{S}^{d-1})$, the Poisson integral of f is defined by

$$P_r f(\xi) := (f * P_r)(\xi), \quad \xi \in \mathbb{S}^{d-1}, \tag{2.2.5}$$

where the kernel $P_r(\langle x, \cdot \rangle)$ is given by, for $0 < r < 1$,

$$P_r(t) := \frac{1-r^2}{(1-2rt+r^2)^{d/2}}. \tag{2.2.6}$$

Lemma 2.2.4. *For $0 < r < 1$, the Poisson kernel satisfies the following properties:*

(1) For $x,y \in \mathbb{S}^{d-1}$, $P_r(\langle x,y\rangle) = \sum_{n=0}^{\infty} Z_n(x,y) r^n$.
(2) $P_r f = \sum_{n=0}^{\infty} r^n \operatorname{proj}_n f$.
(3) $P_r(\langle x,y\rangle) \geq 0$ *and* $\omega_d^{-1} \int_{\mathbb{S}^{d-1}} P_r(\langle x,y\rangle) \mathrm{d}\sigma(y) = 1$.

Proof. The first item follows from (1.2.7) and the Poisson kernel of the Gegenbauer polynomials in (B.2.8). The second item follows from the first. The infinite series converges uniformly, since $r < 1$. Integration term by term shows that $\omega_d^{-1} \int_{\mathbb{S}^{d-1}} P_r(\langle x,y\rangle) \mathrm{d}\sigma(y) = 1$. □

Theorem 2.2.5. *Let f be a continuous function on $\mathbb{S}^{d-1}$. For $0 \leq r < 1$, $u(r\xi) := P_r f(\xi)$ is a harmonic function in $x = r\xi$ and* $\lim_{r\to 1^-} u(r\xi) = f(\xi)$, $\forall \xi \in \mathbb{S}^{d-1}$.

Proof. The proof is standard, and we shall be brief. By Lemma 2.2.4,

$$|u(r\xi) - f(\xi)| = \frac{1}{\omega_d} \left| \int_{\mathbb{S}^{d-1}} [f(y) - f(\xi)] P_r(\langle \xi, y\rangle) \mathrm{d}\sigma(y) \right|$$
$$\leq \sup_{\|\xi - y\| \leq \delta} |f(y) - f(\xi)| + 2\|f\|_\infty \int_{\|\xi - y\| \geq \delta} P_r(\langle \xi, y\rangle) \mathrm{d}\sigma(y)$$

for every $\delta > 0$. If $\|\xi - y\| \geq \delta$, then $2(1 - \langle \xi, y\rangle) = \|\xi - y\|^2 \geq \delta^2$ for $y \in \mathbb{S}^{d-1}$, so that $P_r(\langle \xi, y\rangle) \leq (1 - r^2)/((1-r)^2 + r\delta^2) \to 0$, as $r \to 1^-$. Thus, taking $r \to 1^-$ and then $\delta \to 0$, the proof follows because f is continuous. □

In other words, u is the solution of the Dirichlet problem $\Delta u = 0$ inside the unit ball with the boundary condition $u = f$ on the unit sphere.

Corollary 2.2.6. *If* $f, g \in L^1(\mathbb{S}^{d-1})$ *and* $\operatorname{proj}_n f = \operatorname{proj}_n g$ *for all* $n = 0, 1, \ldots$, *then* $f = g$.

Proof. If $\operatorname{proj}_n f = \operatorname{proj}_n g$ for all $n \in \mathbb{N}_0$, then $P_r f = P_r g$ for $0 \leq r < 1$. The desired conclusion $f = g$ then follows from Theorem 2.2.5 and the uniqueness of the limit in L^1. □

Next, we define multiplier operators of spherical harmonic expansions. The operator norm of an operator $T : L^p(\mathbb{S}^{d-1}) \to L^p(\mathbb{S}^{d-1})$, $1 \leq p \leq \infty$, is defined by

$$\|T\|_{(p,p)} := \sup_{\|f\|_p \leq 1} \|Tf\|_p,$$

where in the case of $p = \infty$, we assume $f \in C(\mathbb{S}^{d-1})$.

Definition 2.2.7. A linear operator T on $L^p(\mathbb{S}^{d-1})$ for some $1 \leq p \leq \infty$ is called a multiplier operator if there exists a sequence $\{\mu_n\}$ of real numbers such that

$$\operatorname{proj}_n Tf = \mu_n \operatorname{proj}_n f, \qquad \forall f \in L^p(\mathbb{S}^{d-1}), \quad \forall n \in \mathbb{N}_0.$$

It is a bounded multiplier operator on $L^p(\mathbb{S}^{d-1})$ if $\|T\|_{(p,p)}$ is finite.

Because of Theorem 2.1.3, the convolution operator $f \mapsto f * g$ is a multiplier operator for every $g : [-1,1] \mapsto \mathbb{R}$. In particular, the translation operator T_θ in (2.1.6) is an example of a multiplier operator, as is the Cesàro operator defined later. We are interested in the L^p bounded multiplier operators. It is clear that a multiplier operator is bounded on $L^2(\mathbb{S}^{d-1})$ if and only if its associated sequence $\{\mu_j\}$ is bounded.

Proposition 2.2.8. *If T is a bounded multiplier operator on $L^p(\mathbb{S}^{d-1})$ for some p and $1 \le p \le \infty$, then it extends to a bounded operator on $L^q(\mathbb{S}^{d-1})$ with $\|T\|_{(q,q)} = \|T\|_{(q',q')}$ for all $|\frac{1}{q} - \frac{1}{2}| \le |\frac{1}{p} - \frac{1}{2}|$, where $\frac{1}{q} + \frac{1}{q'} = 1$.*

Proof. Recall the inner product $\langle f, g\rangle = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(x)g(x)\,\mathrm{d}\sigma(x)$, $f,g \in L^2(\mathbb{S}^{d-1})$. By the orthogonality of spherical harmonics, if f and g are polynomials, then it is easy to see that $\langle Tf, g\rangle = \langle f, Tg\rangle$. By the density of spherical polynomials and the Riesz representation theorem, we deduce that

$$\|T\|_{(q',q')} = \sup_{\|f\|_{q'}\le 1}\sup_{\|g\|_q\le 1} |\langle Tf, g\rangle| = \sup_{\|f\|_{q'}\le 1}\sup_{\|g\|_q\le 1} |\langle f, Tg\rangle|$$
$$= \sup_{\|g\|_q\le 1} \|Tg\|_q = \|T\|_{(q,q)}.$$

Thus, if T is a bounded multiplier operator on L^p, then it is bounded on $L^{p'}$ as well. Using the Riesz–Thorin interpolation theorem, we further conclude that T is bounded on $L^q(\mathbb{S}^{d-1})$ for all $|\frac{1}{q} - \frac{1}{2}| \le |\frac{1}{p} - \frac{1}{2}|$. This completes the proof. □

Our next proposition gives a characterization of multiplier operators on the sphere. Recall the operator T_Q defined by $T_Q f(x) = f(Q^{-1}x)$ for $Q \in O(d)$.

Proposition 2.2.9. *A bounded linear operator T on $L^2(\mathbb{S}^{d-1})$ is a multiplier operator if and only if it is invariant under the group of rotations, that is, if and only if $TT_Q = T_Q T$ for all $Q \in O(d)$.*

Proof. We begin with the proof of the necessity. Let T be a multiplier operator associated with a bounded sequence $\{\mu_j\}$. By (2.1.4) and the rotation invariance of the Lebesgue measure $\mathrm{d}\sigma(x)$, $T_Q \operatorname{proj}_n = \operatorname{proj}_n T_Q$ for all $Q \in O(d)$. Thus, for each $f \in L^2(\mathbb{S}^{d-1})$ and $n \in \mathbb{N}_0$, □

$$\operatorname{proj}_n(T_Q T f) = T_Q \operatorname{proj}_n(Tf) = \mu_n T_Q \operatorname{proj}_n f = \mu_n \operatorname{proj}_n(T_Q f) = \operatorname{proj}_n(TT_Q f).$$

It then follows by Corollary 2.2.6 that $TT_Q = T_Q T$. Next, we prove the sufficiency. Assume that T is bounded on L^2 and that $TT_Q = T_Q T$ for all $Q \in O(d)$. By definition, it suffices to show that there exists a sequence of real numbers μ_n such that $\operatorname{proj}_n(Tf) = \mu_n \operatorname{proj}_n f$ for each $f \in \mathscr{H}_n^d$ and all $n \in \mathbb{N}_0$. However, by (2.1.4), it suffices to show that

$$T[Z_n(\langle \cdot, y\rangle)](x) = \mu_n Z_n(\langle x, y\rangle), \quad x, y \in \mathbb{S}^{d-1}.$$

Since the reproducing kernel $Z_n(\langle x,y\rangle)$ is invariant under simultaneous rotation in both variables, the rotation invariance of T shows that for all $Q \in O(d)$,

$$T\left[Z_n\left(\langle\cdot,Qy\rangle\right)\right](Qx) = T\left[Z_n\left(\langle Q(\cdot),Qy\rangle\right)\right](x) = T\left[Z_n\left(\langle\cdot,y\rangle\right)\right](x),$$

which implies, as in the proof of Lemma 1.2.5, that $T\left[Z_n\left(\langle\cdot,y\rangle\right)\right](x)$ is a zonal function $F_n(\langle x,y\rangle)$ of $\langle x,y\rangle$. On the other hand, for each fixed $x \in \mathbb{S}^{d-1}$, it follows directly by (1.2.3) that the function $y \mapsto T\left[Z_n\left(\langle\cdot,y\rangle\right)\right](x) = F_n\left(\langle x,y\rangle\right)$ is a spherical harmonic of degree n. Thus, using (1.2.2) and the Funk–Hecke formula (1.2.11), we conclude that for some real number μ_n,

$$F_n(\langle x,y\rangle) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} F_n(\langle x,z\rangle)Z_n\big(\langle y,z\rangle\big)\,\mathrm{d}\sigma(z) = \mu_n Z_n\big(\langle x,y\rangle\big),$$

which completes the proof. □

2.3 The Hardy–Littlewood Maximal Function

For $x \in \mathbb{S}^{d-1}$ and $\theta > 0$, we define a spherical cap $\mathsf{c}(x,\theta)$ centered at x by

$$\mathsf{c}(x,\theta) := \{y \in \mathbb{S}^{d-1} : \langle x,y\rangle \geq \cos\theta\}. \tag{2.3.1}$$

Let $|\mathsf{c}(x,\theta)|$ denote the surface area of $\mathsf{c}(x,\theta)$, that is,

$$|\mathsf{c}(x,\theta)| := \int_{\mathsf{c}(x,\theta)} \mathrm{d}\sigma(y) = \omega_{d-1}\int_0^\theta (\sin\phi)^{d-2}\mathrm{d}\phi, \tag{2.3.2}$$

where the second equation follows from (A.5.1), which is independent of x.

Definition 2.3.1. For $f \in L^1(\mathbb{S}^{d-1})$, we define the Hardy-Littlewood maximal function

$$Mf(x) := \sup_{0<\theta\leq\pi} \frac{1}{|\mathsf{c}(x,\theta)|}\int_{\mathsf{c}(x,\theta)} |f(y)|\mathrm{d}\sigma(y).$$

An alternative definition of Mf is given in the following lemma.

Lemma 2.3.2. *For a nonnegative function $f \in L^2(\mathbb{S}^{d-1})$,*

$$Mf(x) = \sup_{0<\theta\leq\pi} \frac{\omega_d}{|\mathsf{c}(x,\theta)|}(f * \chi_{[\cos\theta,1]})(x) \tag{2.3.3}$$

$$= \sup_{0<\theta\leq\pi} \frac{\int_0^\theta T_\phi f(x)(\sin\phi)^{d-2}\mathrm{d}\phi}{\int_0^\theta (\sin\phi)^{d-2}\mathrm{d}\phi}, \tag{2.3.4}$$

where $\chi_{[a,1]}$ denotes the characteristic function of the interval $[a,1]$.

Proof. The first equation follows from (2.3.2) and

$$\frac{1}{\omega_d}\int_{\mathsf{c}(x,\theta)} f(y)\mathrm{d}\sigma(y) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)\chi_{[\cos\theta,1]}(\langle x,y\rangle)\mathrm{d}\sigma(y) = (f*\chi_{[\cos\theta,1]})(x),$$

whereas the second equation follows from the above equation and (2.1.8). □

Given $E\subset\mathbb{S}^{d-1}$, we denote by $\mathrm{meas}(E)$ the Lebesgue measure $\int_E \mathrm{d}\sigma(x)$ of E. The maximal function satisfies a weak estimate for L^1 functions given by the following weak type-$(1,1)$ inequality:

Theorem 2.3.3. *For $f \in L^1(\mathbb{S}^{d-1})$ and $\alpha > 0$,*

$$\mathrm{meas}\left\{x\in\mathbb{S}^{d-1} : Mf(x)\ge\alpha\right\} \le c\frac{\|f\|_1}{\alpha}.$$

Proof. As in the case of the maximal function defined for functions in $\mathbb{R}^d$, the proof relies on a covering lemma. Let $E_\alpha := \{x\in\mathbb{S}^{d-1} : Mf(x) > \alpha\}$. For every $x\in E_\alpha$, there exists a spherical cap $\mathsf{c}(x,\theta)$ such that $\int_{\mathsf{c}(x,\theta)}|f(y)|d\sigma > \alpha|\mathsf{c}(x,\theta)|$ by the definition of Mf. The collection of such $\mathsf{c}(x,\theta)$ for $x\in E_\alpha$ clearly covers E_α. The covering lemma, which can be proved exactly as in the classical case of $\mathbb{R}^d$ (cf. [159]), states that given an arbitrary compact subset E of E_α, there exists a sequence of spherical caps $\mathsf{c}(x_k,\theta_k)$ that are mutually disjoint such that $\sum_k|\mathsf{c}(x_k,\theta_k)| \ge c\,\mathrm{meas}\,E$, where c is a constant depending only on the dimension. Hence, it follows that

$$\|f\|_1 \ge \int_{\cup_k \mathsf{c}(x_\kappa,\theta_k)} |f(y)|\mathrm{d}\sigma(y) \ge \alpha\sum_k|\mathsf{c}(x_k,\theta_k)| \ge c\,\alpha\,\mathrm{meas}\,E.$$

Taking the supremum over all compact subsets E of E_α, we deduce the desired inequality, which completes the proof. □

Corollary 2.3.4. *Let $f\in L^p(\mathbb{S}^{d-1})$, $1<p<\infty$, or $f\in C(\mathbb{S}^{d-1})$ for $p=\infty$. Then the maximal function is a strong type-(p,p) operator for $1<p\le\infty$; that is,*

$$\|Mf\|_p \le c\|f\|_p, \qquad 1<p\le\infty.$$

Proof. The definition of Mf shows immediately that $\|Mf\|_\infty \le \|f\|_\infty$. Thus, M is of weak type $(1,1)$ and strong type (∞,∞), from which the result for $1<p<\infty$ follows from the Marcinkiewicz interpolation theorem [159]. □

As an application of the boundedness of maximal functions, let us consider

$$f_\theta(x) := \frac{1}{|\mathsf{c}(x,\theta)|}\int_{\mathsf{c}(x,\theta)}|f(y)|\mathrm{d}\sigma(y), \quad 0<\theta\le 1, \quad x\in\mathbb{S}^{d-1}.$$

Lemma 2.3.5. *If $f \in L^1(\mathbb{S}^{d-1})$, then $\lim_{\theta \to 0^+} f_\theta(x) = f(x)$ for almost every $x \in \mathbb{S}^{d-1}$.*

Proof. By (2.3.4), we can write

$$f_\theta(x) - f(x) = \frac{\int_0^\theta (T_\phi f(x) - f(x))(\sin\phi)^{d-2}\mathrm{d}\phi}{\int_0^\theta (\sin\phi)^{d-2}\mathrm{d}\phi},$$

which implies $\|f_\theta - f\|_1 \le \sup_{0\le\phi\le\theta} \|T_\phi f - f\|_1$, and by Lemma 2.1.7, f_θ converges to f in $L^1(\mathbb{S}^{d-1})$. To prove the almost-everywhere convergence, we show that

$$\Omega f(x) := \left| \limsup_{\theta\to 0^+} f_\theta(x) - \liminf_{\theta\to 0^+} f_\theta(x) \right| = 0$$

for almost every $x \in \mathbb{S}^{d-1}$. Since $C(\mathbb{S}^{d-1})$ is dense in $L^1(\mathbb{S}^{d-1})$, we can write $f = h + g$ with $h \in C(\mathbb{S}^{d-1})$ and $\|g\|_1$ arbitrarily small. Since $\Omega g(x) \le 2Mg(x)$, the maximal inequality implies

$$\mathrm{meas}\{x : \Omega g(x) \ge \alpha\} \le \mathrm{meas}\{x : 2Mg(x) \ge \alpha\} \le c\frac{\|g\|_1}{\alpha}.$$

Since $\Omega h = 0$, this shows that $\Omega f(x) = 0$ almost everywhere. □

Theorem 2.3.6. *Assume that $g \in L^1([-1,1];w_\lambda)$, $\lambda = \frac{d-2}{2}$, and that $k(\theta) := g(\cos\theta)$ is a continuous, nonnegative, and decreasing function on $[0,\pi]$. Then for $f \in L^1(\mathbb{S}^{d-1})$,*

$$|(f * g)(x)| \le cM(|f|)(x), \quad x \in \mathbb{S}^{d-1},$$

where $c = \int_0^\pi k(\theta)(\sin\theta)^{d-2}\mathrm{d}\theta$.

Proof. For fixed x, define $F_x(\theta) = \int_0^\theta T_\phi f(x)(\sin\phi)^{d-2}\mathrm{d}\phi$. By (2.1.8) and (2.3.3), an integration by parts shows that

$$\begin{aligned}
|f * g(x)| &= \frac{\omega_{d-1}}{\omega_d} \left| \int_0^\pi g(\cos\theta) T_\theta f(x)(\sin\theta)^{d-2}\mathrm{d}\theta \right| \\
&= \frac{\omega_{d-1}}{\omega_d} \left| F_x(\pi)k(\pi) - \int_0^\pi k'(\phi)F_x(\phi)\mathrm{d}\phi \right| \\
&\le Mf(x)\left[k(\pi)\int_0^\pi (\sin\phi)^{d-2}\mathrm{d}\phi - \int_0^\pi k'(\theta)\int_0^\theta (\sin\phi)^{d-2}\mathrm{d}\phi\mathrm{d}\theta \right],
\end{aligned}$$

where we have used the fact that $k(\theta)$ is nonnegative and $k'(\theta) \le 0$. Consequently, integrating by parts again, we see that $|f * g(x)| \le cMf(x)$ follows from the fact that $\int_0^\pi k(\theta)(\sin\theta)^{d-2}\mathrm{d}\theta$ is bounded. □

2.4 Spherical Harmonic Series and Cesàro Means

By Theorem 2.2.2, the spherical harmonic series converges in $L^2(\mathbb{S}^{d-1})$. In particular, for $f \in L^2(\mathbb{S}^{d-1})$, the partial sum operator $S_n f$ converges to f in the $\|\cdot\|_2$ norm, and the operator norm $\|S_n\|_2$ is uniformly bounded. The operator norm of S_n in $L^p(\mathbb{S}^{d-1})$ is defined by

$$\|S_n\|_p := \sup_{\|f\|_p=1} \|S_n f\|_p.$$

Theorem 2.4.1. *Let $d > 2$. Then $\|S_n\|_\infty = \|S_n\|_1 = \Lambda_n$, where*

$$\Lambda_n := \max_{x\in\mathbb{S}^{d-1}} \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} |K_n(x,y)| \mathrm{d}\sigma(y) \sim n^{\frac{d-2}{2}}.$$

Proof. That $\|S_n\|_\infty = \|S_n\|_1 = \Lambda_n$ follows from a standard argument for linear integral operators. By the closed form of $K_n(x,y)$ in (2.2.4) and the integral relation (A.5.1), we have

$$\Lambda_n = \frac{(2\lambda+1)_n}{(\lambda+\frac{1}{2})_n} c_\lambda \int_{-1}^{1} |P_n^{(\lambda+\frac{1}{2},\lambda-\frac{1}{2})}(t)|(1-t^2)^{\lambda-\frac{1}{2}} \mathrm{d}t, \quad \lambda = \frac{d-2}{2},$$

from which the asymptotic relation follows from (B.1.8). □

In the case of $d = 2$, we have $\|S_n\|_\infty \sim \log n$, as shown in classical Fourier analysis.

The constant Λ_n is often called the Lebesgue constant. Since it is unbounded as $n \to \infty$, the uniform boundedness principle implies that there is a function $f \in C(\mathbb{S}^{d-1})$ for which $S_n f$ does not converge to f in the uniform norm. We then look for summability methods for the spherical harmonic series that will ensure convergence. One important class of such methods is that of Cesàro means.

Definition 2.4.2. For $\delta \in \mathbb{R}$, the Cesàro, or (C,δ), means of the sequence $\{a_k\}_{k=0}^\infty$ are defined by

$$s_n^\delta := \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^\delta a_k, \quad n = 0,1,\ldots, \tag{2.4.1}$$

where

$$A_k^\delta = \binom{k+\delta}{k} = \frac{(\delta+k)(\delta+k-1)\ldots(\delta+1)}{k!}. \tag{2.4.2}$$

The sequence is said to be (C,δ) summable if s_n^δ converges as $n \to \infty$.

The properties of A_k^δ and s_k^d are collected in Appendix A. By (A.4.4), a simple exercise shows that if s_n^δ converges to s, then $s_n^{\delta+\tau}$ converges to s for all $\tau > 0$.

Denote by $S_n^\delta f$ the (C,δ) means of the spherical harmonic series,

$$S_n^\delta f := \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^\delta \operatorname{proj}_k f. \tag{2.4.3}$$

If $\delta = 0$, then $S_n^\delta f = S_n f$. By (2.1.4), $S_n^\delta f$ can be written as a convolution operator

$$S_n^\delta f = f * K_n^\delta, \qquad K_n^\delta(t) := \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^\delta \frac{k+\lambda}{\lambda} C_k^\lambda(t), \tag{2.4.4}$$

where $\lambda = (d-2)/2$. This kernel is closely connected to the Cesàro means $s_n^\delta(w_\lambda; f)$ of the Fourier orthogonal series in the Gegenbauer polynomials. Indeed, let $k_n^\delta(w_\lambda; \cdot, \cdot)$ denote the kernel of $s_n^\delta(w_\lambda; f)$,

$$s_n^\delta(w_\lambda; f, x) = c_\lambda \int_{-1}^{1} f(y) k_n^\delta(w_\lambda; x, y) w_\lambda(y) \mathrm{d}y.$$

Then it is easy to verify that

$$K_n^\delta(t) = k_n^\delta(w_\lambda; 1, t), \qquad \lambda = \frac{d-2}{2}. \tag{2.4.5}$$

As a consequence, some of the convergence results of spherical harmonic series can be deduced from those of the Gegenbauer series. Here is one example:

Theorem 2.4.3. *For $\delta \ge d-1$, S_n^δ is a nonnegative operator; that is, $S_n^\delta f(x) \ge 0$ if $f(x) \ge 0$ for all $x \in \mathbb{S}^{d-1}$.*

Proof. By (2.4.4), we need only show that $K_n^{d-1}(t) \ge 0$ for $t \in [-1,1]$, which follows, by (2.4.5), from the classical result of Kogbetiliantz that $k_n^\delta(w_\lambda; 1, t) \ge 0$ if $\delta \ge 2\lambda + 1$ ([5, p. 389]). □

Moreover, the relation (2.4.5) shows that the Lebesgue constant of S_n^δ can be deduced from the Lebesgue function of $s_n^\delta(w_\lambda)$ evaluated at the point $x = 1$.

Theorem 2.4.4. *Let $\lambda = \frac{d-2}{2}$. Then $\|S_n^\delta\|_\infty = \|S_n^\delta\|_1 = \Lambda_n^\delta$, where*

$$\Lambda_n^\delta := \max_{x \in \mathbb{S}^{d-1}} \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} |K_n^\delta(\langle x, y \rangle)| \mathrm{d}\sigma(y) \sim \begin{cases} n^{\lambda-\delta}, & 0 < \delta < \lambda, \\ \log n, & \delta = \lambda, \\ 1 & \delta > \lambda. \end{cases}$$

In particular, $\|S_n^\delta\|_\infty$ and $\|S_n^\delta\|_1$ are bounded if and only if $\delta > \frac{d-2}{2}$.

Proof. Again, that $\|S_n^\delta\|_\infty = \|S_n^\delta\|_1 = \Lambda_n^\delta$ follows from a standard argument. By the integral relation (A.5.1),

$$\Lambda_n^\delta = c_\lambda \int_{-1}^{1} \left| k_n^\delta(w_\lambda; 1, t) \right| (1-t^2)^{\lambda - \frac{1}{2}} \mathrm{d}t.$$

The asymptotic behavior of this integral, as $n \to \infty$, is given in [162, Sect. 9.4]. □

Corollary 2.4.5. *If $\delta > \frac{d-2}{2}$, then for $f \in L^p(\mathbb{S}^{d-1})$ and $1 \le p \le \infty$, or $f \in C(\mathbb{S}^{d-1})$ and $p = \infty$,*

$$\sup_n \|S_n^\delta f\|_p \le c\|f\|_p \quad \textit{and} \quad \lim_{n\to\infty} \|S_n^\delta f - f\|_p = 0.$$

Furthermore, for $p = 1$ or ∞, the convergence fails in general if $\delta = \frac{d-2}{2}$.

Proof. The boundedness of $\|S_n^\delta f\|_p$ for $\delta > \frac{d-2}{2}$ follows from Theorem 2.4.4 and the Riesz–Thorin interpolation theorem. Since $S_n P = P$ for every polynomial P of fixed degree m and $S_n = S_n^0$, it follows that $S_n^\delta P$ converges to P as $n \to \infty$ for all $\delta > 0$. Hence, the norm convergence of $S_n^\delta f$ follows from the boundedness of the norm. If $\delta = \frac{d-2}{2}$, then $\|S_n^\delta\|_p$ is unbounded for $p = 1$ and $p = \infty$, and the convergence fails in general by the uniform boundedness principle. □

The index $\lambda = \frac{d-2}{2}$ is often called the critical index for the (C,δ) means of the spherical harmonic series on $\mathbb{S}^{d-1}$.

A pointwise estimate for the kernel of the Cesàro means of the Jacobi series is given in Lemma B.1.2, from which the relation (2.4.5) gives the following:

Lemma 2.4.6. *Let $\lambda = \frac{d-2}{2} \ge 0$. If $0 \le \delta \le \lambda + 1$, then*

$$|K_n^\delta(t)| \le c n^{\lambda-\delta} \left[(1-t+n^{-2})^{-(\delta+\lambda+1)/2} + (1+t+n^{-2})^{-\lambda/2} \right].$$

If $\lambda + 1 \le \delta \le 2\lambda + 1$, then

$$|K_n^\delta(t)| \le c n^{-1} \left[(1-t+n^{-2})^{-(\lambda+1)} + (1+t+n^{-2})^{-(2\lambda+1-\delta)/2} \right].$$

If $\delta \ge 2\lambda + 1$, then

$$|K_n^\delta(t)| \le c n^{-1} (1-t+n^{-2})^{-(\lambda+1)}.$$

These estimates of the kernel functions can be used to establish the upper bound of Λ_n^δ in Theorem 2.4.4. We use them to study the almost-everywhere convergence of the Cesàro means. For $\delta \ge 0$, we define the maximal Cesàro (C,δ) operator by

$$S_*^\delta f(x) := \sup_{N \ge 0} |S_N^\delta f(x)|, \qquad x \in \mathbb{S}^{d-1}.$$

It turns out that the maximal Cesàro operator S_*^δ can be controlled pointwise by the Hardy–Littlewood maximal function whenever $\delta > \frac{d-2}{2}$.

Theorem 2.4.7. *If $\delta > \frac{d-2}{2}$ and $f \in L^1(\mathbb{S}^{d-1})$, then for every $x \in \mathbb{S}^{d-1}$,*

$$S_*^\delta f(x) \le c \left[Mf(x) + Mf(-x) \right]. \tag{2.4.6}$$

If, in addition, $\delta \ge d-1$, then the term $Mf(-x)$ in (2.4.6) can be dropped.

Proof. For the proof of (2.4.6), it suffices to consider the case $\lambda < \delta \le \lambda + 1$, where $\lambda = \frac{d-2}{2}$, since by (A.4.4), $S_*^{\delta+\tau} f(x) \le S_*^{\delta}(f)(x)$ for every $\tau > 0$. Setting

$$G_{n,1}^{\delta}(\cos\theta) := n^{\lambda-\delta}(n^{-1}+\theta)^{-(\delta+\lambda+1)}\chi_{[0,\frac{\pi}{2}]}(\theta),$$
$$G_{n,2}^{\delta}(\cos\theta) := n^{\lambda-\delta}(n^{-1}+\theta)^{-\lambda}\chi_{[0,\frac{\pi}{2}]}(\theta),$$

we obtain from Lemma 2.4.6 that for $\lambda < \delta \le \lambda + 1$,

$$K_n^{\delta}(\cos\theta) \le c\left[G_{n,1}^{\delta}(\cos\theta) + G_{n,2}^{\delta}(\cos(\pi-\theta))\right].$$

It is easy to see that $g(t) = G_{n,i}^{\delta}(t)$ satisfies the conditions of Theorem 2.3.6, so that by $T_\theta(-x) = T_{\pi-\theta}(x)$, (2.4.4), and Theorem 2.3.6, we have then

$$\begin{aligned}|S_n^{\delta} f(x)| &\le \left[(|f| * G_{n,1}^{\delta})(x) + (|f| * G_{n,2}^{\delta})(-x)\right]\\ &\le c\,[Mf(x) + Mf(-x)].\end{aligned}$$

Furthermore, if $\delta > 2\lambda + 1$, then Lemma 2.4.6 shows that $|K_n^{\delta}(\cos\theta)|$ is bounded by a single term, and the same proof yields $|S_n^{\delta} f(x)| \le cMf(x)$. □

From Theorem 2.4.7, Theorem 2.3.3, and the density argument in the proof of Lemma 2.3.5, we deduce the following corollary.

Corollary 2.4.8. *If $\delta > \frac{d-2}{2}$ and $f \in L^1(\mathbb{S}^{d-1})$, then $\lim_{N\to\infty} S_N^{\delta} f(x) = f(x)$ for almost every $x \in \mathbb{S}^{d-1}$, and moreover,*

$$\operatorname{meas}\{x \in \mathbb{S}^{d-1} : S_*^{\delta} f(x) > \alpha\} \le c\frac{\|f\|_1}{\alpha}, \quad \forall \alpha > 0.$$

2.5 Convergence of Cesàro Means: Further Results

According to Corollary 2.4.5, the (C,δ) means $S_n^{\delta} f$ converge to f in the $L^1(\mathbb{S}^{d-1})$ norm or in the uniform norm if and only if $\delta > \frac{d-2}{2}$. We also know, since $S_n^0 f = S_n f$, that convergence holds for $\delta \ge 0$ in the $L^2(\mathbb{S}^{d-1})$ norm. The case $1 < p < \infty$ is more delicate and far more difficult to resolve.

Below, we shall state several results for $1 < p < \infty$ without proofs. Some of these results will be proved in the more general setting of weighted approximation in Chap. 9. Throughout this section, we set, for $1 \le p \le \infty$,

$$\delta(p) := \max\left\{0, (d-1)\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}\right\}. \tag{2.5.1}$$

We start with a negative result of Bonami and Clerc [18, Theorem 5.1].

Theorem 2.5.1. *If $1 \le p \le \infty$ and $0 \le \delta \le \delta(p)$, then there exists a function $f \in L^p(\mathbb{S}^{d-1})$ such that $S_n^\delta f$ does not converge in $L^p(\mathbb{S}^{d-1})$. In particular, if $\delta = 0$ and $p \ne 2$, then there exists a function $f \in L^p(\mathbb{S}^{d-1})$ such that $S_n f$ does not converge in $L^p(\mathbb{S}^{d-1})$.*

Theorem 2.5.1 also implies that if $1 \le p \le \infty$ and $0 \le \delta \le \delta(p)$, then $\{\|S_n^\delta\|_p\}_{n=1}^\infty$ is unbounded.

In the positive direction, the convergence of $S_n^\delta f$ depends on a sharp bound of the projection operator. Such a bound was established by Sogge [155].

Theorem 2.5.2. *If $1 \le p \le \infty$ and $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{d}$, then*

$$\|\operatorname{proj}_n f\|_2 \le c_d n^{\delta(p)} \|f\|_p. \tag{2.5.2}$$

The connection between (2.5.2) and the convergence of S_n^δ, revealed in the proof of Theorem 5.2 in [18], leads to the following theorem in [155].

Theorem 2.5.3. *If $1 \le p < \infty$ and $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{d}$, then $\lim_{n\to\infty} \|S_n^\delta f - f\|_p = 0$ holds for all $f \in L^p(\mathbb{S}^{d-1})$ if and only if $\delta > \delta(p)$.*

Theorems 2.5.2 and 2.5.3 will be proved in Chap. 9 in the more general setting of weighted approximation on the sphere.

In the case of $d = 3$, Sogge [155] further proved that the conclusion of Theorem 2.5.3 remains true without the assumption $|1/2 - 1/p| \ge 1/3$.

For the maximal Cesàro operator S_*^δ, the following result was proved, using Stein's interpolation theorem for analytic families of operators, in [18].

Theorem 2.5.4. *If $1 < p \le 2$, $\delta > (d-2)(\frac{1}{p} - \frac{1}{2})$, and $f \in L^p(\mathbb{S}^{d-1})$, then*

$$\|S_*^\delta f\|_p \le C_p \|f\|_p.$$

Together with Corollary 2.4.8, Theorem 2.5.4 implies the following corollary.

Corollary 2.5.5. *If $1 \le p \le 2$, $f \in L^p(\mathbb{S}^{d-1})$, and $\delta > (d-2)(\frac{1}{p} - \frac{1}{2})$, then*

$$\lim_{n\to\infty} S_n^\delta f(x) = f(x)$$

for almost every $x \in \mathbb{S}^{d-1}$.

2.6 Near-Best Approximation Operators and Highly Localized Kernels

For a given function $f \in L^p(\mathbb{S}^{d-1})$, its Cesàro means $S_n^\delta f$ provide a sequence of polynomials that approximate f. These means are useful for our further study, but they are not ideal for quantitative results in approximation theory.

Definition 2.6.1. Let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. For $n \ge 0$, the error of the best approximation to f by polynomials of degree at most n is defined by

$$E_n(f)_p := \inf_{g \in \Pi_n(\mathbb{S}^{d-1})} \|f - g\|_p, \qquad 1 \le p \le \infty. \tag{2.6.1}$$

The best-approximation element exists, since $\Pi_n(\mathbb{S}^{d-1})$ is a finite-dimensional space, by a general theorem on Banach spaces [54, p. 59]. Finding such a polynomial, however, is not easy. For most applications, it fortunately is sufficient to find a polynomial that is a near-best approximation.

Definition 2.6.2. Let η be a C^∞-function on $[0,\infty)$ such that $\eta(t) = 1$ for $0 \le t \le 1$ and $\eta(t) = 0$ for $t \ge 2$. Define

$$L_n f(x) := f * L_n(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) L_n(\langle x, y\rangle)\, \mathrm{d}\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{2.6.2}$$

for $n = 0, 1, 2, \ldots$, where

$$L_n(t) := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \frac{k+\lambda}{\lambda} C_k^\lambda(t), \quad \lambda = \frac{d-2}{2}, \quad t \in [-1,1]. \tag{2.6.3}$$

In the following, if a function satisfies the properties of the η function in the theorem, we shall call it a C^∞ cutoff function, or simply a cutoff function.

Since η is supported on $[0,2]$, the summation in $L_n f$ can be terminated at $k = 2n-1$, so that $L_n f$ is a polynomial of degree at most $2n-1$. It approximates f as well as the best-approximation polynomial of degree n.

Theorem 2.6.3. *Let $f \in L^p$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then*

(1) *$L_n f \in \Pi_{2n-1}(\mathbb{S}^{d-1})$ and $L_n f = f$ for $f \in \Pi_n(\mathbb{S}^{d-1})$.*
(2) *For $n \in \mathbb{N}$, $\|L_n f\|_p \le c\|f\|_p$.*
(3) *For $n \in \mathbb{N}$,*

$$\|f - L_n f\|_p \le (1+c) E_n(f)_p. \tag{2.6.4}$$

Proof. We have already shown that $L_n f$ is a polynomial of degree at most $2n-1$. Using the projection operator proj_n of $\mathscr{H}_n^d$, we can write

$$L_n f = \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \mathrm{proj}_k f.$$

Since the definition of η shows that $\eta(\frac{k}{n}) = 1$ for $0 \le k \le n$, it follows readily that $L_n f = \sum_{k=0}^{n} \mathrm{proj}_k f = f$ if $f \in \Pi_n(\mathbb{S}^{d-1})$. This proves (1).

By Young's inequality, Theorem 2.1.2, $\|L_n f\|_p \le \|f\|_p \|L_n\|_{\lambda,1}$, where $\lambda = \frac{d-2}{2}$. The proof of (2) reduces to showing that $\|L_n\|_{\lambda,1}$ is bounded. Let σ be a positive integer such that $\sigma \ge d-1$ so that the (C,σ) means $K_n^\sigma(t)$ of the sequence $\frac{k+\lambda}{\lambda}C_k^\lambda(t)$ are nonnegative on $[-1,1]$ (see Theorem 2.4.3). Let $\triangle$ denote the difference operator defined in (A.3.1). Using summation by parts repeatedly, we can write

$$L_n(t) = \sum_{k=1}^{\infty} \triangle^{\sigma+1}\eta\left(\frac{k}{n}\right)\binom{k+\sigma}{k}K_k^\sigma(t),$$

where $\triangle^m$ acts on the function $t \mapsto \eta(\frac{t}{n})$. Since $\eta \in C^\infty[0,+\infty)$ implies that $|\triangle^{\sigma+1}\eta(k/n)| \le cn^{-\sigma-1}$ and $\binom{k+\sigma}{k} \le ck^\sigma$, it follows that

$$\|L_n\|_{\lambda,1} \le cn^{-\sigma-1}\sum_{k=1}^{2n} k^\sigma \|K_k^\sigma\|_{\lambda,1} \le cn^{-\sigma-1}\sum_{k=1}^{2n} k^\sigma \le c,$$

since the support of η is on $[0,2]$. This completes the proof of (2).

The proof of (3) is an easy consequence of (1) and (2). Indeed, let p_n be the best-approximation polynomial of degree n. Then (1) shows that $L_n p_n = p_n$, so that

$$\|f - L_n f\|_p \le \|f - p_n\| + \|L_n(f-p_n)\| \le (1+c)\|f-p_n\|_p = (1+c)E_n(f)_p,$$

by the triangle inequality and (2). □

Recall that the Laplace–Beltrami operator Δ_0 can be decomposed, by (1.8.3), in terms of the angular derivative $D_{i,j}$ defined in (1.8.1).

Proposition 2.6.4. *The operator $D_{i,j}$ commutes with convolution. More precisely, for $f \in C^1(\mathbb{S}^{d-1})$ and $g \in C^1[-1,1]$, $D_{i,j}(f*g) = (D_{i,j}f)*g$. In particular, the operator $D_{i,j}$, hence Δ_0, commutes with the operator L_n.*

Proof. Directly from the definition,

$$D_{i,j}^{(x)} g(\langle x,y\rangle) = g'(\langle x,y\rangle)(x_i y_j - y_i x_j) = -D_{i,j}^{(y)} g(\langle x,y\rangle),$$

where $D_{i,j}^{(x)}$ means that $D_{i,j}$ acts on the x variable. Hence,

$$\begin{aligned} D_{i,j}(f*g)(x) &= \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y) D_{i,j}^{(x)} g(\langle x,y\rangle)\mathrm{d}\sigma(y) \\ &= -\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y) D_{i,j}^{(y)} g(\langle x,y\rangle)\mathrm{d}\sigma(y) = (D_{i,j}f)*g(x), \end{aligned}$$

where the last step follows from (1.8.7). □

The fact that $L_n f$ reproduces polynomials of degree up to n and that it is a polynomial of degree at most $2n-1$ itself makes it a fundamental tool in polynomial approximation. Even more, its kernel, $L_n(t)$, possesses the remarkable property that

L_n and its derivatives $L_n^{(j)}$ are highly localized at $t = 1$. More precisely, we state the following theorem.

Theorem 2.6.5. *Let ℓ be a positive integer. For $n \ge 1$ and $\theta \in [0,\pi]$,*

$$|L_n^{(j)}(\cos\theta)| \le c_{\ell,j} \left\| \eta^{(3\ell-1)} \right\|_\infty n^{d-1+2j}(1+n\theta)^{-\ell}, \quad j = 0,1,\ldots. \tag{2.6.5}$$

By choosing ℓ large but fixed, the theorem shows that L_n and its derivatives decay faster than any polynomial of a fixed degree. This desirable property will be used in a number of occasions in this book. As an application, we prove the following corollary first.

Corollary 2.6.6. *Let ℓ be a positive integer and let $\delta > 0$. Then*

$$\sup_{z \in \mathtt{c}(y,\frac{\delta}{n})} |L_n(\langle x,y\rangle) - L_n(\langle x,z\rangle)| \le c\delta n^{d-1}(1+n\mathtt{d}(x,y))^{-\ell} \tag{2.6.6}$$

for all $x,y \in \mathbb{S}^{d-1}$ that satisfy $\mathtt{d}(x,y) \ge 4\delta/n$.

Proof. If $z \in \mathtt{c}(y,\frac{\delta}{n})$ and $\mathtt{d}(x,y) \ge 4\delta/n$, then $1+n\mathtt{d}(x,z) \sim 1+n\mathtt{d}(x,y)$ by the triangle inequality. Applying the estimate (2.6.5) with $j = 1$ and $\ell+1$ instead of ℓ, we obtain, by the mean value theorem,

$$|L_n(\langle x,y\rangle) - L_n(\langle x,z\rangle)| \le c|\langle x,y\rangle - \langle x,z\rangle| n^{d+1}(1+n\mathtt{d}(x,y))^{-\ell-1}.$$

Since $\langle x,y\rangle = \cos d(x,y)$, it follows by the triangle inequality that

$$\begin{aligned} |\langle x,y\rangle - \langle x,z\rangle| &= 2\sin\frac{\mathtt{d}(x,z)-\mathtt{d}(x,y)}{2}\sin\frac{\mathtt{d}(x,z)+\mathtt{d}(x,y)}{2} \\ &\le c\mathtt{d}(z,y)(\mathtt{d}(x,y)+n^{-1}) \le c\delta n^{-2}(1+n\mathtt{d}(x,y)). \end{aligned}$$

Putting the two inequalities together proves the stated result. ◻

Recall that the Gegenbauer polynomials C_n^λ are special cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}$. For further uses, we shall prove a more general result than Theorem 2.6.5. For this, we define, for $\alpha \ge \beta \ge -\frac{1}{2}$,

$$G_n^{(\alpha,\beta)}(t) := \sum_{k=0}^{\infty} \varphi\left(\frac{k}{n}\right) \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)} P_k^{(\alpha,\beta)}(t) \tag{2.6.7}$$

for some smooth cutoff function φ. If $\varphi = \eta$, then by (B.2.1) in Appendix B,

$$L_n(t) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d-1)} G_n^{\left(\frac{d-3}{2},\frac{d-3}{2}\right)}(t),$$

so that Theorem 2.6.5 follows directly from the estimates of the kernels $G_n^{(\alpha,\beta)}(t)$. Furthermore, the assumption about the cutoff function in the theorem below is considerably weaker than that of Theorem 2.6.5.

Theorem 2.6.7. *Let ℓ be a positive integer and let $\varphi \in C^{3\ell-1}[0,\infty)$ satisfy supp $\varphi \subset [0,2]$ and $\varphi^{(j)}(0) = 0$ for $j = 1,2,\dots,3\ell-2$. Then the kernel function $G_n \equiv G_n^{(\alpha,\beta)}$ defined in (2.6.7), with $\alpha \ge \beta \ge -1/2$, satisfies, for $\theta \in [0,\pi]$ and $n \in \mathbb{N}$,*

$$\left|G_n^{(j)}(\cos\theta)\right| \le c_{\ell,j,\alpha}\left\|\varphi^{(3\ell-1)}\right\|_\infty n^{2\alpha+2j+2}(1+n\theta)^{-\ell}, \quad j=0,1,\dots. \tag{2.6.8}$$

Proof. Taking derivatives and using (B.1.5) in Appendix B, it follows that

$$G_n^{(j)}(t) = \sum_{k=0}^{\infty}\varphi\left(\frac{k+j}{n}\right)\frac{(2k+\alpha+\beta+2j+1)\Gamma(k+\alpha+\beta+2j+1)}{2^j\Gamma(k+\beta+j+1)}P_k^{(\alpha+j,\beta+j)}(t).$$

Because φ is supported on $[0,2)$, the summation terminates at $2n-j$. Summing by parts ℓ times and using (B.1.10) with $(\alpha+j+i,\beta+j)$ in place of (α,β) at the ith time for $i=1,\dots,\ell$, we obtain

$$G_n^{(j)}(t) = 2^{-j}\sum_{k=0}^{\infty}a_{n,\ell}(k)\frac{\Gamma(k+\alpha+\beta+2j+\ell+1)}{\Gamma(k+\beta+j+1)}P_k^{(\alpha+j+\ell,\beta+j)}(t), \tag{2.6.9}$$

where $\{a_{n,\ell}\}_{\ell=0}^{\infty}$ is a sequence of functions on $[0,\infty)$ defined recursively by

$$a_{n,0}(s) := (2s+\alpha+\beta+2j+1)\varphi\left(\frac{s+j}{n}\right),$$

$$a_{n,\ell+1}(s) := \frac{a_{n,\ell}(s)}{2s+\alpha+\beta+2j+\ell+1} - \frac{a_{n,\ell}(s+1)}{2s+\alpha+\beta+2j+\ell+3}.$$

We claim that if $m+j \le \ell$ and $\ell \ge 1$, then

$$\left|a_{n,\ell}^{(m)}(s)\right| \le c_{\ell,j}(s+1)^{-m-2j+1}\left(\frac{s+1}{n}\right)^{2\ell-1}\left\|\varphi^{(2\ell+m+j-1)}\right\|_{L^\infty\left[0,\frac{s+j+\ell}{n}\right]}, \tag{2.6.10}$$

which implies, in particular, with $m=0$ and $j=\ell$,

$$|a_{n,\ell}(k)| \le c_{\ell,\ell}\left\|\varphi^{(3\ell-1)}\right\|_\infty n^{-2\ell+1}. \tag{2.6.11}$$

For the moment, we take (2.6.10) for granted and proceed with the proof of (2.6.8). Using (2.6.11) and (2.6.9), we obtain

$$\left|G_n^{(j)}(\cos\theta)\right| \le c_{\ell,j}\left\|\varphi^{(3\ell-1)}\right\|_\infty n^{-2\ell+1}\sum_{k=0}^{2n}(k+1)^{\alpha+j+\ell}\left|P_k^{(\alpha+j+\ell,\beta+j)}(\cos\theta)\right|,$$

which implies, by (B.1.7), that for $\theta \in [0, \pi/2]$,

$$\left|G_n^{(j)}(\cos\theta)\right| \le c_{\ell,j} \left\|\varphi^{(3\ell-1)}\right\|_\infty n^{-2\ell+1} \left[\sum_{0\le k\le \max\{\theta^{-1},2n\}} (k+1)^{2\alpha+2j+2\ell} \right.$$
$$\left. + \sum_{\max\{\theta^{-1},2n\}<k\le 2n} (k+1)^{\alpha+j+\ell-\frac{1}{2}} \theta^{-\alpha-j-\ell-\frac{1}{2}} \right]$$
$$\le c_{\ell,j} \left\|\varphi^{(3\ell-1)}\right\|_\infty n^{2\alpha+2j+2} (1+n\theta)^{-(\alpha+j+\ell+\frac{1}{2})},$$

and that for $\theta \in [\frac{\pi}{2}, \pi]$,

$$\left|G_n^{(j)}(\cos\theta)\right| \le c_{\ell,j} \left\|\varphi^{(3\ell-1)}\right\|_\infty n^{-2\ell+1} \sum_{k=0}^{2n} (k+1)^{\alpha+j+\ell} (k+1)^{\beta+j}$$
$$\le c_{\ell,j} \left\|\varphi^{(3\ell-1)}\right\|_\infty n^{2\alpha+2j+2} n^{-\ell},$$

where the last step uses the assumption $\alpha \ge \beta$. Putting the above together and recalling that $\alpha \ge -\frac{1}{2}$, we have proved the desired estimate (2.6.8).

It remains to prove the claim (2.6.10). We first observe that by Taylor's theorem,

$$\left\|\varphi^{(m)}\right\|_{L^\infty[0,t]} \le \frac{t^k}{k!} \left\|\varphi^{(m+k)}\right\|_{L^\infty[0,t]}, \quad t \ge 0, \tag{2.6.12}$$

whenever $m \ge 1$ and $m+k \le 3\ell-1$. Next we prove the case $j=1$ of (2.6.10). Since $a_{n,1}$ is supported on $[0,2n]$, we may assume, without loss of generality, that $0 \le s \le 2n$. Directly from the definition, we have

$$a_{n,1}(s) = \varphi\left(\frac{s+j}{n}\right) - \varphi\left(\frac{s+j+1}{n}\right) = -\int_{\frac{j}{n}}^{\frac{j+1}{n}} \varphi'\left(\frac{s}{n}+t\right) \mathrm{d}t,$$

which implies, if we combine (2.6.12) with $k = 2\ell-1$, that

$$\left|a_{n,1}^{(m)}(s)\right| \le n^{-m} \left| \int_{\frac{j}{n}}^{\frac{j+1}{n}} \varphi^{(m+1)}\left(\frac{s}{n}+t\right) \mathrm{d}t \right|$$
$$\le n^{-m-1} \frac{1}{(2\ell-1)!} \left(\frac{s+j+1}{n}\right)^{2\ell-1} \left\|\varphi^{(m+2\ell)}\right\|_{L^\infty[0,\frac{s+j+1}{n}]}$$
$$\le c_{\ell,j} (s+1)^{-m-1} \left(\frac{s+1}{n}\right)^{2\ell-1} \left\|\varphi^{(m+2\ell)}\right\|_{L^\infty[0,\frac{s+j+1}{N}]},$$

where the last step uses the assumption $0 \le s \le 2n$. This proves (2.6.10) when $\ell = 1$. We now proceed by induction. Assuming that (2.6.10) has been proven for some $\ell \ge 1$ and observing that

$$a_{n,\ell+1}^{(m)}(s) = -\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} \left(\frac{a_{n,\ell}(s+t)}{2s+2t+\alpha+\beta+2j+\ell+1}\right) \mathrm{d}t,$$

we obtain, by the product formula of derivatives, that for $m+j+1 \le \ell$,

$$\begin{aligned} \left|a_{n,\ell+1}^{(m)}(s)\right| &\le \int_0^1 \max_{k_1+k_2=m+1} \left|a_{n,j}^{(k_1)}(s+t)\right| (s+1)^{-k_2-1}\,\mathrm{d}t \\ &\le c_{\ell,j}(s+1)^{-m-2j-1}\left(\frac{s+1}{n}\right)^{2\ell-1} \left\|\varphi^{(2\ell+m+j)}\right\|_{L^\infty[0,\frac{s+j+\ell+1}{n}]}. \end{aligned}$$

This proves (2.6.10) in the case of $\ell+1$ and completes the induction. □

2.7 Notes and Further Results

The translation operator T_θ is also called the spherical mean in the literature. Most of the properties of T_θ stated in the first section can be found in [14, 135, 148]; see also [133, 174]. The following useful representation of T_θ using the Haar measure on $SO(d)$ was proved in [42] in the case of even d:

$$T_\theta f(x) = \int_{SO(d)} f(Q^{-1}M_\theta Q x)\,dQ, \quad x \in \mathbb{S}^{d-1}, \ f \in L(\mathbb{S}^{d-1}),$$

where dQ denotes the Haar measure on $SO(d)$ normalized by $\int_{SO(d)} dQ = 1$, and M_θ is a $d \times d$ orthogonal matrix given by

$$M_\theta := \begin{pmatrix} \cos\theta & \sin\theta & \cdots & 0 & 0 \\ -\sin\theta & \cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos\theta & \sin\theta \\ 0 & 0 & \cdots & -\sin\theta & \cos\theta \end{pmatrix}.$$

A general theory of convolutions and multipliers on $\mathbb{S}^{d-1}$ can be found in [63].

Sharp estimates of the Cesáro kernels and their derivatives are often needed in analysis. The kernels K_n^δ are the (C,δ) means of the Gegenbauer polynomials, a special case of the Jacobi polynomials. For the estimates of these kernels and their derivatives, we refer to [18,30,34]. A more refined result is the following asymptotic expression for the Cesàro kernels in [109, 155]: for $\lambda = \frac{d-2}{2}$ and $\frac{\pi}{2n+2} \le \theta \le \pi - \frac{\pi}{2n+2}$,

$$K_n^\delta(\cos\theta) = \frac{2A_n^\lambda}{A_n^\delta} \frac{\sin\left(\left(n+\lambda+\frac{\delta+1}{2}\right)\theta - \frac{\lambda+\delta}{2}\pi\right)}{(2\sin\theta)^\lambda (2\sin\frac{\theta}{2})^{1+\delta}}$$
$$+ \frac{(n+1)^{\lambda-\delta-1}\eta_n^{\lambda,\delta}(\theta)}{(\sin\theta)^{\lambda+1}(\sin\frac{\theta}{2})^{1+\delta}} + \frac{\xi_n^{\lambda,\delta}(\theta)}{(n+1)(\sin\frac{\theta}{2})^{2+2\lambda}},$$

where $|\eta_n^{\lambda,\delta}(\theta)| + |\xi_n^{\lambda,\delta}(\theta)| \le c$. Precise constants in the main term of the asyptotics of the Lebesgue constants of the Cesàro means were obtained in [109]. For a detailed discussion of Cesàro summability in L^p and H^p spaces, the reader is referred to [18, 34, 155].

A complement of Corollary 2.4.8 is a counterexample of [163]: there exists a function $f \in L(\mathbb{S}^{d-1})$ for which $\lim_{n\to\infty} \sup_{j\ge n} |S_j^{\frac{d-2}{2}}(f)(x)| = \infty$ for almost every $x \in \mathbb{S}^{d-1}$. Furthermore, a complement of Corollary 2.4.5 is the following result in [24] on the convergence of the Cesàro means at the critical index $\delta = \frac{d-2}{2}$: if $\int_{\mathbb{S}^{d-1}} |f(x)| \log^2(1+|f(x)|)\, d\sigma(x) < \infty$, then $\lim_{n\to\infty} S_n^{\frac{d-2}{2}}(f)(x) = f(x)$ for almost every $x \in \mathbb{S}^{d-1}$.

The operator $L_n f$ was used in [148], but the use of such an operator on the sphere appeared already in [95]. The fast decay of the kernel was established in [23, 120, 129, 138]. Under additional assumptions on the cutoff function, the rate of decay can be improved to the subexponential estimate [91]

$$\left|L_n^{(j)}(\cos\theta)\right| \le c_1 n^{2\alpha+2j+2} \exp\left\{-\frac{c_2 n\theta}{[\ln(e+n\theta)]^{1+\varepsilon}}\right\}, \quad 0 \le \theta \le \pi, \qquad (2.7.1)$$

where $c_2 = c'\varepsilon$ with $c' > 0$ an absolute constant and $c_1 = c''8^j$ with $c'' > 0$ depending only on α, β, and ε.

Fast Fourier spherical transforms are available for numerical computation of the spherical harmonic expansions; see [62, 101, 143].

Chapter 3
Littlewood–Paley Theory and the Multiplier Theorem

Two fundamental results in harmonic analysis are the Fefferman–Stein theorem on maximal functions and the Claderón–Zygmund decomposition theorem, which we need not only in $L^p(\mathbb{S}^{d-1})$ but also in the weighted L^p spaces on spheres and other regular domains, such as balls, in later chapters. In the first section, we state these theorems in the more general setting of homogeneous spaces, in anticipation of later applications. A Littlewood–Paley theory on the sphere is developed in the second section, where the Cesàro means appear as a powerful tool and the Fefferman–Stein inequality plays a key role. As an application of the Littlewood–Paley theory, a Marcinkiewicz-type multiplier theorem is proved in the third section, which gives a sufficient condition for the L^p boundedness of multiplier operators. Fractional powers of the Laplace–Beltrami operator are defined in the fourth section, where the boundedness of the square root of the Laplace–Beltami operator is compared with that of first-order angular differential operators. The result is used to define and study the Riesz operators on the sphere.

3.1 Analysis on Homogeneous Spaces

Both the Euclidean space $\mathbb{R}^d$ and the sphere $\mathbb{S}^{d-1}$ are homogeneous spaces. We start with the definition of homogeneous spaces in general.

Definition 3.1.1. A homogeneous space is a measure space (X,μ,ρ) with a positive measure μ and a metric ρ such that all open balls $B(x,r) := \{y \in X : \rho(x,y) < r\}$ are measurable, and μ is a regular measure satisfying the doubling property

$$\mu(B(x,2r)) \le C\mu(B(x,r)), \quad \forall x \in X, \quad \forall r > 0, \tag{3.1.1}$$

where C is independent of x and r. The least constant C in Eq. (3.1.1) is called the doubling constant; it is referred to as the geometric constant of the space.

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Many of the measures in analysis satisfy the doubling condition. In this book, we are mainly concerned with X that are regular domains such as a spheres and balls, and our measures are absolutely continuous with respect to the usual Lebesgue measure. For examples of our measures, see the Chap. 5.

In the rest of this section, we assume that (X,μ,ρ) is a fixed homogeneous space and state several results without proof. Our main reference for this section is [158], where the full proofs can be found.

Definition 3.1.2. If f is a locally integrable function with respect to $d\mu$ on X, its Hardy–Littlewood maximal function is defined by

$$M_\mu f(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\,\mathrm{d}\mu(y), \quad x \in X.$$

The Vitali-type covering lemma extends to the setting of homogeneous spaces [158, p. 12, Lemma 1], where it yields the following maximal inequality [158, p. 13, Theorem 1].

Theorem 3.1.3. *Let f be a function defined on X.*

(*a*) *If $f \in L^p(d\mu)$ and $1 \le p \le \infty$, then $M_\mu f$ is finite almost everywhere.*
(*b*) *If $f \in L(d\mu)$, then for every $\alpha > 0$,*

$$\mu\left(\left\{x \in X : \; M_\mu f(x) > \alpha\right\}\right) \le c_1 \frac{\|f\|_{L^1(d\mu)}}{\alpha},$$

where c_1 depends only on the geometric constant of the space.
(*c*) *If $f \in L^p(d\mu)$ and $1 < p \le \infty$, then $M_\mu f \in L^p(d\mu)$ and*

$$\|M_\mu f\|_{L^p(d\mu)} \le c_p \|f\|_{L^p(d\mu)},$$

where the bound depends only on p and c_1.

The maximal function $M_\mu f$ satisfies the Fefferman–Stein inequality [74], the proof of which can be found in [158, pp. 51–55].

Theorem 3.1.4 (Fefferman–Stein). *If $1 < p,q < \infty$ and $\{f_j\}$ is a sequence of functions on X, then*

$$\left\| \left(\sum_{j=1}^{\infty} |M_\mu(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L^p(d\mu)} \le C_{p,q} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(d\mu)}.$$

Since we assume that μ is a regular measure, the space of continuous functions with bounded support is dense in $L^1(d\mu)$, and as a result, the Lebesgue differential theorem holds.

Corollary 3.1.5 ([158, p. 13, Corollary]). *If f is locally integrable with respect to $d\mu$, then for almost every x in X,*

$$\lim_{r\to 0+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\,\mathrm{d}\mu(y) = f(x).$$

The classical Calderón–Zygmund decomposition theorem also holds in homogeneous spaces.

Theorem 3.1.6 (Calderón–Zygmund decomposition). *Suppose we are given a function $f \in L^1(d\mu)$ and a number α with $\alpha > \frac{1}{\mu(X)} \int_X |f(y)|\,\mathrm{d}\mu(y)$ ($\alpha > 0$ if $\mu(X) = \infty$). Then there exist a decomposition $f = g + b$ of f with $b = \sum_k b_k$ and a sequence of balls $\{B_k\}$ such that for some positive c depending only on the geometric constant of the space,*

(*i*) $|g(x)| \le c\alpha$ *for a.e.* $x \in X$;
(*ii*) *each* b_k *is supported in* B_k,

$$\frac{1}{\mu(B_k)} \int_{B_k} |b_k(x)|\,\mathrm{d}\mu(x) \le c\alpha \quad \textit{and} \quad \int_{B_k} b_k(x)\,\mathrm{d}\mu(x) = 0;$$

(*iii*) $\sum_k \mu(B_k) \le \frac{c}{\alpha} \int_X |f(x)|\,\mathrm{d}\mu(x)$.

The proof of this theorem can be found in [158, p. 17, p. 37, 8.1].

We end this section with two general results on semigroups of operators. The definition of such operators is given in [157, p. 2].

Definition 3.1.7. Let (X,μ) be a measure space with a positive σ-finite measure μ. A family of operators $\{T^t\}_{t\ge 0}$ is said to form a symmetric diffusion semigroup if

$$T^{t_1}T^{t_2} = T^{t_1+t_2}, \qquad T^0 = \mathrm{id},$$

and it satisfies the following assumptions:

(*i*) T^t are contractions on $L^p(X,\mu)$, i.e., $\|T^t f\|_{L^p(d\mu)} \le \|f\|_{L^p(d\mu)}$, $1 \le p \le \infty$;
(*ii*) T^t are symmetric, i.e., each T^t is self-adjoint on $L^2(X,d\mu)$;
(*iii*) T^t are positive, i.e., $T^t f \ge 0$ if $f \ge 0$;
(*iv*) $T^t f_0 = f_0$ if $f_0(x) = 1$.

Theorem 3.1.8. *Suppose that $\{T^t\}_{t\ge 0}$ is a symmetric diffusion semigroup on (X,μ). Then the function*

$$Mf(x) = \sup_{s>0} \left(\frac{1}{s} \int_0^s T^t f(x)\mathrm{d}t \right)$$

satisfies the inequalities

(*a*) $\|Mf\|_{L^p(d\mu)} \le c_p \|f\|_{L^p(d\mu)}$ *for each* p *with* $1 < p \le \infty$;
(*b*) $\mu(\{x \in X : Mf(x) > \alpha\}) \le (c/\alpha)\|f\|_{L^1(d\mu)}$ *for each* $\alpha > 0$ *and* $f \in L^1(X,\mu)$, *where* c *is independent of* f *and* α.

This statement can be found in [157, p. 48], and it is a special case of the Hopf–Dunford–Schwartz ergodic theorem.

Finally, given $f \in L^p(X,\mu)$, its Littlewood–Paley g-function in terms of $\{T^t\}$ is defined by

$$\tilde{g}(f) := \left(\int_0^\infty t \left| \frac{\partial}{\partial t} T^t f \right|^2 \mathrm{d}t \right)^{\frac{1}{2}}. \tag{3.1.2}$$

We then have the following theorem [157, Theorem 10, p. 111].

Theorem 3.1.9. *For $f \in L^p(d\mu)$, $1 < p < \infty$,*

$$c_p^{-1} \|f\|_{L^p(d\mu)} \le \|\tilde{g}(f)\|_{L^p(d\mu)} \le c_p \|f\|_{L^p(d\mu)},$$

where the first inequality holds under the additional assumption $\int_X f \,\mathrm{d}\mu = 0$, and the constant c_p is independent of f.

3.2 Littlewood–Paley Theory on the Sphere

In this section, we develop a Littlewood–Paley theory on the sphere, which will be used to establish the multiplier theorem in the following section.

Recall the Poisson integral $P_r f$ in Definition 2.2.3. We define

$$g(f) := \left(\int_0^1 (1-r) \left| \frac{\partial}{\partial r} P_r f \right|^2 \mathrm{d}r \right)^{\frac{1}{2}}, \tag{3.2.1}$$

which is a function $x \mapsto g(f)(x)$ defined on $\mathbb{S}^{d-1}$. The boundedness of $g(f)$ in $L^p(\mathbb{S}^{d-1})$ follows as a consequence of Theorem 3.1.9.

Theorem 3.2.1. *If $1 < p < \infty$ and $f \in L^p(\mathbb{S}^{d-1})$, then*

$$c_p^{-1} \|f\|_p \le \|g(f)\|_p \le c_p \|f\|_p, \tag{3.2.2}$$

where the condition $\int_{\mathbb{S}^{d-1}} f(x) \,\mathrm{d}\sigma(x) = 0$ is required in the first inequality.

Proof. By Lemma 2.2.4, it is easy to see that $\{P_{e^{-t}} : t \ge 0\}$ is a semigroup of operators. For example, using (i) in the lemma and the orthogonality of spherical harmonics, we have $T^r T^s = T^{r+s}$, where $T^t = P_{e^{-t}}$. For this semigroup, the function $\tilde{g}$ in Eq. (3.1.2) becomes

$$\tilde{g}(f) = \left(\int_0^\infty \left| \frac{\partial}{\partial t} (P_{e^{-t}} f) \right|^2 t \,\mathrm{d}t \right)^{\frac{1}{2}} = \left(\int_0^1 \left| \frac{\partial}{\partial r} P_r f \right|^2 r |\log r| \,\mathrm{d}r \right)^{\frac{1}{2}},$$

and Theorem 3.1.9 shows that $\|\tilde{g}(f)\|_p \sim \|f\|_p$. Since $r|\log r| \le c(1-r)$ on $[0,1]$, we have $\tilde{g}(f)(x) \le g(f)(x)$, and the lower bound of $\|g(f)\|_p$ in Eq. (3.2.2) follows from $c\|f\|_p \le \|\tilde{g}(f)\|_p$. To prove the upper bound, we split the integral in $g(f)$ into two parts:

$$g(f) \le \left(\int_0^{\frac{1}{2}} (1-r) \left| \frac{\partial}{\partial r} P_r f \right|^2 \mathrm{d}r \right)^{\frac{1}{2}} + \left(\int_{\frac{1}{2}}^1 (1-r) \left| \frac{\partial}{\partial r} P_r f \right|^2 \mathrm{d}r \right)^{\frac{1}{2}}.$$

Using Lemma 2.2.4 and $\| \operatorname{proj}_k \|_{(p,p)} \le ck^{(d-2)/2}$, which follows from Corollary 1.2.7, the $L^p(\mathbb{S}^{d-1})$ norm of the first term is bounded by

$$\sup_{0<r<\frac{1}{2}} \left\| \frac{\partial}{\partial r} P_r f \right\|_p \le \sum_{k=1}^{\infty} k 2^{-k+1} \| \operatorname{proj}_k f \|_p \le c\|f\|_p \sum_{k=1}^{\infty} 2^{-k} k^{d/2} \le c\|f\|_p,$$

while the $L^p(\mathbb{S}^{d-1})$ norm of the second term is bounded, since $r|\log r| \sim (1-r)$ for $r \in [1/2, 1]$, by $\|\tilde{g}(f)\|_p \le c\|f\|_p$. The proof is complete. □

For the proof of the multiplier theorem in the following section, we need a refined version of the Littlewood–Paley g-function $g(f)$ defined via the Cesàro means of the spherical harmonic series: for $\delta \ge 0$ and $f \in L^p(\mathbb{S}^{d-1})$, define

$$g_\delta(f) := \left(\sum_{n=1}^{\infty} \left| S_n^{\delta+1} f - S_n^{\delta} f \right|^2 n^{-1} \right)^{\frac{1}{2}}. \tag{3.2.3}$$

The two g-functions $g(f)$ and $g_\delta(f)$ are closely connected. In the following, we adopt the convention of writing $f \le g$ if $f(x) \le g(x)$ for almost all $x \in \mathbb{S}^{d-1}$.

Lemma 3.2.2. *For $\delta \ge 0$, $g(f) \le c g_\delta(f)$. In particular, if $f \in L^p(\mathbb{S}^{d-1})$ satisfies $\int_{\mathbb{S}^{d-1}} f(y) \mathrm{d}\sigma(y) = 0$, then for all $\delta \ge 0$,*

$$\|f\|_p \le c\|g_\delta(f)\|_p, \quad 1 < p < \infty.$$

Proof. From Lemma 2.2.4, we obtain

$$\begin{aligned} \frac{\partial}{\partial r} P_r f &= (1-r)^{\delta+1} (1-r)^{-\delta-1} \sum_{k=0}^{\infty} k r^{k-1} \operatorname{proj}_k f \\ &= (1-r)^{\delta+1} \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{n} k A_{n-k}^{\delta} \operatorname{proj}_k f \Big) r^{n-1}. \end{aligned} \tag{3.2.4}$$

On the other hand, a straightforward computation shows that

$$S_n^{\delta+1} f - S_n^{\delta} f = -(n+\delta+1)^{-1} (A_n^{\delta})^{-1} \sum_{k=0}^{n} k A_{n-k}^{\delta} \operatorname{proj}_k f. \tag{3.2.5}$$

These two identities connect $P_r f$ to $S_n^\delta f$, which implies, in particular, that

$$\left|\frac{\partial}{\partial r} P_r f\right| \le c(1-r)^{\delta+1} \sum_{n=1}^{\infty} n A_n^\delta \left|S_n^{\delta+1} f - S_n^\delta f\right| r^{n-1},$$

which, by the Cauchy–Schwarz inequality, implies

$$\begin{aligned}\left|\frac{\partial}{\partial r} P_r f\right|^2 &\le c(1-r)^{2\delta+2} \sum_{n=1}^{\infty} n A_n^\delta \left|S_n^{\delta+1} f - S_n^\delta f\right|^2 r^{n-1} \sum_{m=1}^{\infty} m A_m^\delta r^{m-1} \\ &= c(1+\delta)(1-r)^{\delta} \sum_{n=1}^{\infty} n A_n^\delta \left|S_n^{\delta+1} f - S_n^\delta f\right|^2 r^{n-1}.\end{aligned}$$

Consequently, we conclude that

$$\begin{aligned}|g(f)|^2 &\le c \sum_{n=1}^{\infty} n A_n^\delta \left|S_n^{\delta+1} f - S_n^\delta f\right|^2 \int_0^1 (1-r)^{1+\delta} r^{n-1} \mathrm{d}r \\ &\le c \sum_{n=1}^{\infty} n^{-1} \left|S_n^{\delta+1} f - S_n^\delta f\right|^2 = |g_\delta(f)|^2,\end{aligned}$$

where we have used $\int_0^1 (1-r)^{\delta+1} r^{n-1}\,\mathrm{d}r = \Gamma(\delta+2)\Gamma(n)/\Gamma(n+\delta+2) \sim n^{-\delta-2}$. □

The other direction of the inequality, $\|g_\delta(f)\|_p \le c\|f\|_p$, holds as well, and its proof is much more involved. In addition, we shall work with a more general g-function, which will be needed in the proof of the multiplier theorem. Let $\{v_k\}_{k=1}^\infty$ be a sequence of positive numbers satisfying

$$\sup_{n\ge 1} \frac{1}{n} \sum_{k=1}^{n} v_k =: A < \infty. \tag{3.2.6}$$

The new g-function is defined in terms of $\{v_k\}$ by

$$g_\delta^*(f) := \left(\sum_{n=1}^{\infty} |S_n^{\delta+1} f - S_n^\delta f|^2 n^{-1} v_n\right)^{\frac{1}{2}}.$$

In the special case of $v_n \equiv 1$, $g_\delta^*(f)$ becomes $g_\delta(f)$.

Theorem 3.2.3. *Assume that $\delta > (d-2)/2$ and $1 < p < \infty$. Then for every sequence $\{n_k\}$ with $n_\kappa \in \mathbb{N}_0$ and $f_k \in L^p(\mathbb{S}^{d-1})$,*

$$\left\|\left(\sum_{k=1}^{\infty} \left|S_{n_k}^\delta f_k\right|^2\right)^{\frac{1}{2}}\right\|_p \le c_p \left\|\left(\sum_{k=1}^{\infty} |f_k|^2\right)^{\frac{1}{2}}\right\|_p. \tag{3.2.7}$$

Proof. This is an immediate consequence of Eq. (2.4.6) and the Fefferman–Stein inequality in Theorem 3.1.4. □

The boundedness of g_δ^* is given in the following theorem.

Theorem 3.2.4. *Assume that* $1 < p < \infty$ *and that the inequality* (3.2.7) *holds for all* $\{n_k\} \subset \mathbb{N}_0$ *and* $f_k \in L^p(\mathbb{S}^{d-1})$*. Then* $\|g_\delta^*(f)\|_p \le c_p \|f\|_p$ *with a constant* $c_p = Ac_p'$ *independent of* f *and* v_k*.*

The proof of Theorem 3.2.4 amounts to showing that $\|g_\delta^*(f)\|_p$ is bounded by $\|g(f)\|_p$ for $g(f)$ in Eq. (3.2.1). This will require several lemmas, which we prove first. The first two lemmas relate $S_n^\delta f$ to $P_r f$.

Lemma 3.2.5. *If* $\delta \ge 0$, $1 - \frac{1}{n} \le r < 1$, *and* $f \in L^1(\mathbb{S}^{d-1})$, *then*

$$S_n^\delta f = r^{-n} P_r\left(S_n^\delta f\right) + \sum_{j=0}^{n-1} a_{j,n}^\delta P_r\left(S_j^\delta f\right), \tag{3.2.8}$$

where the coefficients $a_{j,n}^\delta$ *satisfy, with a constant* c *independent of* f *and* n*,*

$$\max_{0 \le j \le n-1} |a_{j,n}^\delta| \le c(1-r). \tag{3.2.9}$$

Proof. From Eq. (A.4.4), we obtain

$$(1-s)^{-\delta-1} P_s f = \sum_{n=0}^{\infty} \left(A_n^\delta S_n^\delta f\right) s^n. \tag{3.2.10}$$

On the other hand, since $P_s f = P_{s/r} P_r f$ for $0 < s < r < 1$, we obtain, by Eq. (A.4.3),

$$\begin{aligned}
(1-s)^{-\delta-1} P_s f &= (1-s)^{-\delta-1}\left(1 - \frac{s}{r}\right)^{1+\delta}\left(1 - \frac{s}{r}\right)^{-1-\delta} P_{s/r}(P_r f) \\
&= \left(\sum_{j=0}^{\infty} A_j^\delta s^j\right)\left(\sum_{j=0}^{\infty} A_j^{-\delta-2}\left(\frac{s}{r}\right)^j\right)\left(\sum_{j=0}^{\infty} A_j^\delta P_r(S_j^\delta f)\left(\frac{s}{r}\right)^j\right) \\
&= \sum_{n=0}^{\infty} s^n \left[\sum_{j=0}^{n} A_j^\delta P_r(S_j^\delta f) r^{-j} \sum_{k+\ell=n-j} A_k^\delta A_\ell^{-\delta-2} r^{-\ell}\right],
\end{aligned}$$

where the second step uses Eq. (3.2.10) with s/r in place of s. Comparing the coefficients of s^n in the above two identities yields (3.2.8) with

$$a_{j,n}^\delta = (A_n^\delta)^{-1} A_j^\delta r^{-n} \sum_{\ell=0}^{n-j} A_{n-j-\ell}^\delta A_\ell^{-\delta-2} r^{n-j-\ell}.$$

For $1-n^{-1} \le r \le 1$, we have $r^{-n} \le (1-n^{-1})^{-n} \le e^{-1}$. Hence setting $m = n-j$ yields

$$|a_{j,N}^{\delta}| \le c \left| \sum_{k=0}^{m} A_{m-k}^{\delta} A_k^{-\delta-2} r^{m-k} \right| \le c(1-r)\left(1+(m(1-r))^k\right),$$

where the second step follows from the inequality (A.4.10) proved in Appendix A.4. Since $1-n^{-1} \le r \le 1$, this proves (3.2.9). □

Lemma 3.2.6. *If $\delta \ge 0$, $0<r<1$, and $f \in L^1(\mathbb{S}^{d-1})$, then*

$$P_r(S_n^{\delta} f) = \sum_{j=0}^{n} b_{j,n}^{\delta} S_j^{\delta} f, \tag{3.2.11}$$

where the coefficients $b_{j,n}^{\delta}$ satisfy, with a constant c_{δ} independent of f and n,

$$\sum_{j=0}^{n} |b_{j,n}^{\delta}| \le c_{\delta}. \tag{3.2.12}$$

Proof. Using Eq. (3.2.10), we have, for $0 < s, r < 1$,

$$(1-s)^{-\delta-1} P_s P_r f = \sum_{n=0}^{\infty} A_n^{\delta} \left(P_r S_n^{\delta} f\right) s^n.$$

On the other hand, since $P_s P_r = P_{sr}$, we obtain, by Eq. (A.4.3),

$$\begin{aligned}(1-s)^{-1-\delta} P_s P_r f &= (1-s)^{-\delta-1}(1-sr)^{1+\delta}(1-sr)^{-\delta-1} P_{sr} f \\ &= \left(\sum_{j=0}^{\infty} A_j^{\delta} s^j\right)\left(\sum_{j=0}^{\infty} A_j^{-\delta-2} s^j r^j\right)\left(\sum_{j=0}^{\infty} A_j^{\delta}\left(S_j^{\delta} f\right)(sr)^j\right) \\ &= \sum_{n=0}^{\infty} s^n \left(\sum_{j=0}^{n} A_j^{\delta}\left(S_j^{\delta} f\right) r^j \sum_{k+\ell=n-j} A_k^{\delta} A_{\ell}^{-\delta-2} r^{\ell}\right).\end{aligned}$$

Comparing the coefficients of s^n of the above two identities yields (3.2.11) with

$$b_{j,n}^{\delta} := (A_n^{\delta})^{-1} A_j^{\delta} r^j \sum_{\ell=0}^{n-j} A_{n-j-\ell}^{\delta} A_{\ell}^{-\delta-2} r^{\ell}.$$

The sum in $b_{j,n}^{\delta}$ can be estimated by a slight modification of Eq. (A.4.10), which shows that

$$\left|b_{j,n}^{\delta}\right| \le c (A_n^{\delta})^{-1} A_j^{\delta} r^j (1-r)\left(1+((n-j)(1-r))^{\delta}\right).$$

Consequently, it follows that

$$\sum_{j=0}^{n-1}\left|b_{j,n}^{\delta}\right| \le c(1-r)\sum_{j=0}^{\infty} r^j + c(1-r)^{\delta+1}\sum_{j=0}^{\infty} A_j^{\delta} r^j \le c,$$

where the last step follows from Eq. (A.4.3). This completes the proof. □

In the next lemma, the length of a given interval $I \subset \mathbb{R}$ is denoted by $|I|$.

Lemma 3.2.7. *Assume that $1 < p < \infty$ and that the inequality (3.2.7) holds. If $r_j \in (0,1)$ and I_j is a subinterval of $[r_j, 1)$ for $j = 1, 2, \ldots$, then the inequality*

$$\left\|\left(\sum_{k=1}^{\infty}\left|S_{n_k}^{\delta} P_{r_k} f_k\right|^2\right)^{\frac{1}{2}}\right\|_p \le c_p \left\|\left(\sum_{k=1}^{\infty}\frac{1}{|I_k|}\int_{I_k}|P_r f_k|^2\,\mathrm{d}r\right)^{\frac{1}{2}}\right\|_p \tag{3.2.13}$$

holds for all sequences $\{n_k\}_{k=1}^{\infty}$ with $n_k \in \mathbb{N}_0$ and $f_k \in L^1(\mathbb{S}^{d-1})$ with a constant c_p independent of $\{r_k\}$, $\{I_k\}$, $\{n_k\}$, and $\{f_k\}$.

Proof. First we use Lemma 3.2.6 to obtain

$$\left|S_{n_k}^{\delta} P_{r_k} f_k\right|^2 \le c\sum_{\ell=0}^{n_k}\left|b_{\ell,n_k}^{\delta}\right|\left|S_{\ell}^{\delta} f_k\right|^2, \quad k = 1, 2, \ldots.$$

Summing over k and invoking (3.2.7), we deduce

$$\begin{aligned}\left\|\left(\sum_{k=1}^{\infty}\left|S_{n_k}^{\delta} P_{r_k} f_k\right|^2\right)^{\frac{1}{2}}\right\|_p &\le c_p \left\|\left(\sum_{k=1}^{\infty}\sum_{\ell=0}^{n_k}\left|b_{\ell,n_k}^{\delta}\right| |f_k|^2\right)^{\frac{1}{2}}\right\|_p \\ &\le c_p \left\|\left(\sum_{k=1}^{\infty}|f_k|^2\right)^{\frac{1}{2}}\right\|_p. \end{aligned} \tag{3.2.14}$$

Next, we show that the desired inequality (3.2.13) follows from Eq. (3.2.14). To see this, for each $k \ge 0$ and $n \ge 1$, we let $\{r_{k,i}\}_{i=0}^{2^n} \subset I_k$ be such that $r_{k,i} - r_{k,i-1} = 2^{-n}|I_k|$ for all $1 \le i \le 2^n$. Then for each $n \in \mathbb{N}$, $R_n := 2^{-n}\sum_{i=1}^{2^n}|P_{r_{k,i}} f_k|^2$ is a Riemann sum of the integral $\frac{1}{|I_k|}\int_{I_k}|P_r f_k|^2\,\mathrm{d}r$. Thus,

$$\left\|\left(\sum_{k=1}^{\infty}\frac{1}{|I_k|}\int_{I_k}|P_r f_k|^2\,\mathrm{d}r\right)^{\frac{1}{2}}\right\|_p = \lim_{n\to\infty}\left\|\left(2^{-n}\sum_{k=1}^{\infty}\sum_{i=1}^{2^n}|P_{r_{k,i}} f_k|^2\right)^{\frac{1}{2}}\right\|_p.$$

On the other hand, since for each fixed $n \in \mathbb{N}$, $r_k < r_{k,i}$ for all $1 \le i \le n$ and $k \in \mathbb{N}$, using Eq. (3.2.14), we have

$$\left\|\left(\sum_{k=1}^{\infty}\left|S_{n_k}^{\delta}P_{r_k}f_k\right|^2\right)^{\frac{1}{2}}\right\|_p=\left\|\left(2^{-n}\sum_{i=1}^{2^n}\sum_{k=1}^{\infty}\left|S_{n_k}^{\delta}P_{r_k/r_{k,i}}(P_{r_{k,i}}f_k)\right|^2\right)^{\frac{1}{2}}\right\|_p$$
$$\leq c_p\left\|\left(2^{-n}\sum_{i=1}^{2^n}\sum_{k=1}^{\infty}\left|P_{r_{k,i}}f_k\right|^2\right)^{\frac{1}{2}}\right\|_p.$$

Letting $n\to\infty$ proves (3.2.13) and completes the proof of the lemma. □

We are now in a position to prove Theorem 3.2.4.

Proof of Theorem 3.2.4. Without loss of generality, we may assume that $n\leq\sum_{k=1}^{n}\nu_k\leq 2n$, since otherwise, we may consider the sequence $\widetilde{\nu_j}=M^{-1}\nu_j+1$. For convenience, we define, for $n=1,2,\ldots$,

$$E_nf=-(n+1+\delta)^{-1}\sum_{k=0}^{n}k\,\mathrm{proj}_k f.$$

It follows from Eq. (3.2.5) that for $0\leq j\leq n$,

$$S_j^{\delta}(E_nf)=\frac{j+\delta+1}{n+\delta+1}\left(S_j^{\delta+1}f-S_j^{\delta}f\right). \tag{3.2.15}$$

Using Lemma 3.2.5, we obtain that for every $r\in[1-n^{-1},1)$,

$$S_n^{\delta+1}f-S_n^{\delta}f=S_n^{\delta}(E_nf)=r^{-n}P_r(S_n^{\delta}(E_nf))+\sum_{j=1}^{n-1}a_{j,n}^{\delta}P_r\left(S_j^{\delta}(E_nf)\right),$$

where we have used the fact that $P_rS_j^{\delta}=S_j^{\delta}P_r$, which can be easily verified. Applying Eq. (3.2.15) one more time shows that

$$S_n^{\delta+1}f-S_n^{\delta}f=r^{-n}\left[S_n^{\delta+1}(P_rf)-S_n^{\delta}(P_rf)\right]$$
$$+\sum_{j=1}^{n-1}\frac{j+\delta+1}{n+\delta+1}a_{j,n}^{\delta}\left[S_j^{\delta+1}(P_rf)-S_j^{\delta}(P_rf)\right]. \tag{3.2.16}$$

By Eq. (3.2.9), $|a_{j,n}^{\delta}|\leq c(1-r)\leq cn^{-1}$. Now let $\mu_1=1$ and $\mu_n=1+\sum_{i=1}^{n-1}\nu_i$ for $n>1$. Clearly, $r_n:=1-\frac{1}{\mu_n}\in[1-n^{-1},1-(2n-1)^{-1}]$. Thus, applying Eq. (3.2.16) with $r=r_n$ and setting $f_n:=P_{r_n}f$, we obtain

$$\left|S_n^{\delta+1}f-S_n^{\delta}f\right|\leq c\left|S_n^{\delta+1}f_n-S_n^{\delta}f_n\right|+cn^{-2}\sum_{j=1}^{n-1}j\left|S_j^{\delta+1}f_n-S_j^{\delta}f_n\right|.$$

It then follows from the Cauchy–Schwarz inequality that

$$\left|S_n^{\delta+1}f - S_n^\delta f\right|^2 \le c\left|S_n^{\delta+1}f_n - S_n^\delta f_n\right|^2 + cn^{-3}\sum_{j=1}^{n-1} j^2\left|S_j^{\delta+1}f_n - S_j^\delta f_n\right|^2.$$

By this inequality and Theorem 3.2.1, the proof of Theorem 3.2.4 will follow, as seen in the definition of $g_\delta^*(f)$, from the following two inequalities:

$$\left\|\left(\sum_{n=1}^\infty n^{-1}\left|S_n^{\delta+1}f_n - S_n^\delta f_n\right|^2 v_n\right)^{\frac{1}{2}}\right\|_p \le c_p\|g(f)\|_p \tag{3.2.17}$$

and

$$\left\|\left(\sum_{n=1}^\infty \frac{v_n}{n^4}\sum_{j=1}^{n-1} j^2\left|S_j^{\delta+1}f_n - S_j^\delta f_n\right|^2\right)^{\frac{1}{2}}\right\|_p \le c_p\|g(f)\|_p. \tag{3.2.18}$$

In order to prove the last two inequalities, let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$. For $n = 1, 2, \ldots$, we define the operators L_n and Q_n by

$$L_n f := \sum_{j=0}^\infty \eta\left(\frac{j}{n}\right)\operatorname{proj}_j f, \qquad Q_n f := -\sum_{j=0}^{2n} j\eta\left(\frac{j}{n}\right)\operatorname{proj}_j f.$$

Summation by parts $d+1$ times by Eq. (A.4.8) shows that

$$L_N f = \sum_{j=0}^{2N} \triangle^{d+1}\eta\left(\frac{j}{N}\right)A_j^d S_j^d f,$$

where $\triangle^{d+1}$ acts on the variable j, so that $|\triangle^{d+1}\eta(\frac{j}{N})| \le cN^{-d-1}$ by Eq. (A.3.4). As a consequence, by Theorem 2.4.7, we have

$$\sup_{N\ge 0}|L_N f| \le c\sup_{j\ge 0}\left|S_j^d f\right| \le cM(f),$$

where M is the Hardy–Littlewood maximal function. The two operators L_N and Q_N are related by the relation $P_r(Q_N f)| = rL_N(\frac{\partial}{\partial r}P_r f)$. Consequently, it follows that

$$|P_r(Q_N f)| = r\left|L_N\left(\frac{\partial}{\partial r}P_r f\right)\right| \le cM\left(\frac{\partial}{\partial r}P_r f\right).$$

Now, by Eq. (3.2.15), $f_n = P_{r_n}f$, and $S_j^\delta(E_N f) = S_j^\delta(Q_N f)$, we see that

$$S_j^{\delta+1}f_n - S_j^\delta f_n = (j+\delta+1)^{-1}P_{r_n}\left(S_j^\delta(Q_N f)\right) \tag{3.2.19}$$

for $1 \le j \le n \le N$. Using this identity with $j = n$ and Lemma 3.2.7, we obtain

$$\left\|\left(\sum_{n=1}^{N}\left|S_n^{\delta+1}f_n-S_n^{\delta}f_n\right|^2\right)^{\frac12}\right\|_p\le c\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^3}\left|P_{r_n}(S_n^{\delta}(Q_nf))\right|^2\right)^{\frac12}\right\|_p$$
$$\le c\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^3}\frac{1}{|I_n|}\int_{r_n}^{r_{n+1}}|P_r(Q_Nf)|^2\,\mathrm{d}r\right)^{\frac12}\right\|_p$$
$$\le c\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^3}\frac{1}{|I_n|}\int_{r_n}^{r_{n+1}}\left(M\left(\frac{\partial}{\partial r}P_rf\right)\right)^2\mathrm{d}r\right)^{\frac12}\right\|_p,$$

where $|I_n|=r_{n+1}-r_n$. Applying the Fefferman–Stein inequality to the Riemann sums of the integrals $\int_{r_n}^{r_{n+1}}$, we then obtain

$$\left\|\left(\sum_{n=1}^{N}\left|S_n^{\delta+1}f_n-S_n^{\delta}f_n\right|^2\right)^{\frac12}\right\|_p\le c\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^3}\frac{1}{|I_n|}\int_{r_n}^{r_{n+1}}\left|\left(\frac{\partial}{\partial r}P_rf\right)\right|^2\mathrm{d}r\right)^{\frac12}\right\|_p$$
$$\le c\left\|\left(\sum_{n=1}^{N}\frac{1}{n}\int_{r_n}^{r_{n+1}}\left|\left(\frac{\partial}{\partial r}P_rf\right)\right|^2\mathrm{d}r\right)^{\frac12}\right\|_p$$
$$\le c_p\|g(f)\|_p,$$

where we have used $|I_n|=\frac{v_n}{\mu_n\mu_{n+1}}\sim\frac{v_n}{n^2}$ in the second step and $1-r\sim\frac{1}{n}$ for all $r\in[r_n,r_{n+1}]$ in the last step. Letting $N\to\infty$ proves the inequality (3.2.17).

The proof of Eq. (3.2.18) is similar. In fact, using Lemma 3.2.7 and Eq. (3.2.19), we have

$$\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^4}\sum_{j=1}^{n-1}j^2\left|S_\ell^{\delta+1}f_n-S_\ell^{\delta}f_n\right|^2\right)^{\frac12}\right\|_p\le c\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^4}\sum_{j=1}^{n-1}\left|P_{r_n}(S_j^{\delta}Q_Nf)\right|^2\right)^{\frac12}\right\|_p$$
$$\le c_p\left\|\left(\sum_{n=1}^{\infty}\frac{v_n}{n^3}\frac{1}{|I_n|}\int_{r_n}^{r_{n+1}}\left|\frac{\partial}{\partial r}P_rf\right|^2\mathrm{d}r\right)^{\frac12}\right\|_p,$$

which already appeared in the proof of Eq. (3.2.17). □

3.3 The Marcinkiewicz Multiplier Theorem

The Marcinkiewicz multiplier theorem for spherical harmonic expansions, stated below, gives a sufficient condition for the L^p boundedness of multiplier operators. Recall the difference operator $\triangle^n$, which, when applied to a given sequence $\{a_k\}_{k=0}^{\infty}$ of complex numbers, is defined by

$$\triangle^0a_k=a_k,\quad\triangle a_k=a_k-a_{k+1},\quad\triangle^{n+1}a_k=\triangle^n(\triangle a_k),\quad n=1,2,\ldots.$$

Theorem 3.3.1. *Let $\{\mu_j\}_{j=0}^{\infty}$ be a bounded sequence of real numbers such that*

$$(B_k) \qquad \sup_{j\ge 0} 2^{j(k-1)} \sum_{\ell=2^j+1}^{2^{j+1}} |\triangle^k \mu_\ell| \le M < \infty$$

for some positive integer $k > \frac{d}{2}$. Then $\{\mu_j\}$ defines a bounded multiplier on $L^p(\mathbb{S}^{d-1})$ for $1 < p < \infty$; that is, for $f \in L^p(\mathbb{S}^{d-1})$, the inequality

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j f \right\|_p \le c_p \|f\|_p, \quad 1 < p < \infty,$$

holds with a constant $c_p = c_p' M$ independent of f and $\{\mu_j\}$.

The proof of Theorem 3.3.1 relies on two lemmas. The first one explains further the condition (B_k).

Lemma 3.3.2. *If $\{\mu_j\}$ is a bounded sequence of real numbers satisfying (B_k) for some positive integer k, then $\{\mu_j\}$ satisfies (B_i) for all $1 \le i \le k$, with a possible change of the absolute constant M.*

Proof. It suffices to show that (B_k) implies (B_{k-1}) for every $k \ge 2$. It is evident that (B_k) implies $\sum_{\ell=1}^{\infty} |\triangle^k \mu_\ell| < \infty$, which implies, by $\triangle^{k-1}\mu_j - \triangle^{k-1}\mu_{j+m} = \sum_{i=j}^{j+m-1} \triangle^k \mu_i$, that $\triangle^{k-1}\mu_j$ is a Cauchy sequence, hence converges, and the limit $\lim_{j\to\infty} \triangle^{k-1}\mu_j = 0$ holds as a consequence of $|\sum_{i=j}^{2j-1} \triangle^{k-1}\mu_i| \le c_k \sup_j |\mu_j| < \infty$. It follows then that

$$\sum_{\ell=2^j+1}^{2^{j+1}} |\triangle^{k-1}\mu_\ell| \le \sum_{\ell=2^j+1}^{2^{j+1}} \sum_{i=\ell}^{\infty} |\triangle^k \mu_i| \le 2^j \sum_{i=2^j+1}^{\infty} |\triangle^k \mu_i| \le 2M 2^{-j(k-2)},$$

where the last step used (B_k), so that (B_k) which proves (B_{k-1}). □

Lemma 3.3.3. *Let $\{a_j\}_{j=0}^{\infty}$ be a sequence of real numbers and let $\{\mu_j\}_{j=0}^{\infty}$ be a bounded sequence of real numbers satisfying $(B_{\delta+1})$. Let s_n^δ and σ_n^δ denote the Cesàro (C,δ) means of the sequences $\{a_j\}_{j=0}^{\infty}$ and $\{a_j\mu_j\}_{j=0}^{\infty}$, respectively. If δ is a nonnegative integer, then*

$$\sigma_n^\delta = \mu_n s_n^\delta + \sum_{\ell=0}^{n-1} C_{\ell,n}^\delta s_\ell^\delta, \tag{3.3.1}$$

where the constants $C_{\ell,n}^\delta$ are independent of $\{a_j\}_{j=0}^{\infty}$ and satisfy

$$\left| C_{\ell,n}^\delta \right| \le c \sum_{k=1}^{\delta+1} (\ell+1)^{k-1} \left| \triangle^k \mu_\ell \right|, \qquad \ell = 0, 1, \ldots, n-1. \tag{3.3.2}$$

Proof. Using summation by parts, we obtain, for $0 < r < 1$,

$$\sum_{n=0}^{\infty} A_n^{\delta}\sigma_n^{\delta} r^n = (1-r)^{-\delta-1}\sum_{n=0}^{\infty}\mu_n a_n r^n = (1-r)^{-\delta-1}\sum_{n=0}^{\infty}\triangle^{\delta+1}(\mu_n r^n)A_n^{\delta}s_n^{\delta}.$$

Since $\triangle^{\delta+1-k}r^{n+k} = r^{n+k}(1-r)^{\delta+1-k}$, the product formula (A.3.3) of the difference operator shows that

$$\triangle^{\delta+1}(\mu_n r^n) = \sum_{k=0}^{\delta+1}(-1)^k A_k^{-\delta-2}(\triangle^k\mu_n)r^{n+k}(1-r)^{\delta+1-k}.$$

Using $(1-r)^{-k} = \sum_{j=0}^{\infty}A_j^{k-1}r^j$, the above two identities lead to

$$\begin{aligned}\sum_{n=0}^{\infty} A_n^{\delta}\sigma_n^{\delta} r^n &= \sum_{k=0}^{\delta+1}(-1)^k A_k^{-\delta-2}(1-r)^{-k}\sum_{n=0}^{\infty}(\triangle^k\mu_n)A_n^{\delta}S_n^{\delta}r^{n+k}\\ &= \sum_{m=0}^{\infty}\left[\sum_{k=0}^{\delta+1}(-1)^k A_k^{-\delta-2}\sum_{i=0}^{m-k}A_{m-k-i}^{k-1}(\triangle^k\mu_i)A_i^{\delta}S_i^{\delta}\right]r^m.\end{aligned}$$

Comparing the coefficients of s^m yields (3.3.1) with, for $0 \le \ell \le n-1$,

$$C_{\ell,n}^{\delta} = (A_n^{\delta})^{-1}A_{\ell}^{\delta}\sum_{k=1}^{\min\{n-\ell,\delta+1\}}(-1)^k A_k^{-\delta-2}A_{n-\ell-k}^{k-1}\triangle^k\mu_{\ell},$$

where we have used the fact that $A_{m-\ell}^{-1} = 0$ for $\ell < m$ in the last step. Finally, for $0 \le \ell \le n-1$, we have

$$|C_{\ell,n}^{\delta}| \le c\ell^{\delta}n^{-\delta}\sum_{k=1}^{\delta+1}(n-\ell)^{k-1}|\triangle^k\mu_{\ell}| \le c\sum_{k=1}^{\delta+1}\ell^{k-1}|\triangle^k\mu_{\ell}|,$$

which proves the desired inequality (3.3.2). □

We are now in a position to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Without loss of generality, we may assume $\mu_0 = 0$ and $M = 1$. Let δ be the smallest integer such that $\delta > \frac{d-2}{2}$. Then Eq. (3.2.7) holds by Theorem 3.2.3. Let $F = \sum_{j=1}^{\infty}\mu_j \operatorname{proj}_j f$. By Theorem 3.2.4, it suffices to show that the inequality

$$g_{\delta}(F) \le c\left(\sum_{n=1}^{\infty}\left|S_n^{\delta+1}f - S_n^{\delta}f\right|^2 v_n n^{-1}\right)^{\frac{1}{2}} \tag{3.3.3}$$

holds for some sequence $\{v_n\}$ of positive numbers satisfying $\sup_n n^{-1}\sum_{j=1}^{n}v_j < \infty$. To establish (3.3.3), we apply Lemma 3.3.3 with s_n^{δ} and σ_n^{δ} the (C,δ) means of $\{k\operatorname{proj}_k f\}_{k=0}^{\infty}$ and $\{\mu_k k\operatorname{proj}_k f\}_{k=0}^{\infty}$, respectively, which gives, by Eq. (3.2.5),

$$S_n^{\delta+1}F - S_n^{\delta}F = \frac{-1}{n+\delta+1}\sigma_n^{\delta} = \frac{-1}{n+\delta+1}\left(\mu_n s_n^{\delta} + \sum_{\ell=0}^{n-1} C_{\ell,n}^{\delta} s_\ell^{\delta}\right),$$

where, using Eq. (3.2.5) again,

$$s_\ell^{\delta} = (A_\ell^{\delta})^{-1}\sum_{j=0}^{\ell} A_{\ell-j}^{\delta} j \operatorname{proj}_j f = -(\ell+\delta+1)\left(S_\ell^{\delta+1}f - S_\ell^{\delta}f\right).$$

It then follows by Eq. (3.3.2) that

$$|S_n^{\delta+1}F - S_n^{\delta}F| \le |\mu_n||S_n^{\delta+1}f - S_n^{\delta}f| + cn^{-1}\sum_{j=1}^{\delta+1}\sum_{\ell=1}^{n-1}\ell^j|\triangle^j\mu_\ell||S_\ell^{\delta+1}f - S_\ell^{\delta}f|.$$

Using Lemma 3.3.2 and $(B_{\delta+1})$, we conclude that

$$\sum_{j=1}^{\delta+1}\sum_{\ell=1}^{n-1}\ell^j|\triangle^j\mu_\ell| \le cn. \tag{3.3.4}$$

Hence, by the Cauchy–Schwarz inequality, we deduce

$$\begin{aligned} |g_\delta(F)|^2 &\le c|g_\delta(f)|^2 + c\sum_{j=1}^{\delta+1}\sum_{n=1}^{\infty} n^{-2}\left(\sum_{\ell=1}^{n-1}\ell^j\left|\triangle^j\mu_\ell\right|\left|S_\ell^{\delta+1}f - S_\ell^{\delta}f\right|^2\right) \\ &\le c|g_\delta(f)|^2 + c\sum_{\ell=1}^{\infty}\left|S_\ell^{\delta+1}f - S_\ell^{\delta}f\right|^2\sum_{j=1}^{\delta+1}\ell^{j-1}\left|\triangle^j\mu_\ell\right| \\ &\le c\sum_{n=1}^{\infty}\left|S_n^{\delta+1}f - S_n^{\delta}f\right|^2 v_n n^{-1}, \end{aligned}$$

where $v_n = 1 + \sum_{j=1}^{\delta+1}|\triangle^j\mu_n|n^j$. Finally, it follows directly from Eq. (3.3.4) that

$$n^{-1}\sum_{\ell=1}^{n} v_\ell = 1 + \sum_{j=1}^{\delta+1} n^{-1}\sum_{\ell=1}^{n}\ell^j|\triangle^j\mu_\ell| \le c.$$

This proves (3.3.3) and completes the proof of Theorem 3.3.1. □

3.4 The Littlewood–Paley Inequality

As an application of the Marcinkiewicz multiplier theorem, we prove a Littlewood–Paley-type inequality in this section. Let us start with the following definition.

Definition 3.4.1. Given a compactly supported continuous function $\theta : [0,\infty) \to \mathbb{R}$, we define a sequence of operators $\Delta_{\theta,j}$ by $\Delta_{\theta,0}(f) = \operatorname{proj}_0(f)$ and

$$\Delta_{\theta,j}(f) := \sum_{n=0}^{\infty} \theta\left(\frac{n}{2^j}\right) \operatorname{proj}_n(f), \quad j = 1,2,\ldots.$$

Theorem 3.4.2. *Let m be the smallest integer greater than $d/2$. If θ is a compactly supported function in $C^m[0,\infty)$ with $\operatorname{supp}\theta \subset (a,b)$ for some $0 < a < b < \infty$, then for all $f \in L^p(\mathbb{S}^{d-1})$ with $1 < p < \infty$,*

$$\left\| \left(\sum_{j=0}^{\infty} \left|\Delta_{\theta,j} f\right|^2 \right)^{1/2} \right\|_p \le c\|f\|_p, \tag{3.4.1}$$

where c depends only on p, d, a, and b. If, in addition,

$$0 < A_1 \le \sum_{j=0}^{\infty} |\theta(2^{-j}t)|^2 \le A_2 < \infty, \quad \forall t > 0, \tag{3.4.2}$$

and $\int_{\mathbb{S}^{d-1}} f(x)\,\mathrm{d}\sigma(x) = 0$, then

$$\left\| \left(\sum_{j=0}^{\infty} \left|\Delta_{\theta,j} f\right|^2 \right)^{1/2} \right\|_p \sim \|f\|_p, \quad 1 < p < \infty. \tag{3.4.3}$$

Proof. First, we prove the inequality (3.4.1). Let $\{\xi_j\}_{j=0}^{\infty}$ be a sequence of independent random variables that take values ± 1 and have mean zero. Then by the Khinchine inequality, for every real sequence $\{a_j\}$,

$$\left(\mathbb{E} \left| \sum_{j=0}^{\infty} a_j \xi_j \right|^p \right)^{1/p} \sim \left(\sum_{j=0}^{\infty} |a_j|^2 \right)^{\frac{1}{2}}, \quad 0 < p < \infty, \tag{3.4.4}$$

where $\mathbb{E}$ denotes the expectation of random variables. Now consider the (random) linear operator

$$Tf = \sum_{j=0}^{\infty} \xi_j \Delta_{\theta,j} f. \tag{3.4.5}$$

Directly from the definition of $\Delta_{\theta,j} f$, Tf can be rewritten in the form

$$Tf = \sum_{k=1}^{\infty} A(k) \operatorname{proj}_k f, \quad A(u) := \sum_{j=0}^{\infty} \theta\left(\frac{u}{2^j}\right) \xi_j.$$

Since $\theta \in C^m[0,\infty)$ is supported in a finite interval $(a,b) \subset (0,\infty)$, it follows by a straightforward computation that

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r A(u)\right| \le c_r u^{-r}, \ \ u \ge 1, \ \ r = 0,1,\dots,m,$$

which, in particular, implies that

$$|\triangle^r A(k)| \le c_r' k^{-r}, \ \ r = 0,1,\dots,m, \ \ k \ge 1,$$

where the constants c_r and c_r' are independent of the random variables ξ_j. We now apply the Marcinkiewicz multiplier theorem, Theorem 3.3.1, with $\mu_k = A(k)$ to deduce that

$$\|Tf\|_p \le c_p \|f\|_p, \ \ 1 < p < \infty, \tag{3.4.6}$$

where c_p is a constant depending only on p and d. Combining (3.4.4) and (3.4.5) with Eq. (3.4.6), we conclude that

$$\left\|\left(\sum_{j=0}^{\infty} \left|\varDelta_{\theta,j} f\right|^2\right)^{\frac12}\right\|_p \sim \left(\mathbb{E}\|Tf\|_p^p\right)^{1/p} \le c_p \|f\|_p,$$

which proves the desired inequality (3.4.1).

Second, we prove the inverse inequality

$$\left\|\left(\sum_{j=1}^{\infty} \left(\varDelta_{\theta,j} f\right)^2\right)^{1/2}\right\|_p \ge c_p' \|f\|_p \tag{3.4.7}$$

for $f \in L^p$ with $1 < p < \infty$ and $\int_{\mathbb{S}^{d-1}} f(x)\,\mathrm{d}\sigma(x) = 0$ under the additional assumption that

$$\sum_{j=0}^{\infty} \left|\theta(2^{-j}x)\right|^2 = 1, \ \ x > 0. \tag{3.4.8}$$

This assumption implies that for every spherical polynomial g,

$$\sum_{j=0}^{\infty} (\varDelta_{\theta,j} \circ \varDelta_{\theta,j}) g = g - \operatorname{proj}_0 g. \tag{3.4.9}$$

Now for $f \in L^p$ with $\int_{\mathbb{S}^{d-1}} f(x)\,\mathrm{d}\sigma(x) = 0$ and $\varepsilon > 0$, there is a $g \in L^q(\mathbb{S}^{d-1})$ with $\|g\|_q = 1$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that $\|f\|_p - \varepsilon/2 \le \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(x)g(x)\mathrm{d}\sigma$. Let g_n be a spherical polynomial such that $\|g - g_n\|_q < \varepsilon/2$. Then it follows readily that $\|f\|_p - \varepsilon \le \frac{1}{\omega_d} |\int_{\mathbb{S}^{d-1}} f g_n \mathrm{d}\sigma(x)|$. Using Eq. (3.4.9), we have

$$\begin{aligned}\left|\int_{\mathbb{S}^{d-1}} f g_n \mathrm{d}\sigma(x)\right| &= \frac{1}{\omega_d}\left|\int_{\mathbb{S}^{d-1}} \sum_{j=0}^{\infty} \Delta_{\theta,j} f(x) \Delta_{\theta,j} g_n(x)\, \mathrm{d}\sigma(x)\right| \\ &\le c \left\| \left(\sum_{j=0}^{\infty} |\Delta_{\theta,j} f|^2\right)^{\frac12} \right\|_p \left\| \left(\sum_{j=0}^{\infty} |\Delta_{\theta,j} g_n|^2\right)^{\frac12} \right\|_q \\ &\le c\|g_n\|_q \left\| \left(\sum_{j=0}^{\infty} |\Delta_{\theta,j} f|^2\right)^{\frac12} \right\|_p \le c \left\| \left(\sum_{j=0}^{\infty} |\Delta_{\theta,j} f|^2\right)^{\frac12} \right\|_p.\end{aligned}$$

This proves the inverse inequality (3.4.7) under the additional condition (3.4.8).

Finally, we show that for the validity of the inverse inequality (3.4.7), the condition (3.4.8) can be relaxed to Eq. (3.4.2). To this end, we define

$$\tilde{\theta}(x) := \frac{\theta(x)}{\left(\sum_{j=0}^{\infty} |\theta(2^{-j}x)|^2\right)^{\frac12}}.$$

It is evident that $\tilde{\theta} \in C^m[0,\infty)$, $\operatorname{supp}\tilde{\theta} \subset (a,b) \subset (0,\infty)$, and

$$\sum_{j=0}^{\infty} \tilde{\theta}(2^{-j}x) = 1, \quad \forall x > 0.$$

Thus, using the already proven case of inequality (3.4.7), we have

$$\|f\|_p \le c_p \left\| \left(\sum_{j=1}^{\infty} \left(\Delta_{\tilde{\theta},j} f\right)^2\right)^{1/2} \right\|_p. \tag{3.4.10}$$

Next, let $\phi \in C^\infty[0,\infty)$ be such that $\phi(x) = 1$ for $x \in [a,b]$, and $\operatorname{supp}\phi \subset (a_1,b_1)$ for some $0 < a_1 < a < b < b_1 < \infty$. Define

$$\psi(x) = \frac{\phi(x)}{\left(\sum_{j=0}^{\infty} |\theta(2^{-j}x)|^2\right)^{\frac12}}.$$

Then $\tilde{\theta}(x) = \theta(x)\psi(x)$, and hence $\Delta_{\tilde{\theta},j} = \Delta_{\psi,j} \circ \Delta_{\theta,j}$. Thus, we may rewrite (3.4.10) in the form

$$\|f\|_p \le c_p \left\| \left(\sum_{j=0}^{\infty} |\Delta_{\psi,j} g_j|^2\right)^{\frac12} \right\|_p, \tag{3.4.11}$$

with $g_j := \Delta_{\theta,j} f$. On the other hand, since $\psi \in C^m[0,\infty)$ has compact support, and $m-1 > \lambda = \frac{d-2}{2}$, using Theorem 2.4.7 and summation by parts m times, it follows that

$$\sup_{j\in\mathbb{N}} |\Delta_{\psi,j} g(x)| \le C \sup_{k\in\mathbb{Z}_+} |S_k^{m-1}(g)(x)| \le CMg(x) + CMg(-x).$$

Thus, by Eq. (3.4.11) and the Fefferman–Stein inequality in Theorem 3.1.4, we deduce that

$$\|f\|_p \le c_p \left\| \left(\sum_{j=0}^{\infty} |Mg_j|^2 \right)^{\frac{1}{2}} \right\|_p \le c_p \left\| \left(\sum_{j=0}^{\infty} |g_j|^2 \right)^{\frac{1}{2}} \right\|_p = \left\| \left(\sum_{j=0}^{\infty} |\Delta_{\theta,j} f|^2 \right)^{\frac{1}{2}} \right\|_p,$$

which establishes the desired inverse inequality. □

3.5 The Riesz Transform on the Sphere

Our definition of the Riesz transform on the sphere depends on the fractional Laplace–Beltrami operator. The main results on the Riesz transforms are proved in the first subsection, assuming a technical lemma on the square of the Laplace–Beltrami operator, which is proved in the second subsection.

3.5.1 Fractional Laplace–Beltrami Operator and Riesz Transform

Our definition of the fractional Laplace–Beltrami operator is motivated by the fact that $\Delta_0 Y = -k(k+d-2)Y$ for all $Y \in \mathscr{H}_k^d$ and $k = 0, 1, \ldots$.

Definition 3.5.1. Let r be a nonzero real number and let $f \in L^1(\mathbb{S}^{d-1})$. A function $g \in L(\mathbb{S}^{d-1})$ is called the rth-order Laplace–Beltrami derivative of f if

$$\int_{\mathbb{S}^{d-1}} g(x)\, \mathrm{d}\sigma(x) = 0, \quad \text{and} \quad \operatorname{proj}_k g = (k(k+d-2))^r \operatorname{proj}_k f, \quad k = 1, 2, \ldots,$$

and we write $g = (-\Delta_0)^r f$.

Since a function is uniquely determined by its orthogonal projections, by Corollary 2.2.6, $(-\Delta_0)^r f$ is uniquely defined. Making use of the operator $L_n f$ defined via a cutoff function in Sect. 2.6, we work mostly with polynomials and pass to the limit for general functions. Recall that $D_{i,j}$ denotes an angular derivative. We note that $D_{i,j}$ commutes with $(-\Delta_0)^r$ for $r \in \mathbb{R}$, since polynomials can be decomposed into spherical harmonics, while by Lemma 1.8.3, $D_{i,j}$ maps $\mathscr{H}_n^d$ into $\mathscr{H}_n^d$. Our main technical lemma is the following lemma.

Lemma 3.5.2. *If f is a spherical polynomial and $1 < p < \infty$, then*

$$\max_{1\le i<j\le d} \left\| D_{i,j}(-\Delta_0)^{-\frac{1}{2}} f \right\|_p \le c_p \|f\|_p, \tag{3.5.1}$$

and furthermore, for every $\alpha > 0$,

$$\mathrm{meas}\left\{x \in \mathbb{S}^{d-1} : \left| D_{i,j}(-\Delta_0)^{-\frac{1}{2}} f(x) \right| > \alpha \right\} \le c\frac{\|f\|_1}{\alpha}. \tag{3.5.2}$$

Lemma 3.5.2 will be proved in the following subsection. It is the essential ingredient in the proof of the following theorem.

Theorem 3.5.3. *If $1 < p < \infty$ and $f \in C^1(\mathbb{S}^{d-1})$, then*

$$\max_{1\le i<j\le d} \left\| D_{i,j} f \right\|_p \sim \left\| (-\Delta_0)^{\frac{1}{2}} f \right\|_p, \tag{3.5.3}$$

with the constant of equivalence depending only on d and p.

Proof. We first claim that it is sufficient to prove (3.5.3) for spherical polynomials. Indeed, let L_n be the operator defined in Eq. (2.6.2), so that $L_n f \in \Pi_{2n}(\mathbb{S}^{d-1})$ and $L_n f = f$ if $f \in \Pi_n(\mathbb{S}^{d-1})$. If Eq. (3.5.3) has been established for polynomials, then for each $f \in C^1(\mathbb{S}^{d-1})$,

$$\max_{1\le i<j\le d} \left\| D_{i,j} L_n f \right\|_p \sim \left\| (-\Delta_0)^{\frac{1}{2}} L_n f \right\|_p, \quad n = 1, 2, \ldots. \tag{3.5.4}$$

However, since $D_{i,j} L_n = L_n D_{i,j}$ and $L_n D_{i,j} f$ converges uniformly to $D_{i,j} f$ as $n \to \infty$ for each $f \in C^1(\mathbb{S}^{d-1})$, we conclude, by Eq. (3.5.4), that $(-\Delta_0)^{\frac{1}{2}} L_n f$ converges uniformly on $\mathbb{S}^{d-1}$. Thus, for some $g \in C(\mathbb{S}^{d-1})$,

$$\mathrm{proj}_k\, g = \lim_{n\to\infty} \mathrm{proj}_k (-\Delta_0)^{\frac{1}{2}} L_n f = (k(k+d-2))^{\frac{1}{2}} \mathrm{proj}_k\, f, \quad k = 0, 1, \ldots.$$

By Definition 3.5.1, this shows that $g = (-\Delta_0)^{\frac{1}{2}} f \in C(\mathbb{S}^{d-1})$. Consequently, taking the limit of Eq. (3.5.4), we conclude that Eq. (3.5.3) holds for all $f \in C^1(\mathbb{S}^{d-1})$.

We now establish (3.5.3) for a polynomial, $f \in \Pi_n(\mathbb{S}^{d-1})$. Since $(-\Delta_0)^{\frac{1}{2}} f$ is also a polynomial by definition, applying (3.5.1) on $(-\Delta_0)^{\frac{1}{2}} f$ gives one direction of the equivalence. The other direction can be deduced via a duality argument. Let $\langle f, g \rangle := \int_{\mathbb{S}^{d-1}} f(x) g(x) \mathrm{d}\sigma$ and let L_n be defined as above. Then for $p' = p/(p-1)$,

$$\begin{aligned}\left\| (-\Delta_0)^{\frac{1}{2}} f \right\|_p &= \sup_{\|g\|_{p'}\le 1} \left| \left\langle (-\Delta_0)^{\frac{1}{2}} f, g \right\rangle \right| = \sup_{\|g\|_{p'}\le 1} \left| \left\langle (-\Delta_0)^{\frac{1}{2}} f, L_n g \right\rangle \right| \\ &= \sup_{\|g\|_{p'}\le 1} \left| \left\langle \Delta_0 f, (-\Delta_0)^{-\frac{1}{2}} L_n g \right\rangle \right|,\end{aligned}$$

where we have used the fact that for every $h \in \Pi_n(\mathbb{S}^{d-1})$, $\langle h, g\rangle = \langle h, L_n g\rangle$ in the second equality and that both f and $L_n g$ are polynomials in the last equality. Using the decomposition (1.8.3) of Δ_0 and Eq. (1.8.7), we further deduce that

$$\begin{aligned}\left\|(-\Delta_0)^{\frac{1}{2}} f\right\|_p &\le \sum_{1\le i<j\le d} \sup_{\|g\|_{p'}\le 1} \left|\left\langle D_{i,j} f, D_{i,j}(-\Delta_0)^{-\frac{1}{2}} L_n g\right\rangle\right| \\ &\le \sum_{1\le i<j\le d} \sup_{\|g\|_{p'}\le 1} \left\|D_{i,j} f\right\|_p \left\|D_{i,j}(-\Delta_0)^{-\frac{1}{2}} L_n g\right\|_{p'} \\ &\le c \max_{1\le i<j\le d} \left\|D_{i,j} f\right\|_p,\end{aligned}$$

where the last step follows from Eq. (3.5.1) and the boundedness of $\|L_n\|_p$. This establishes the other direction of the equivalence and completes the proof. □

If f is a polynomial, then so is $(-\Delta_0)^{-\frac{1}{2}} f$ by definition. Applying Lemma 3.5.2 to the polynomial $(-\Delta_0)^{-\frac{1}{2}} f$ shows that

$$\|D_{i,j}(-\Delta_0)^{-\frac{1}{2}} f\|_p \le c_p \|f\|_p, \quad 1 < p < \infty,$$

for every spherical polynomial f. Since the space of spherical polynomials is dense in L^p, it follows that $D_{i,j}(-\Delta_0)^{-\frac{1}{2}}$ extends uniquely to a bounded operator on L^p for all $1 < p < \infty$. Furthermore, using Eq. (3.5.2), we can extend the definition of $D_{i,j}(-\Delta_0)^{-\frac{1}{2}}$ to the whole L^1 space. This justifies the following definition.

Definition 3.5.4. For $1 \le i < j \le d$, we define the Riesz transform $R_{i,j} f$ by

$$R_{i,j} f := D_{i,j}(-\Delta_0)^{-\frac{1}{2}} f.$$

The main properties of the Riesz transforms are summarized below.

Theorem 3.5.5. *The Riesz transforms satisfy the following properties:*

(*i*) *If* $1 \le i < j \le d$ *and* $f \in L^1(\mathbb{S}^{d-1})$, *then for every* $\alpha > 0$,

$$\operatorname{meas}\left\{x \in \mathbb{S}^{d-1} : |(R_{i,j} f)(x)| > \alpha\right\} \le c \frac{\|f\|_1}{\alpha}.$$

(*ii*) *If* $f \in L^p(\mathbb{S}^{d-1})$, $1 < p < \infty$, *and* $\int_{\mathbb{S}^{d-1}} f(x)\, d\sigma(x) = 0$, *then*

$$\|f\|_p \sim \sum_{1\le i<j\le d} \|R_{i,j} f\|_p.$$

(*iii*) *Let* I *denote the identity operator on* $L^1(\mathbb{S}^{d-1})$. *Then*

$$\sum_{1\le i<j\le d} R_{i,j}^2 = I.$$

Proof. Using $L_n f$ and passing to the limit, we need to establish the result only for polynomials. Thus, (i) follows from Eq. (3.5.2), and (ii) is a consequence of Theorem 3.5.3, since evidently $\sum_{1\le i<j\le d} \|D_{i,j}f\|_p \sim \max_{1\le i<j\le d} \|D_{i,j}f\|_p$. Statement (iii) follows directly from the decomposition (1.8.3) of Δ_0 and the commutativity of $D_{i,j}$ and $(-\Delta_0)^{-\frac{1}{2}}$. □

Lemma 3.5.6. *The operator $R_{i,j}$ satisfies*

$$\int_{\mathbb{S}^{d-1}} R_{i,j}f(x)g(x)\mathrm{d}\sigma(x) = -\int_{\mathbb{S}^{d-1}} f(x)R_{i,j}g(x)\mathrm{d}\sigma(x), \tag{3.5.5}$$

and $R_{i,j}$ commutes with the convolution operator, $(R_{i,j}f)*g = R_{i,j}(f*g)$.

Proof. Directly from the definition, $(-\Delta_0)^r$ is self-adjoint, so that Eq. (3.5.5) follows from Eq. (1.8.7). Furthermore, by Eq. (2.1.5), $((-\Delta_0)^r f)*g = (-\Delta_0)^r(f*g)$ also follows directly from the definition of $(-\Delta_0)^r$. Hence, $(R_{i,j}f)*g = R_{i,j}(f*g)$ follows from Proposition 2.6.4. □

3.5.2 Proof of Lemma 3.5.2

We first define an analogue of $L_n f$ for $(-\Delta_0)^{\frac{1}{2}} f$. Let $\eta \in C^\infty[0,\infty)$ be such that $\eta(x) = 1$ for $x \in [0,1]$, and $\eta(x) = 0$ for $x \ge 2$. For $m = 1,2,\ldots$ and $\lambda = \frac{d-2}{2}$, define

$$H_m(t) := \sum_{k=1}^{2^{m+1}} \eta\left(\frac{k}{2^m}\right)(k(k+d-2))^{-\frac{1}{2}}\frac{k+\lambda}{\lambda}C_k^\lambda(t), \quad t\in[-1,1].$$

Then as can be easily seen,

$$(-\Delta_0)^{-\frac{1}{2}} f = H_m * f, \quad \forall f \in \Pi_{2^m}(\mathbb{S}^{d-1}).$$

We need sharp estimates on H_m and its derivatives, which, however, do not follow directly from the estimate of $L_n(t)$ in Sect. 2.6, owing to $(k(k+d-2))^{-\frac{1}{2}}$.

Lemma 3.5.7. *For $r = 0,1,2,\ldots$, $\theta \in [0,\pi]$, and $m \in \mathbb{N}$, we have*

$$\left|H_m^{(r)}(\cos\theta)\right| \le c_{r,d}(2^{-m}+\theta)^{-(d-2+2r)}.$$

Proof. The support of η implies $H_{-1}(\cos\theta) = 0$; hence we can write

$$H_m(\cos\theta) = \sum_{j=0}^{m}(H_j(\cos\theta) - H_{j-1}(\cos\theta)) =: \sum_{j=0}^{m} A_j(\cos\theta),$$

where by the properties of η,

$$A_j(\cos\theta) = 2^{-j} \sum_{k=2^{j-1}}^{2^{j+1}} \psi_j(2^{-j}k) \frac{k+\lambda}{\lambda} C_k^\lambda(\cos\theta)$$

with $\psi_j(x) := (\eta(x) - \eta(2x))(x(x + 2^{-j+1}\lambda))^{-\frac{1}{2}}$. Since ψ_j is a C^∞ function supported in $[\frac{1}{2}, 2]$, which implies that $\left\| \psi_j^{(3\ell-1)} \right\|_\infty \le c_{\ell,\eta}$ for every $\ell \in \mathbb{N}$, it follows by Proposition 2.6.7 that

$$\left| A_j^{(r)}(\cos\theta) \right| \le c_{\ell,r} 2^{j(d-2+2r)} (1 + 2^j\theta)^{-\ell}.$$

Letting $\tau = d - 2 + 2r$ and $\ell = \tau + 1$, we deduce that

$$\left| H_m^{(r)}(\cos\theta) \right| \le \sum_{j=0}^{m} \left| A_j^{(r)}(\cos\theta) \right| \le c_{\ell,r} \sum_{j=0}^{m} 2^{j\tau}(1 + 2^j\theta)^{-\tau-1}.$$

If $0 \le \theta \le 2^{-m}$, then the last sum is bounded by $2^{m\tau} \le c(2^{-m} + \theta)^{-\tau}$, which is the desired upper bound. Assume now that $2^{-m} \le 2^{-k} \le \theta < 2^{-k+1} \le 2\pi$. Then the last sum is bounded by a constant multiple of

$$\sum_{j=0}^{k} 2^{j\tau} + \sum_{j=k}^{m} 2^{j\tau}(2^j\theta)^{-\tau-1} \le c\left(2^{m\tau} + 2^{-k}\theta^{-\theta-1}\right) \le c(2^{-m} + \theta)^{-\tau},$$

which completes the proof. □

Next, we define, for $1 \le i < j \le d$ and $m = 1, 2, \ldots$, the kernels

$$H_{i,j,m}(x,y) := D_{i,j}\left[H_m(\langle \cdot, y \rangle)\right](x) = H_m'(\langle x, y \rangle)(x_i y_j - x_j y_i),$$

where the second equality follows from the definition of $D_{i,j}$. The estimates of these kernels are given in terms of the geodesic distance $\mathtt{d}(x,y) = \arccos\langle x, y \rangle$, which is comparable to $\|x - y\|$ by Eq. (A.1.1).

Lemma 3.5.8. *Let* $1 \le i < j \le d$ *and* $m = 1, 2, \ldots$.

(a) *If* $x, y \in \mathbb{S}^{d-1}$, *then*

$$\left| H_{i,j,m}(x,y) \right| \le \frac{c}{(2^{-m} + \mathtt{d}(x,y))^{d-1}}.$$

(b) *If in addition,* $z \in \mathbb{S}^{d-1}$ *and* $\mathtt{d}(y,z) \le \frac{1}{2}(2^{-m} + \mathtt{d}(x,y))$, *then*

$$\left| H_{i,j,m}(x,y) - H_{i,j,m}(x,z) \right| \le c \frac{\mathtt{d}(y,z)}{(2^{-m} + \mathtt{d}(x,y))^{d}}.$$

Proof. By Lemma 3.5.7 with $r=1$, we obtain

$$\begin{aligned}\left|H_{i,j,m}(x,y)\right| &= \left|H_m'(\langle x,y\rangle)\right| |x_iy_j - x_jy_i| \le \left|H_m'(\langle x,y\rangle)\right| (|x_i-y_i|+|x_j-y_j|) \\ &\le c\left(2^{-m}+\mathtt{d}(x,y)\right)^{-(d-2+2)} \mathtt{d}(x,y) \le c\left(2^{-m}+\mathtt{d}(x,y)\right)^{-(d-1)},\end{aligned}$$

which proves (a). To prove (b), we first note that if $\mathtt{d}(y,z) \le \frac{1}{2}(2^{-m}+\mathtt{d}(x,y))$, then by the triangle inequality, $\mathtt{d}(x,z)+2^{-m} \sim \mathtt{d}(x,y)+2^{-m}$. Thus, using the intermediate value theorem and Lemma 3.5.7 with $r=2$, we deduce that for some θ between $\mathtt{d}(x,y)$ and $\mathtt{d}(x,z)$,

$$\begin{aligned}|H_m'(\langle x,y\rangle) - H_m'(\langle x,z\rangle)| &= \left|H_m''(\cos\theta)\right| |\langle x,y\rangle - \langle x,z\rangle| \\ &\le c\left(2^{-m}+\theta\right)^{-d-2} \left|\sin\left(\tfrac{\mathtt{d}(x,y)-\mathtt{d}(x,z)}{2}\right)\sin\left(\tfrac{\mathtt{d}(x,y)+\mathtt{d}(x,z)}{2}\right)\right| \\ &\le c\left(2^{-m}+\mathtt{d}(x,y)\right)^{-d-1} \mathtt{d}(y,z).\end{aligned}$$

Furthermore, by the triangle inequality and Eq. (A.1.1),

$$|x_iy_j - x_jy_i| \le |x_i-y_i||y_j| + |y_j-x_j||y_i| \le 2\mathtt{d}(x,y)$$

and

$$|x_iy_j - x_jy_i - x_iz_j + x_jz_i| \le |y_j-z_j| + |y_i-z_i| \le 2\mathtt{d}(y,z).$$

It follows that

$$\begin{aligned}|H_{i,j,m}(x,y) - H_{i,j,m}(x,z)| \le\ & \left|H_m'(\langle x,y\rangle) - H_m'(\langle x,z\rangle)\right| |x_iy_j - x_jy_i| \\ &+ \left|H_m'(\langle x,z\rangle)\right| |x_iy_j - x_jy_i - x_iz_j + x_jz_i| \\ \le\ & c\left(2^{-m}+\mathtt{d}(x,y)\right)^{-d} \mathtt{d}(y,z),\end{aligned}$$

which proves (b) and completes the proof. □

We note that $H_{i,j,m}(x,y) = -H_{i,j,m}(y,x)$, so that (b) holds if x and y are exchanged. With these preparations, we are in a position to prove Lemma 3.5.2.

Proof of Lemma 3.5.2. Assume that $f \in \Pi_{2^m}(\mathbb{S}^{d-1})$ for some $m \in \mathbb{N}$. For $1 \le i < j \le d$, retain the notation $R_{i,j}f = D_{i,j}(-\Delta_0)^{-\frac{1}{2}}f$. Then $R_{i,j} : \Pi_{2^m}(\mathbb{S}^{d-1}) \to \Pi_{2^m}(\mathbb{S}^{d-1})$. For $p=2$, using the fact that $D_{i,j}$ commutes with $\Delta_0^{-\frac{1}{2}}$, we obtain immediately from Eqs. (1.8.7) and (3.5.5) that

$$\|R_{i,j}f\|_2^2 = \langle R_{i,j}f, R_{i,j}f\rangle = \langle D_{i,j}^2 f, \Delta_0^{-1} f\rangle.$$

In particular, the right-hand side is positive, so that by Eq. (1.8.3),

$$\|R_{i,j}f\|_2^2 \le \sum_{1\le i'<j'\le d} \langle D_{i',j'}^2 f, \Delta_0^{-1} f\rangle = \langle \Delta_0 f, \Delta_0^{-1} f\rangle \le \|f\|_2^2.$$

Thus, we have proved (3.5.1) for the case $p=2$. □

By the Riesz–Thorin theorem, the proof of Eq. (3.5.1) follows from the weak $(1,1)$ estimate (3.5.2), which we prove next using the Calderón–Zygmund decomposition in Theorem 3.1.6. Since for $f \in \Pi_{2^m}(\mathbb{S}^{d-1})$,

$$(-\Delta_0)^{-\frac{1}{2}} f(x) = (f * H_m)(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) H_m(\langle x,y\rangle)\, \mathrm{d}\sigma(y),$$

it follows that

$$R_{i,j} f = D_{i,j}(f * H_m)(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) H_{i,j,m}(x,y)\, \mathrm{d}\sigma(y).$$

Since $\mathbb{S}^{d-1}$ has finite measure, Eq. (3.5.2) is obvious if $0 < \alpha < \|f\|_1$. Assume now $\alpha > \|f\|_1$, so that Theorem 3.1.6 is applicable. Then there exists a Calderón–Zygmund decomposition $f = g + b$ at the level α such that $\|g\|_1 \le c\|f\|_1$ and $\|b\|_1 \le c\|f\|_1$, and the functions g and b satisfy the properties specified in Theorem 3.1.6, where the balls are spherical caps in this particular case. For the function g, using the already proven case of Eq. (3.5.1) for $p = 2$ and the fact that $|g(x)| \le c\alpha$ for almost every $x \in \mathbb{S}^{d-1}$, we obtain

$$\operatorname{meas}\left\{x \in \mathbb{S}^{d-1} : |(R_{i,j} g)(x)| > c\alpha\right\} \le \frac{\|R_{i,j} g\|_2^2}{c^2\alpha^2} \le c' \frac{\|g\|_2^2}{\alpha^2} \le cc' \frac{\|g\|_1}{\alpha} \le c'' \frac{\|f\|_1}{\alpha}.$$

Thus, the proof of Eq. (3.5.2) is reduced to proving the inequality

$$\operatorname{meas}\left\{x \in \mathbb{S}^{d-1} : \left|(R_{i,j} b)(x)\right| > c\alpha\right\} \le c' \frac{\|f\|_1}{\alpha}. \tag{3.5.6}$$

For each k, we can write $B_k = \mathtt{c}(x_k, r_k)$ with $x_k \in \mathbb{S}^{d-1}$ and $r_k > 0$. Let $B_k^* = \mathtt{c}(x_k, 4r_k)$. Then for $x \notin B_k^*$, using Lemma 3.5.8, we obtain

$$\begin{aligned} |(R_{i,j} b_k)(x)| &= \left| \int_{B_k} (H_{i,j,m}(x,y) - H_{i,j,m}(x,x_k))\, b_k(y)\, \mathrm{d}\sigma(y) \right| \\ &\le c' \int_{B_k} \frac{\mathtt{d}(y,x_k)}{(\mathtt{d}(x,x_k))^d} |b_k(y)|\, \mathrm{d}\sigma(y) \le c' \frac{r_k}{(\mathtt{d}(x,x_k))^d} \int_{B_k} |b_k(y)|\, \mathrm{d}\sigma(y) \\ &\le c' \operatorname{meas}(B_k) \frac{r_k \alpha}{(\mathtt{d}(x,x_k))^d}, \end{aligned}$$

which implies, with an appropriate parameterization of the integral over $(B_k^*)^c$,

$$\begin{aligned} \int_{(B_k^*)^c} |(R_{i,j} b_k)(x)|\, \mathrm{d}\sigma(x) &\le c\alpha \operatorname{meas}(B_k) \int_{(B_k^*)^c} \frac{r_k}{(\mathtt{d}(x,x_k))^d} \mathrm{d}\sigma(x) \\ &\le c\alpha r_k \operatorname{meas}(B_k) \int_{4r_k}^{\frac{\pi}{2}} \frac{(\sin\theta)^{d-2}}{\theta^d} \mathrm{d}\theta \le c' \alpha \operatorname{meas}(B_k). \end{aligned}$$

Consequently, setting $B^* := \bigcup_k B_k^*$, we deduce

$$\begin{aligned}\operatorname{meas}\left\{x : |(R_{i,j}b)(x)| > c\alpha\right\} &\le \operatorname{meas} B^* + \operatorname{meas}\{x \in (B^*)^c : |(R_{i,j}b)(x)| > c\alpha\} \\ &\le \sum_k \operatorname{meas}(B_k^*) + \frac{1}{c\alpha}\sum_k \int_{(B_k^*)^c} |(R_{i,j}b_k)(x)|\mathrm{d}\sigma \\ &\le c' \sum_k \operatorname{meas}(B_k) \le c' \frac{\|f\|_1}{\alpha},\end{aligned}$$

where the last step follows from (iii) of Theorem 3.1.6. This proves the inequality (3.5.6) and hence, Eq. (3.5.2), which further implies, by the Riesz–Thorin theorem, the inequality (3.5.1) for $1 < p \le 2$.

The remaining case of Eq. (3.5.1) for $2 < p < \infty$ follows by duality. Indeed, let again $L_n f$ be the operator defined via a cutoff function as in Sect. 2.6, which, by Lemma 3.5.6, commutes with $R_{i,j}$. Then for $f \in \Pi_n(\mathbb{S}^{d-1})$ and $p' = \frac{p}{p-1} \in (1,2)$, we deduce, using Eq. (3.5.5) and the boundedness of $\|L_n\|_p$,

$$\begin{aligned}\|R_{i,j}f\|_p &= \sup_{\|g\|_{p'}\le 1} \langle R_{i,j}f, g\rangle = \sup_{\|g\|_{p'}\le 1} \langle R_{i,j}f, L_n g\rangle = \sup_{\|g\|_{p'}\le 1} -\langle f, L_n R_{i,j} g\rangle \\ &\le \sup_{\|g\|_{p'}\le 1} \|f\|_p \|L_n R_{i,j} g\|_{p'} \le c \sup_{\|g\|_{p'}\le 1} \|f\|_p \|R_{i,j} g\|_{p'} \le c_p \|f\|_p,\end{aligned}$$

where the last step follows from the already proven case of Eq. (3.5.1). □

3.6 Notes and Further Results

The main reference for our Sect. 3.1 is [158]. Another good reference is [33]. The proof of the Fefferman–Stein inequality can be found in [74] and [158, pp. 51–55].

The Littlewood–Paley theory in Sect. 3.2 and the proof of Theorem 3.3.1 follow essentially the argument in Bonami and Clerc [18]. The proof has been substantially simplified by making use of the operators defined via a cutoff function. A classical reference on general Littlewood–Paley theory is [157].

Several Riesz transforms on the sphere are defined in [3], including $R_{i,j}$. The proof in Sect. 3.5 is new and has never before been published.

Chapter 4
Approximation on the Sphere

A central problem in approximation theory is to characterize the best approximation of a function by polynomials, or other classes of simple functions, in terms of the smoothness of the function. In this chapter, we study the characterization of the best approximation by polynomials on the sphere. In the classical setting of one variable, the smoothness of a function on $\mathbb{S}^1$ is described by the modulus of smoothness, defined via the forward difference of the function. A main challenge for the sphere $\mathbb{S}^{d-1}$ with $d \geq 3$ is to define a modulus of smoothness that will characterize the smoothness, since multiplication on the higher-dimensional sphere is not commutative. It became clear only recently that a satisfactory modulus of smoothness can be defined as the maximum of the moduli of smoothness of one variable in $\theta_{i,j}$, the angle of the polar coordinate on (x_i, x_j), over all possible choices of (i, j). Conspicuously, the number $d(d-1)/2$ of such angles is the dimension of $\mathrm{SO}(d)$, while $\mathbb{S}^{d-1} = \mathrm{SO}(d)/\mathrm{SO}(d-1)$. This modulus of smoothness allows us to tap into the rich resources of trigonometric approximation theory for ideas and tools, and effectively reduces many problems in approximation theory on $\mathbb{S}^{d-1}$ to problems of trigonometric approximation.

A succinct review of trigonometric approximation theory on $\mathbb{S}^1$ is given in the first section, which serves as a model and provides building blocks for our results on the sphere. The modulus of smoothness on the sphere is discussed in the second section, and a main lemma for its application in approximation theory is given in the third section. An operator that provides an explicit construction, up to a constant, to the best-approximation polynomial of a fixed degree is presented in the fourth section. The characterization of the best approximation by polynomials on the sphere in terms of the modulus of smoothness is given in the fifth section. A K-functional is defined in the sixth section and shown to be equivalent to the modulus of smoothness, so that it provides another characterization. Several examples of functions for which the order of the modulus of smoothness, hence the order of best approximation, can be explicitly determined are given in the seventh section. Finally, other moduli of smoothness are discussed in the eighth section.

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4.1 Approximation by Trigonometric Polynomials

Under polar coordinates $(x_1, x_2) = r(\cos\theta, \sin\theta)$, the functions on the circle $\mathbb{S}^1$ can be identified with 2π-periodic functions in θ. The spherical harmonics of degree n on $\mathbb{S}^1$ are $\cos n\theta$ and $\sin n\theta$, so that the space $\Pi_n(\mathbb{S}^1)$ becomes the space of trigonometric polynomials of degree at most n,

$$\Pi_n(\mathbb{S}^1) = \operatorname{span}\{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta\}.$$

Hence, the polynomial approximation on $\mathbb{S}^1$ is the same as the trigonometric polynomial approximation for 2π-periodic functions.

For $f \in L^p(\mathbb{S}^1)$, if $1 \le p < \infty$ and $f \in C(\mathbb{S}^1)$ if $p = \infty$, the error of best approximation by trigonometric polynomials of degree at most n is defined by

$$E_n(f)_p := \inf_{g \in \Pi_n(\mathbb{S}^1)} \|f - g\|_p, \qquad 1 \le p \le \infty.$$

The central problem in trigonometric approximation theory is to characterize $E_n(f)_p$ in terms of the smoothness of the function f. For this purpose, we need the notion of modulus of smoothness, usually defined through the forward difference of f.

Let I denote the identity operator and S_h the translation operator defined by $S_h f(x) := f(x+h)$. For $r = 1, 2, \ldots$, the forward difference operator $\overrightarrow{\triangle}_h^r$ is defined by

$$\overrightarrow{\triangle}_h := S_h - I \quad \text{and} \quad \overrightarrow{\triangle}_h^r := \overrightarrow{\triangle}_h^{r-1} \overrightarrow{\triangle}_h,$$

or $\overrightarrow{\triangle}_h = (S_h - I)^r$. The binomial theorem implies that

$$\overrightarrow{\triangle}_h^r f(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x + hj). \tag{4.1.1}$$

The usual difference operator $\triangle$ is related to $\overrightarrow{\triangle}_h$ by $\triangle = -\overrightarrow{\triangle}_1$.

Definition 4.1.1. For $f \in L^p(\mathbb{S}^1)$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^1)$ if $p = \infty$, $r = 1, 2, \ldots$ and $t > 0$,

$$\omega_r(f;t)_p := \sup_{|\theta| \le t} \left\| \overrightarrow{\triangle}_\theta^r f \right\|_p, \qquad 1 \le p \le \infty.$$

The modulus of smoothness $\omega_r(f,t)_p$ is a continuous and increasing function of t with $\omega_r(f;t)_p \to 0$, $t \to 0^+$. Furthermore, it satisfies the following properties:

(1) $\omega_r(f+g;t)_p \le \omega_r(f;t)_p + \omega_r(g;t)_p$.
(2) $\omega_r(f;\lambda t)_p \le (\lambda+1)^r \omega_r(f;t)_p$, $\lambda \ge 0$.
(3) $\omega_r(f;t)_p \le c t^r \omega_r(f^{(r)};t)_p$ if $f^{(r)} \in L^p(\mathbb{S}^1)$.

The first of these properties follows from the triangle inequality in $L^p(\mathbb{S}^1)$. For the second one, we use the relation

$$\overrightarrow{\triangle}_{nh}^r f(x) = \sum_{k_1=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} \overrightarrow{\triangle}_h^r f(x+k_1h+\cdots+k_rh), \tag{4.1.2}$$

easily established via induction, to show that $\omega_r(f;nt)_p \le n^r \omega_r(f;t)_p$ for $n \in \mathbb{N}$, and then apply the monotonicity of $\omega_r(f;t)_p$ in t. The third one is a consequence of (i) in Lemma 4.1.4 below. Before stating the lemma, which contains further properties of the forward difference, we state the main theorem of trigonometric best approximation.

Theorem 4.1.2. *For $f \in L^p(\mathbb{S}^1)$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^1)$ if $p = \infty$,*

$$E_n(f)_p \le c\, \omega_r\left(f;n^{-1}\right)_p, \quad 1 \le p \le \infty, \quad n = 1,2,\ldots. \tag{4.1.3}$$

On the other hand,

$$\omega_r(f;n^{-1})_p \le c\, n^{-r} \sum_{k=1}^{n} k^{r-1} E_{k-1}(f)_p, \qquad 1 \le p \le \infty. \tag{4.1.4}$$

The inequality (4.1.3) is usually called the Jackson estimate. Its proof requires constructing a trigonometric polynomial whose error of approximation is bounded by the modulus of smoothness. The inequality (4.1.4) is often called the inverse estimate, its proof relies on the Bernstein inequality in the theorem below. In both cases, a proof will come out as a special case of our approximation on $\mathbb{S}^{d-1}$.

Theorem 4.1.3 (Bernstein inequality). *For $1 \le p \le \infty$ and $T_n \in \Pi_n(\mathbb{S}^1)$,*

$$\|T_n^{(k)}\|_p \le n^{k-r} \|T_n^{(r)}\|_p, \qquad k > r. \tag{4.1.5}$$

Proof. It is sufficient to prove that $\|T_n'\|_p \le n\|T_n\|_p$, from which the general case follows by iteration. We consider the case $p = \infty$ first, which will follow from the stronger Szegő's inequality

$$T_n'(\theta)^2 + n^2 T_n(\theta)^2 \le n^2 \|T_n\|_\infty^2, \quad 0 \le \theta < 2\pi. \tag{4.1.6}$$

To prove this inequality, we can assume, considering $T_n(\cdot + \theta)$ if necessary, that $\theta = 0$ and $T'(0) \ge 0$. Let $\gamma > \|T_n\|_\infty$. We define a real number α, $|\alpha| < \pi/(2n)$, by $\gamma \sin n\alpha = T_n(0)$, and define

$$S_n(x) = \gamma \sin n(x+\alpha) - T_n(x).$$

Then at the points $\theta_k := -\alpha + (k-1/2)\pi/n$, $k = 0, \pm 1, \ldots, \pm n$, we have $\operatorname{sign} S_n(\theta_k) = (-1)^{k+1}$, so that S_n has a unique zero ϕ_k in each of the $2n$ intervals

(θ_k, θ_{k+1}). On the interval (θ_0, θ_1), we have $S_n(0) = 0$ and $S_n(\theta_1) > 0$, so that the inequality $S_n'(0) \le 0$ would give us one more zero in (θ_0, θ_1). Hence $S_n'(0) > 0$ and

$$0 \le T_n'(0) = \gamma n \cos n\alpha - S_n'(0) < \gamma n \cos n\alpha = \gamma n \sqrt{1 - T_n(0)^2},$$

which is $T_n'(0)^2 + n^2 T_n(0)^2 \le n^2\gamma^2$, and setting $\gamma \to \|T_n\|_\infty$, we obtain Eq. (4.1.6).

Next, we consider $p = 1$. For a given $T_n \in \Pi_n(\mathbb{S}^1)$, define

$$Q(\theta) = \frac{1}{2\pi} \int_{\mathbb{S}^1} T_n(\theta + \phi)\, \mathrm{sign}(T_n'(\phi)) \mathrm{d}\phi.$$

It is evident that Q is an element of $\Pi_n(\mathbb{S}^1)$ and $Q_n'(0) = \|T_n'\|_1$. Choose $\theta_0 \in [0, 2\pi)$ such that $|Q(\theta_0)| = \|Q\|_\infty$. Then

$$\|T_n'\|_1 = Q'(0) \le n|Q(\theta_0)| \le \frac{n}{2\pi} \int_{\mathbb{S}^1} |T_n(\theta_0 + \phi)| \mathrm{d}\phi = n\|T_n\|_1.$$

Finally, for the proof of the case $1 < p < \infty$, we refer to [54, p. 102] □

Together, the direct and inverse estimates show that $E_n(f)_p \sim n^{-\alpha}$ if and only if $\omega_r(f;t)_p \sim t^\alpha$ whenever $\alpha < r$. The inverse estimate (4.1.4) is said to be of weak type, since it gives an estimate that contains an additional $\log n$ factor when $E_n(f)_p \sim n^{-r}$. For further results and references on trigonometric approximation, see Sect. 4.8.

In the rest of this section, we state several further properties of the forward difference and the modulus of smoothness that will be needed in the next section.

Lemma 4.1.4. *Let $r \in \mathbb{N}$ and $f \in L^p(\mathbb{S}^1)$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^1)$ if $p = \infty$.*

(i) For $0 < h < 2\pi$ and $f^{(r)} \in L^p(\mathbb{S}^1)$ when needed,

$$\left\| \overrightarrow{\triangle}_h^r f \right\|_p \le c 2^r \|f\|_p \quad \textit{and} \quad \left\| \overrightarrow{\triangle}_h^r f \right\|_p \le c h^r \|f^{(r)}\|_p.$$

(ii) If $T_n \in \Pi_n(\mathbb{S}^1)$ is a trigonometric polynomial of degree at most n, then

$$\left\| T_n^{(r)} \right\|_p \sim h^{-r} \left\| \overrightarrow{\triangle}_h^r T_n \right\|_p, \quad 0 < h \le \pi n^{-1},$$

with the constant of equivalence depending only on r.

(iii) For $0 < t < 2\pi$,

$$\sup_{|\theta| \le t} \left\| \overrightarrow{\triangle}_\theta^r f \right\|_p \sim \left(\frac{1}{t} \int_0^t \left\| \overrightarrow{\triangle}_\theta^r f \right\|_p^p \mathrm{d}\theta \right)^{\frac{1}{p}}.$$

Proof. (i) The forward difference satisfies the integral representation

$$\overrightarrow{\triangle}_h^r f(x) = h^r \int_{\mathbb{R}} f^{(r)}(x+y) M_{r,h}(y) \mathrm{d}y,$$

where $M_{r,h}$ satisfies $0 \le M_{r,h} \le 1$ and $\int_{\mathbb{R}} M_{r,h}(y)\mathrm{d}y = 1$, as can be easily shown by induction, from which the Minkowski inequality completes the proof.

(ii) By (i), we need to prove only that $\|\overrightarrow{\triangle}_h^r T_n\|_p$ is bounded by $\|f^{(r)}\|_p$. By Eq. (4.1.1), expanding T_n in a Taylor expansion shows that

$$\overrightarrow{\triangle}_h^r T_n(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \sum_{k=0}^{n} \frac{(jh)^k}{k!} T^{(k)}(x) = \sum_{k=r}^{n} \frac{h^k}{k!} T_n^{(k)}(x) \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} (jh)^k$$

since the last sum over j is zero if $k \le r-1$ by Eqs. (4.1.1) and (A.3.4). Applying the classical Bernstein inequality (4.1.5), the stated result follows from a simple estimate and the boundedness of nh.

(iii) Directly from its definition, the right-hand side is bounded by the left-hand side. To prove the converse, we use the formula

$$\overrightarrow{\triangle}_h^r f(x) = \sum_{j=1}^{r} (-1)^j \binom{r}{j} \left[\overrightarrow{\triangle}^r_{\frac{j(\theta-h)}{r}} f(x+jh) - \overrightarrow{\triangle}^r_{h+\frac{j(\theta-h)}{r}} f(x) \right]$$

for $\theta > 0$, which can be verified using Eq. (4.1.1). Since $\|g(\cdot + s)\|_p = \|g\|_p$ for every 2π-periodic function g, if we apply the triangle inequality in $\|\cdot\|_p$ and then integrate against θ over $[0,t]$, the above identity implies that

$$t \left\| \overrightarrow{\triangle}_h^r f \right\|_p^p \le \sum_{j=1}^{r} \binom{r}{j} \left[\int_0^t \left\| \overrightarrow{\triangle}^r_{\frac{j(\theta-h)}{r}} f \right\|_p^p \mathrm{d}\theta + \int_0^t \left\| \overrightarrow{\triangle}^r_{h+\frac{j(\theta-h)}{r}} f \right\|_p^p \mathrm{d}\theta \right].$$

Changing variables $\theta \mapsto \phi = \frac{j(\theta-h)}{r}$ and using the identity

$$\overrightarrow{\triangle}_s^r f(x) = (-1)^r \overrightarrow{\triangle}_{-s}^r f(x+rs),$$

we see that the first integral on the right-hand side is equal to

$$\frac{r}{j} \left[\int_0^{\frac{jh}{r}} \left\| \overrightarrow{\triangle}_\phi^r f \right\|_p^p \mathrm{d}\phi + \int_0^{\frac{j(t-h)}{r}} \left\| \overrightarrow{\triangle}_\phi^r f \right\|_p^p \mathrm{d}\phi \right] \le 2r \int_0^t \left\| \overrightarrow{\triangle}_\phi^r f \right\|_p^p \mathrm{d}\phi,$$

and the second integral, treated similarly, satisfies the same upper bound. Putting these together completes the proof. □

Inequality (4.1.7) below is called the Marchaud inequality.

Lemma 4.1.5. *Let $r, s \in \mathbb{N}$ and $1 \le r < s$. If $f \in L^p(\mathbb{S}^1)$, $1 \le p < \infty$, and if $f \in C(\mathbb{S}^1)$ for $p = \infty$, then for $0 < t < 1/2$,*

$$\omega_r(f;t)_p \le c_s t^r \int_t^1 \frac{\omega_s(f;u)_p}{u^{r+1}}\,\mathrm{d}u. \tag{4.1.7}$$

Proof. We start with the identity $(1-x)^r = 2^{-r}(1-x^2)^r + Q(x)(1-x)^{r+1}$, where $Q(x) = (1-(x+1)^r/2^r)/(1-x)$, which gives, on replacing x by the translation operator S_h,

$$\overrightarrow{\triangle}_h^r f = 2^{-r}\overrightarrow{\triangle}_{2h}^r f - Q(S_h)\overrightarrow{\triangle}_h^{r+1} f.$$

Since $Q(x)$ is a polynomial of degree $r-1$ and S_h has norm one in $L^p(\mathbb{S}^1)$, we see that $\|Q(S_h)\|_p$ is bounded. Consequently,

$$\left\|\overrightarrow{\triangle}_h^r f\right\|_p \le 2^{-r}\left\|\overrightarrow{\triangle}_{2h}^r f\right\|_p + c\left\|\overrightarrow{\triangle}_h^{r+1} f\right\|_p \le 2^{-r}\left\|\overrightarrow{\triangle}_{2h}^r f\right\|_p + c\,\omega_{r+1}(f;h)_p.$$

Applying the above inequality iteratively and using (i) in Lemma 4.1.4, we obtain

$$\left\|\overrightarrow{\triangle}_h^r f\right\|_p \le 2^{-rm}\|f\|_p + c\sum_{i=0}^m 2^{-rj}\,\omega_{r+1}(f;2^j h)_p.$$

Taking the supremum over $0 \le h \le t$ and using the monotonicity of $\omega_r(f;t)_p$, we get

$$\begin{aligned}\omega_r(f;t)_p &\le 2^{-rm}\|f\|_p + c\,t^r\sum_{j=0}^m (2^j t)^{-r}\,\omega_{r+1}(f;2^j t)_p\\ &\le 2^{-rm}\|f\|_p + c\,2^r t^r \sum_{j=0}^m \int_{2^j t}^{2^{j+1}t} \frac{\omega_{r+1}(f;2^j u)_p}{u^{r+1}}\,\mathrm{d}u,\end{aligned}$$

which leads to, on choosing m such that $2^{-1} \le 2^{m+1}t < 1$, the inequality

$$\omega_r(f;t)_p \le ct^r\left[\int_t^1 \frac{\omega_{r+1}(f;u)_p}{u^{r+1}}\,\mathrm{d}u + \|f\|_p\right]. \tag{4.1.8}$$

Since $\overrightarrow{\triangle}_h^r$ eliminates the constant whenever $r>0$, we can replace $f(x)$ by $f(x)-C$ on the right-hand side of Eq. (4.1.8). By the Jackson inequality (4.1.3),

$$\inf_{C\in\mathbb{R}}\|f-C\|_p \le c_r\omega_{r+1}(f;1)_p \le c_r\int_{\frac{1}{2}}^1 \frac{\omega_{r+1}(f;u)_p}{u^{r+1}}\mathrm{d}u,$$

since $\omega_{r+1}(f;u) \sim 1$ for $1/2 \le u \le 1$. Consequently, the $\|f\|_p$ term in Eq. (4.1.8) can be dropped, which proves inequality (4.1.7) for $s=r+1$. The general case $s>r+1$ can be deduced easily by induction. □

4.2 Modulus of Smoothness on the Unit Sphere

There are many ways to define a modulus of smoothness on the sphere $\mathbb{S}^{d-1}$. The one we define in this section has the advantage that it relies on the modulus of smoothness on $\mathbb{S}^1$, even though $\mathbb{S}^{d-1}$ is not itself a product space of $\mathbb{S}^1$, which allows us to utilize the results in the previous section.

Let $\{e_1, \ldots, e_d\}$ be the standard basis for $\mathbb{R}^d$: the ith coordinate of e_j is 1 if $i = j$, 0 if $i \neq j$. For $1 \le i \neq j \le d$ and $t \in \mathbb{R}$, recall that $Q_{i,j,t}$ denotes a rotation by the angle t in the (x_i, x_j)-plane, oriented such that the rotation from the vector e_i to the vector e_j is assumed to be positive. As an example, for $(i, j) = (1,2)$ and $(x_1, x_2) = s(\cos\theta, \sin\theta)$, the action of the rotation $Q_{1,2,t} \in SO(d)$ is given by

$$\begin{aligned} Q_{1,2,t}(x_1,\ldots,x_d) &= (x_1\cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t, x_3, \ldots, x_d) \\ &= (s\cos(\theta+t), s\sin(\theta+t), x_3, \ldots, x_d). \end{aligned} \tag{4.2.1}$$

There are $d(d-1)/2$ distinct angles $\theta_{i,j}$, too many for a basis of $\mathbb{S}^{d-1}$, but they are closely related to the Euler angles, defined in Sect. A.2, that form a basis for $SO(d)$.

For $r = 1, 2, \ldots$, we use $Q_{i,j,t}$ to define the difference operator

$$\triangle^r_{i,j,\theta} := \big(I - T(Q_{i,j,\theta})\big)^r, \qquad 1 \le i \neq j \le d, \tag{4.2.2}$$

where $T(Q)$ denotes the rotation operator $T(Q)f(x) = f(Q^{-1}x)$ as in Eq. (1.7.1). Since $Q_{i,j,\theta} = Q_{j,i,-\theta}$, we have $\triangle^r_{i,j,\theta} = \triangle^r_{j,i,-\theta}$. Because of Eq. (4.2.1), $\triangle^r_{i,j,\theta}$ can be expressed in terms of the forward difference. For instance, take $(i, j) = (1,2)$ as an example,

$$\triangle^r_{1,2,\theta} f(x) = \overrightarrow{\triangle}^r_\theta f\,(x_1\cos(\cdot) - x_2\sin(\cdot), x_1\sin(\cdot) + x_2\cos(\cdot), x_3, \ldots, x_d), \tag{4.2.3}$$

where $\overrightarrow{\triangle}^r_\theta$ is acting on the variable $(\cdot)$ and is evaluated at $t = 0$. Our modulus of smoothness on the sphere is defined in terms of these differences.

Definition 4.2.1. For $r \in \mathbb{N}$, $t > 0$, and $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p < \infty$, and $f \in C(\mathbb{S}^{d-1})$ for $p = \infty$, define

$$\omega_r(f;t)_p := \max_{1\le i<j\le d} \sup_{|\theta|\le t} \left\| \triangle^r_{i,j,\theta} f \right\|_p. \tag{4.2.4}$$

For $r = 1$, we write $\omega(f;t)_p := \omega_1(f;t)_p$.

By Eq. (4.2.3), the quantity $\sup_{|\theta|\le t} \|\triangle^r_{i,j,\theta} f\|_p$ resembles the moduli of smoothness on the largest circle of the intersection of the (x_i, x_j)-plane and $\mathbb{S}^{d-1}$, and our modulus of smoothness $\omega_r(f,t)_p$ is the maximum among all possible choices of (i, j).

Just as in the case of $\mathbb{S}^1$, the modulus of smoothness is a continuous and increasing function of t with $\omega_r(f;t)_p \to 0$, $t \to 0^+$, and it satisfies the following properties:

(1) For $s < r$, $\omega_r(f,t)_p \le 2^{r-s}\omega_s(f,t)_p$.
(2) For $\lambda > 0$, $\omega_r(f;\lambda t)_p \le (\lambda+1)^r \omega_r(f;t)_p$.

Several further properties of $\omega_r(f;t)$ are more convenient, and more useful, if stated in terms of $\Delta^r_{i,j,\theta}$. Recall the relation between $Q_{i,j,t}$ and the differential operator $D_{i,j} = x_i\partial_j - x_j\partial_i$ defined in Sect. 1.4, in particular in Eq. (1.8.5).

Lemma 4.2.2. *Let $r \in \mathbb{N}$ and let $f \in L^p(\mathbb{S}^{d-1})$ with $1 \le p < \infty$, and $f \in C(\mathbb{S}^{d-1})$ when $p = \infty$.*

(i) *For every $\lambda > 0$, $t \in (0, 2\pi]$, and $1 \le i < j \le d$, we have*

$$\sup_{|\theta| \le \lambda t} \|\triangle^r_{i,j,\theta} f\|_p \le (\lambda+1)^r \sup_{|\theta|\le t} \|\triangle^r_{i,j,\theta} f\|_p.$$

(ii) *For $1 \le i \ne j \le d$ and $\theta \in [-\pi,\pi]$,*

$$\|\triangle^r_{i,j,\theta} f\|_p \le 2^r \|f\|_p \quad \textit{and} \quad \|\triangle^r_{i,j,\theta} f\|_p \le c\,|\theta|^r \|D^r_{i,j} f\|_p.$$

(iii) *If $f \in \Pi^d_n$ and $1 \le i < j \le d$, then*

$$\|\triangle^r_{i,j,n^{-1}} f\|_p \sim n^{-r} \|D^r_{i,j} f\|_p.$$

(iv) *For $1 \le i < j \le d$ and $t \in (0, 2\pi)$,*

$$\sup_{|\theta|\le t} \|\triangle^r_{i,j,\theta} f\|_p^p \sim \frac{1}{t}\int_0^t \|\triangle^r_{i,j,\theta} f\|_p^p \,\mathrm{d}\theta,$$

with $\|\cdot\|_p^p$ replaced by $\|\cdot\|_\infty$ when $p=\infty$.

Proof. Clearly, we need to consider only the case $(i,j) = (1,2)$. Then case (i) follows from Eq. (4.1.2) as in the case of one variable. For the other cases, we define

$$g_{s,y}(\phi) := f(s\cos\phi, s\sin\phi, \sqrt{1-s^2}y), \quad y \in \mathbb{S}^{d-3}, \quad s \in [0,1], \quad \phi \in [0, 2\pi].$$

For $d > 3$, the integral over $\mathbb{S}^{d-1}$ can be written, by Eq. (A.5.3), as

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} f(y)\,\mathrm{d}\sigma(y) &= \int_{\mathbb{B}^2}\int_{\mathbb{S}^{d-3}} f(x_1,x_2,\sqrt{1-\|x\|^2}y)\,\mathrm{d}\sigma(y)\,(1-\|x\|^2)^{\frac{d-4}{2}}\,\mathrm{d}x \\ &= \int_0^1 s(1-s^2)^{\frac{d-4}{2}} \int_{\mathbb{S}^{d-3}}\int_0^{2\pi} g_{s,y}(\phi)\,\mathrm{d}\phi\,\mathrm{d}\sigma(y)\,\mathrm{d}s, \qquad (4.2.5) \end{aligned}$$

where the second equality follows from changing variables $(x_1,x_2) = s(\cos\phi, \sin\phi)$. By Eq. (4.2.3), the identity (4.2.5) implies immediately

$$\|\triangle^r_{1,2,t} f\|_p^p = \frac{1}{\omega_d}\int_0^1 s(1-s^2)^{\frac{d-4}{2}} \int_{\mathbb{S}^{d-3}} \left[\int_0^{2\pi} \left|\overrightarrow{\triangle}^r_t g_{s,y}(\phi)\right|^p \mathrm{d}\phi\right] \mathrm{d}\sigma(y)\,\mathrm{d}s. \quad (4.2.6)$$

In the case $d = 3$, the formula (4.2.5) degenerates to a form in which the integral over $\mathbb{S}^{d-3}$ is replaced by a sum of two terms; see Eq. (A.5.4). Furthermore, by Eqs. (4.2.1) and (1.8.5), it is easy to see that

$$(-1)^r D^r_{1,2} f(s\cos\phi, s\sin\phi, \sqrt{1-s^2}y) = g^{(r)}_{s,y}(\phi) \quad (4.2.7)$$

and that $g_{s,y}(\phi)$ is a 2π-periodic polynomial in ϕ. With these results, the three cases (ii), (iii), (iv) follow by applying the corresponding results in Lemma 4.1.4 to $g_{s,y}$ in an obvious manner. □

Lemma 4.2.3. *For* $0 < t < \frac{1}{2}$ *and every* $m > r$,

$$\omega_r(f;t)_p \le c_m t^r \int_t^1 \frac{\omega_m(f,u)_p}{u^{r+1}}\,\mathrm{d}u.$$

Proof. This is an analogue of the Marchaud inequality on $\mathbb{S}^{d-1}$, and it follows, by Eq. (4.2.6), from Eq. (4.1.7). □

Let us also state a Bernstein-type inequality for the differential operator $D_{i,j}$.

Lemma 4.2.4. *Let f be a polynomial in $\Pi_n(\mathbb{S}^{d-1})$. For $1 \le i < j \le d$, $r \in \mathbb{N}$,*

$$\|D^r_{i,j} f\|_p \le n^r \|f\|_p, \qquad 1 \le p \le \infty. \quad (4.2.8)$$

Proof. For $f \in \Pi_n^(\mathbb{S}^{d-1})$, $g_{s,y}$ defined in the proof of the previous lemma is a trigonometric polynomial of degree at most n. By Eqs. (4.2.5) and (4.2.6), it is easy to see that the inequality (4.2.8) follows from applying the Bernstein inequality (4.1.5) for the trigonometric polynomials on $g_{s,y}$. □

The following elementary lemma is needed for the proof of our next result.

Lemma 4.2.5. *Suppose that $1 \le i \ne j \le d$ and $x, y \in \mathbb{S}^{d-1}$ differ only at their ith and jth components. Then $y = Q_{i,j,t}\,x$, with the angle t satisfying*

$$\cos t = (x_i y_i + x_j y_j)/s^2 \quad \textit{and} \quad t \sim \|x-y\|/s \quad \textit{with} \quad s := \sqrt{x_i^2 + x_j^2}.$$

Proof. Since x and y differ at exactly two components, they differ by a two-dimensional rotation. Moreover, since $x_i^2 + x_j^2 = y_i^2 + y_j^2$, the formula for $\cos t$ is the classical formula for the angle between two vectors in $\mathbb{R}^2$. We also have

$$t^2 \sim 4\sin^2\frac{t}{2} = 2(1-\cos t) = \|(x_i,x_j)-(y_i,y_j)\|^2/s^2 = \|x-y\|^2/s^2,$$

where the first $\|\cdot\|$ term is the Euclidean norm on $\mathbb{R}^2$, and the second one is the norm on $\mathbb{R}^d$. □

Recall that the distance on $\mathbb{S}^{d-1}$ is defined by $\mathtt{d}(x,y) = \arccos\langle x,y\rangle$.

Lemma 4.2.6. *For $x,y \in \mathbb{S}^{d-1}$,*

$$|f(x)-f(y)| \le c\,\omega(f;\mathtt{d}(x,y))_\infty,$$

where c depends only on the dimension.

Proof. We may assume that $\mathtt{d}(x,y) \le \delta_d := 1/(2d^2)$. Otherwise, we can select an integer m such that $\mathtt{d}(x,y) \le m\delta_d < \pi$. Then m is finite, and we can select points $x = z_0, z_1, \ldots, z_m = y$ on the great circle connecting x and y on $\mathbb{S}^{d-1}$ such that $\mathtt{d}(z_i,z_{i+1}) = \frac{\mathtt{d}(x,y)}{m} \le \delta_d$ for $i = 0,1,\ldots,m-1$, and then use the triangle inequality. Since $\|x\| = 1$ implies that $|x_i| \ge 1/\sqrt{d}$ for at least one i, we can assume without loss of generality, since $\omega(f;t)_\infty$ is independent of the order of $e_1,\ldots,e_d$, that $x_d = \max_{1\le j\le d}|x_j| \ge \frac{1}{\sqrt{d}}$.

For $1 \le j \le d-2$, let $u_j' := (x_1,\ldots,x_j,y_{j+1},\ldots,y_{d-1})$ and $v_j := \sqrt{1-\|u_j'\|^2}$, where by the choice of x_d and δ_d,

$$\|u_j'\|^2 = 1-(x_{j+1}^2-y_{j+1}^2)-\cdots-(x_{d-1}^2-y_{d-1}^2)-x_d^2$$

$$\le 1-\frac{1}{d}+2(d-j-2)\mathtt{d}(x,y) \le 1-\frac{1}{d}+\frac{1}{d} \le 1.$$

We then define $u_0 = y$, $u_j = (u_j',v_j) \in \mathbb{S}^{d-1}$ for $1 \le j \le d-2$, and $u_{d-1} = x$. By definition, u_j and u_{j-1} differ at exactly the jth and dth elements, so that we can write $u_{j-1} = Q_{j,d,t_j}u_j$, where the angle t_j satisfies, by Lemma 4.2.5,

$$t_j \sim \|u_{j-1}-u_j\|/s_j, \quad \text{where} \quad s_j^2 = x_j^2+v_j^2.$$

Our assumption shows that

$$s_j^2 \ge v_j^2 = x_d^2+(x_{j+1}^2-y_{j+1}^2)+\cdots+(x_{d-1}^2-y_{d-1}^2)$$

$$\ge \frac{1}{d}-\|x-y\|^2 \ge \frac{1}{2d},$$

and on the other hand, by Eq. (A.1.1),

$$\|u_j-u_{j-1}\|^2 = |x_j-y_j|^2+\frac{(x_j^2-y_j^2)^2}{(v_{j-1}+v_j)^2} \le (1+8d)|x_j-y_j|^2 \le c\,\mathtt{d}(x,y)^2.$$

Together, the last three displayed equations imply that $t_j \le c\mathtt{d}(x,y)$. Hence,

$$
\begin{aligned}
|f(x)-f(y)| &\le \sum_{j=1}^{d-1} |f(Q_{j,d,t_j}u_j)-f(u_j)| \\
&\le (d-1)\omega(f,c\,\mathtt{d}(x,y))_\infty \le c\,\omega(f,\mathtt{d}(x,y))_\infty,
\end{aligned}
$$

where the last step uses (2) of Proposition 4.2.3. □

4.3 A Key Lemma

This section contains a lemma that plays an essential role in the proof of our main result. The proof, however, is long and technical, and the reader may want to skip it on first reading.

Recall the operator $L_n f$ defined in Definition 2.6.2 and its kernel $L_n(\langle x,y\rangle)$. For a fixed integer $\ell \in \mathbb{N}$, let

$$
G_n(t) = G_{n,\ell}(t) := n^{d-1}(1+nt)^{-\ell}, \qquad t \in [0,\pi].
$$

By Theorem 2.6.5, the kernel function L_n is bounded by

$$
|L_n(\cos\theta)| \le c_\ell G_{n,\ell}(\theta), \qquad 0 \le \theta \le \pi. \tag{4.3.1}
$$

Lemma 4.3.1. *Suppose that* $f \in L^p(\mathbb{S}^{d-1})$ *for* $1 \le p < \infty$ *and* $\ell > p+d$ *in the estimate* (4.3.1)*. Then*

$$
\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} |f(x)-f(y)|^p G_n(\mathtt{d}(x,y))\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y) \le c\,\omega(f;n^{-1})_p^p.
$$

Proof. Let $E_j^+ := \{x \in \mathbb{S}^{d-1} : x_j \ge \frac{1}{\sqrt{d}}\}$ and $E_j^- := \{x \in \mathbb{S}^{d-1} : x_j \le -\frac{1}{\sqrt{d}}\}$ for $1 \le j \le d$. Then $\mathbb{S}^{d-1} = \bigcup_{j=1}^d (E_j^+ \cup E_j^-)$. Hence, it is enough to show that for each $1 \le k \le d$,

$$
\int_{E_k^\pm}\int_{\mathbb{S}^{d-1}} |f(x)-f(y)|^p |G_n(\mathtt{d}(x,y))|\,\mathrm{d}\sigma(y)\,\mathrm{d}\sigma(x) \le c\,\omega(f;n^{-1})_p^p. \tag{4.3.2}
$$

By symmetry, it is enough to consider E_d^+. For $0 < \delta < \pi$ and $x \in \mathbb{S}^{d-1}$, let $c(x,\delta)$ denote the spherical cap defined by

$$
c(x,\delta) := \{y \in \mathbb{S}^{d-1} : \mathtt{d}(x,y) \le \delta\}.
$$

We choose $\delta = 1/(8d)$ and split the integral in Eq. (4.3.2) into two parts:

$$\int_{E_d^+}\int_{c(x,\delta)} \cdots \mathrm{d}\sigma(y)\,\mathrm{d}\sigma(x) + \int_{E_d^+}\int_{\mathbb{S}^{d-1}\setminus c(x,\delta)} \cdots \mathrm{d}\sigma(y)\,\mathrm{d}\sigma(x) =: A + B.$$

We first estimate the integral B:

$$\begin{aligned} B &= \int_{E_d^+}\int_{\{y\in\mathbb{S}^{d-1}:\ \mathtt{d}(x,y)\ge\delta\}} |f(x)-f(y)|^p |G_n(\mathtt{d}(x,y))|\,\mathrm{d}\sigma(y)\,\mathrm{d}\sigma(x) \\ &\le c n^{d-1-\ell}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} |f(x)-f(y)|^p \mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y) \\ &= c n^{d-1-\ell}\int_{SO(d)}\int_{\mathbb{S}^{d-1}} |f(x)-f(Qx)|^p \mathrm{d}\sigma(x)\,dQ, \end{aligned} \tag{4.3.3}$$

where the last step uses the standard realization of $S^{d-1} = SO(d)/SO(d-1)$. It is known that every Q in $SO(d)$ can be decomposed in terms of Euler angles (see Appendix A), $Q = Q_1 Q_2 \cdots Q_{d(d-1)/2}$ with $Q_k = Q_{i_k,i_k+1,t}$ for some $t = t_{i_k,j_k}$ in $[0,2\pi]$ or $[0,\pi]$ and $1 \le i_k < j_k \le d$. It then follows that

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} |f(x)-f(Qx)|^p \mathrm{d}\sigma(x) &\le c\int_{\mathbb{S}^{d-1}} \left|f\left(Q_{d(d-1)/2}x\right)-f(x)\right|^p \mathrm{d}\sigma(x) \\ &\quad + \sum_{k=1}^{\frac{d(d-1)}{2}-1}\int_{\mathbb{S}^{d-1}} \left|f\left(Q_k\cdots Q_{d(d-1)/2}x\right)-f\left(Q_{k+1}\cdots Q_{d(d-1)/2}x\right)\right|^p \mathrm{d}\sigma(x) \\ &\le c \max_{1\le i<j\le d}\sup_{0<\theta\le 2\pi}\int_{\mathbb{S}^{d-1}} \left|f\left(Q_{i,j,\theta}x\right)-f(x)\right|^p \mathrm{d}\sigma(x) \\ &\le c\omega(f;2\pi)_p^p \lesssim n^p\omega(f,n^{-1})_p^p, \end{aligned}$$

which, together with Eq. (4.3.3), gives the desired estimate $B \le c\,\omega(f;n^{-1})_p^p$.

It remains to estimate the integral A. Setting $x = (x', x_d)$ with $x_d = \sqrt{1-\|x'\|^2}$, we deduce from Eq. (A.5.4) that

$$\begin{aligned} A &= \int_{E_d^+}\int_{c(x,\delta)} |f(x)-f(y)|^p G_n(\mathtt{d}(x,y))\,\mathrm{d}\sigma(y)\,\mathrm{d}\sigma(x) \\ &= \int_{\|x'\|\le d^*}\int_{c(x,\delta)} |f(x)-f(y)|^p G_n(\mathtt{d}(x,y))\,\mathrm{d}\sigma(y)\,\frac{\mathrm{d}x'}{\sqrt{1-\|x'\|^2}}, \end{aligned}$$

where $d^* = \sqrt{1-d^{-1}}$. Since $x_d \ge \frac{1}{\sqrt{d}}$, it follows that for every $y = (y', y_d) \in c(x,\delta)$, $y_d \ge x_d - |y_d - x_d| \ge x_d - \mathtt{d}(x,y) \ge x_d - \delta \ge \frac{1}{2\sqrt{d}}$, which further implies, by a simple computation, that

$$\|x' - y'\| \le \|x - y\| \le \|x' - y'\| + |x_d - y_d|$$
$$= \|x' - y'\| + \frac{|\|x'\|^2 - \|y'\|^2|}{x_d + y_d} \le (1 + 2\sqrt{d})\|x' - y'\|..$$

Consequently, setting $g(x') := f\left(x', \sqrt{1 - \|x'\|^2}\right)$, using Eq. (A.5.4) again, and observing that $G_n(\theta_1) \sim G_n(\theta_2)$ whenever $\theta_1 \sim \theta_2$, we obtain

$$A \le c \int_{\|x'\| \le d^*} \int_{\|x' - y'\| \le \delta} |g(x') - g(y')|^p G_n(\|x' - y'\|) \, \mathrm{d}y' \, \mathrm{d}x'$$
$$= c \int_{\|x'\| \le d^*} \int_{\|u\| \le \delta} |g(x') - g(u + x')|^p G_n(\|u\|) \, \mathrm{d}u \, \mathrm{d}x'.$$

Let $b_0(u) := 0$ and $b_j(u) := u_1 e_1 + \cdots + u_j e_j$, $1 \le j \le d-1$. Since

$$g(x') - g(x' + u) = \sum_{j=1}^{d-1} \left(g(x' + b_{j-1}(u)) - g(x' + b_j(u)) \right),$$

it suffices by the triangle inequality to estimate, for $1 \le j \le d-1$,

$$A_j := \int_{\|x'\| \le d^*} \int_{\|u\| \le \delta} |g(x' + b_{j-1}(u)) - g(x' + b_j(u))|^p G_n(\|u\|) \mathrm{d}u \, \mathrm{d}x'$$
$$\le \int_{\|x'\| \le d^* + \delta} \int_{\|u\| \le \delta} |g(x') - g(x' + u_j e_j)|^p G_n(\|u\|) \mathrm{d}u \, \mathrm{d}x',$$

where the second line follows from a change of variables $x' + b_{j-1}(u) \mapsto x'$. By symmetry, it suffices to consider A_1.

Observe that for $u_1 \in \mathbb{R}$ and $u = (u_1, v) \in \mathbb{R}^{d-2}$,

$$G_n(\|u\|) = n^{d-1}(1 + n\|u\|)^{-\ell} \le H_n(|u_1|) n^{d-2} (1 + n\|v\|)^{-d+1},$$

where $H_n(s) = n(1 + ns)^{-\ell + d - 1}$, and we have used the assumption $\ell > d - 1$ in the last step. This implies

$$A_1 \le c \int_{\|x'\| \le d^* + \delta} \int_{-\delta}^{\delta} |g(x') - g(x' + u_1 e_1)|^p$$
$$\times \left[\int_{\{v \in \mathbb{R}^{d-2} : \|v\| \le \sqrt{\delta^2 - |u_1|^2}\}} G_n(\|(u_1, v)\|) \, \mathrm{d}v \right] \mathrm{d}u_1 \, \mathrm{d}x'$$
$$\le c \int_{\|x'\| \le d^* + \delta} \int_{-\delta}^{\delta} |g(x') - g(x' + s e_1)|^p H_n(|s|) \mathrm{d}s \, \mathrm{d}x'. \tag{4.3.4}$$

Set $v_1(t,x') = -x_1 + x_1 \cos t - \sqrt{1-\|x'\|^2} \sin t$. A straightforward calculation shows that

$$\frac{1}{\pi\sqrt{d}} \le \frac{-v_1(t,x')}{t} \le 2 \quad \text{and} \quad \frac{1}{4\sqrt{d}} \le -\frac{\partial}{\partial t} v_1(t,x') \le 2 \tag{4.3.5}$$

whenever $|t| \le \sqrt{d}\delta = \delta^*$ and $\|x'\| \le d^* + \delta$. Thus, performing a change of variable $s = v_1(t,x')$ in Eq. (4.3.4) yields

$$\begin{aligned} A_1 &\le c \int_{\|x'\| \le d^*+\delta} \int_{-\delta^*}^{\delta^*} |g(x') - g(x' + v_1(x',t)e_1)|^p H_n(|v_1(x',t)|) \left| \frac{\partial v_1(t,x')}{\partial t} \right| \mathrm{d}t\, \mathrm{d}x' \\ &\le c \int_{\|x'\| \le \rho} \int_{-\delta^*}^{\delta^*} |g(x') - g(x' + v_1(x',t)e_1)|^p H_n(|t|) \mathrm{d}t\, \mathrm{d}x', \end{aligned}$$

where $\rho := \sqrt{1-(2d)^{-1}} \ge d^* + \delta$, and we used Eq. (4.3.5) and the monotonicity of H_n in the last step. Now observe that for $x = (x', \sqrt{1-\|x'\|^2})$ with $\|x'\| \le \rho$,

$$Q_{1,d,t} x = (x' + v_1(x',t)e_1, z_d), \qquad \forall t \in [-\delta^*, \delta^*],$$

where, using the fact that $\sin t \le t \le 1/(8\sqrt{d})$,

$$z_d = \sqrt{1 - \|x' + v_1(x',t)e_1\|^2} = x_1 \sin t + \sqrt{1-\|x'\|^2} \cos t \ge 1/(4\sqrt{d}) > 0.$$

Thus, using Eq. (A.5.4), we deduce that

$$\begin{aligned} A_1 &\le c \int_{-\delta^*}^{\delta^*} \int_{\{x \in \mathbb{S}^{d-1} : x_d \ge (2d)^{-1}\}} |f(x) - f(Q_{1,d,t}x)|^p \, \mathrm{d}\sigma(x) H_n(|t|) \mathrm{d}t \\ &\le c \int_{-\delta^*}^{\delta^*} \omega(f;|t|)_p^p H_n(|t|) \mathrm{d}t. \end{aligned}$$

Hence, by (2) of Proposition 4.2.3 and the definition of H_n,

$$\begin{aligned} A_1 &\le c\omega(f;n^{-1})_p^p \int_0^{\delta^*} (1+nt)^p H_n(t) \mathrm{d}t = \omega(f;n^{-1})_p^p n \int_0^{\delta^*} (1+nt)^{-\ell+p+d-1} \mathrm{d}t \\ &\le c\omega(f;n^{-1})_p^p \int_0^{\infty} (1+s)^{-\ell+p+d-1} \mathrm{d}s \le c\omega(f;n^{-1})_p^p, \end{aligned}$$

since $\ell > p + d$. This completes the proof. □

4.4 Characterization of Best Approximation

Recall the quantity $E_n(f)_p$ of best approximation by polynomials defined in Definition 2.6.1. Our main result in this section is a characterization of the best approximation by polynomials in terms of the modulus of smoothness. The direct theorem of the characterization uses the operator $L_n f$ defined in Definition 2.6.2, for which we need the commutativity of the operators L_n and $\triangle_{i,j,t}$.

Lemma 4.4.1. *For $1 \le i \ne j \le d$,*

$$\triangle^r_{i,j,t} L_n f = L_n \left(\triangle^r_{i,j,t} f\right).$$

In particular, for $1 \le p \le \infty$ and $t > 0$,

$$\omega_r(f - L_n f, t)_p \le c\, \omega_r(f,t)_p.$$

Proof. Recall that $T(Q)f(x) = f(Q^{-1}x)$ for $Q \in SO(d)$. The definition of $L_n f$ shows that

$$T(Q)L_n f(x) = \int_{\mathbb{S}^{d-1}} f(y) L_n(\langle Q^{-1}x, y\rangle) \mathrm{d}\sigma(y) = \int_{\mathbb{S}^{d-1}} f(y) L_n(\langle x, Qy\rangle) \mathrm{d}\sigma(y)$$
$$= \int_{\mathbb{S}^{d-1}} f(Q^{-1}y) L_n(\langle x, y\rangle) \mathrm{d}\sigma(y) = L_n(T(Q)f)(x)$$

by the rotation invariance of $d\sigma(y)$, which gives the commutativity of L_n and $\triangle^r_{i,j,t}$, since $\triangle^r_{i,j,t} = (I - T_{Q_{i,j,t}})^r$. By (2) of Lemma 2.6.3,

$$\|\triangle^r_{i,j,t}(f - L_n f)\|_p = \|\triangle^r_{i,j,t} f - L_n \triangle^r_{i,j,t} f\|_p \le (1+c)\|\triangle^r_{i,j,t} f\|_p,$$

from which the stated inequality follows. $\square$

We are now in a position to prove our main result on the best approximation on the sphere.

Theorem 4.4.2. *For $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p < \infty$, and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$,*

$$E_n(f)_p \le c\, \omega_r(f; n^{-1})_p, \qquad 1 \le p \le \infty. \tag{4.4.1}$$

On the other hand,

$$\omega_r(f; n^{-1})_p \le c\, n^{-r} \sum_{k=1}^{n} k^{r-1} E_{k-1}(f)_p, \qquad 1 \le p \le \infty, \tag{4.4.2}$$

where $\omega_r(f;t)_p$ and $E_n(f)_p$ are defined in Eqs. (4.2.4) *and* (2.6.1)*, respectively.*

Proof. When $r = 1$ and $1 \le p < \infty$, we use Lemma 2.6.3, Hölder's inequality, and the fact that $\int_{\mathbb{S}^{d-1}} |L_n(\langle x,y\rangle)| \mathrm{d}\sigma(y) \le c$ for all $x \in \mathbb{S}^{d-1}$ to obtain

$$E_n(f)_p \le \|f - L_{\lfloor \frac{n}{2} \rfloor} f\|_p$$
$$\le c \left(\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |f(x) - f(y)|^p |L_{\lfloor \frac{n}{2} \rfloor}(\langle x,y\rangle)| \mathrm{d}\sigma(x) \mathrm{d}\sigma(y) \right)^{\frac{1}{p}},$$

from which Eq. (4.4.1) for $r = 1$ follows from Lemma 4.3.1. For $r = 1$ and $p = \infty$, we use Lemma 4.2.6 and $L_n f$ to conclude that

$$E_n(f)_\infty \le \|f - L_{\lfloor \frac{n}{2} \rfloor} f\|_\infty \le \int_{\mathbb{S}^{d-1}} |f(x) - f(y)| |L_{\lfloor \frac{n}{2} \rfloor}(\langle x,y\rangle)| \mathrm{d}\sigma(y)$$
$$\le c \int_{\mathbb{S}^{d-1}} \omega(f, \mathtt{d}(x,y))_\infty |L_{\lfloor \frac{n}{2} \rfloor}(\langle x,y\rangle)| \mathrm{d}\sigma(y)$$
$$\le c\omega(f; n^{-1})_\infty \int_{\mathbb{S}^{d-1}} (1 + n\mathtt{d}(x,y)) |L_{\lfloor \frac{n}{2} \rfloor}(\langle x,y\rangle)| \mathrm{d}\sigma(y)$$
$$\le c\omega(f, n^{-1})_\infty,$$

where the last inequality follows from Eq. (4.3.1) and the fact that $\langle x,y\rangle = \cos \mathtt{d}(x,y)$, just like the estimate of A_j in the proof of Lemma 4.3.1.

For $r > 1$, we follow the induction procedure on r using $L_n f$ in Lemmas 2.6.3 and 4.4.1. Assume that we have proven Eq. (4.4.1) for some positive integer $r \ge 1$. Let $g = f - L_{\lfloor \frac{n}{2} \rfloor} f$. It suffices to show that $\|g\|_p \le c\,\omega_{r+1}(f, n^{-1})_p$. The definition of $L_n f$ implies that $L_{\lfloor \frac{n}{4} \rfloor} g = 0$, so that

$$\|g\|_p = \|g - L_{\lfloor \frac{n}{4} \rfloor} g\|_p \le c E_{\lfloor \frac{n}{4} \rfloor}(g)_p \le c_1 \omega_r(g; n^{-1})_p.$$

On the other hand, using Lemma 4.2.3, we obtain, for every $m \in \mathbb{N}$,

$$\omega_r(g;t)_p \le c_r t^r \int_t^{2^m t} \frac{\omega_{r+1}(g,u)_p}{u^{r+1}} \mathrm{d}u + c_r 2^{r+1} t^r \|g\|_p \int_{2^m t}^1 u^{-r-1} \mathrm{d}u$$
$$\le c_{m,r} \omega_{r+1}(g,t)_p + c_r' 2^{-mr} \|g\|_p,$$

where c_r' is independent of m. Choosing m such that $4^{-1} \le c_1 c_r' 2^{-mr} < 2^{-1}$, we deduce from these two equations that

$$\|g\|_p \le c\,\omega_{r+1}(g; n^{-1})_p \le c\,\omega_{r+1}(f; n^{-1})_p,$$

where the last step follows from Lemma 4.4.1. This completes the proof of Eq. (4.4.1).

The proof of (4.4.2) relies on the Bernstein inequality. Let P_n denote the polynomial such that $E_n(f)_p = \|f - P_n\|_p$ and set $P_{2^{-1}} = P_0$. Then

$$\omega_r(f;h)_p \le \omega_r(f - P_{2^m}; h)_p + \omega_r(P_{2^m}; h)_p$$

for $m = 1, 2, \ldots$. By (ii) of Lemma 4.2.1,

$$\omega_r(f - P_{2^m}; h)_p \le 2^r \|f - P_{2^m}\|_p = 2^r E_{2^m}(f)_p,$$

whereas by (ii) of Lemma 4.2.1 and the Bernstein inequality (4.2.8),

$$\begin{aligned}
\|\overrightarrow{\triangle}^r_{i,j,h} P_{2^m}\|_p &\le c\, h^r \|D^r_{i,j} P_{2^m}\|_p \le c\, h^r \sum_{k=0}^{m} \|D^r_{i,j}(P_{2^k} - P_{2^{k-1}})\|_p \\
&\le c\, h^r \sum_{k=0}^{m} 2^{kr} \|P_{2^k} - P_{2^{k-1}}\|_p \le c\, h^r \sum_{k=0}^{m} 2^{kr} E_{2^{k-1}}(f)_p,
\end{aligned}$$

since $\|P_{2^k} - P_{2^{k-1}}\|_p \le \|f - P_{2^k}\|_p + \|f - P_{2^{k-1}}\|_p$. Consequently, it follows that

$$\omega_r(f; h)_p \le 2^r E_{2^m}(f)_p + c\, h^r \sum_{k=0}^{m} 2^{kr} E_{2^{k-1}}(f)_p.$$

Directly from its definition, $E_n(f)_p$ is a nonincreasing function of n, whence

$$2^{kr} E_{2^{k-1}}(f)_p \le 2^{2r-1} \sum_{j=2^{k-2}}^{2^{k-1}} (j+1)^{r-1} E_j(f)_p.$$

Combining the last two inequalities proves Eq. (4.4.2) for $n = 2^m$. For a given positive integer n, choose m such that $2^m < n \le 2^{m+1}$. Then Eq. (4.4.2) can be deduced from the special case $n = 2^m$ by the monotonicity of $\omega_r(f; h)$ in h and $E_n(f)_p$ in n. □

Corollary 4.4.3. *Let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. For $0 < \alpha < r$, $E_n(f)_p \sim n^{-\alpha}$ if and only if $\omega_r(f; t)_p \sim t^\alpha$.*

4.5 *K*-Functionals and Approximation in Sobolev Space

Besides the modulus of smoothness, the smoothness of a function can also be described by a *K*-functional, which describes how well the function can be approximated by smooth functions in a certain sense. We define a *K*-functional via the differential operators $D_{i,j} = x_i \partial_j - x_j \partial_i$ that turns out to be equivalent to $\omega_r(f; t)_p$, as is often the case in approximation theory. The *K*-functional is defined via the Sobolev space and is often easier to apply when the function is known to be differentiable.

Definition 4.5.1. For $r \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $\mathscr{W}^r_p \equiv \mathscr{W}^r_p(\mathbb{S}^{d-1})$ consists of functions $f \in L^p(\mathbb{S}^{d-1})$ with distributional derivatives $D^r_{i,j} f$, $1 \le i < j \le$

d, all belonging to $L^p(\mathbb{S}^{d-1})$, where $L^p(\mathbb{S}^{d-1})$ is replaced by $C(\mathbb{S}^{d-1})$ when $p=\infty$. The norm of the space is defined by

$$\|f\|_{\mathscr{W}_p^r(\mathbb{S}^{d-1})} := \|f\|_p + \sum_{1\le i<j\le d} \|D_{i,j}^r f\|_p.$$

By the definition of $D_{i,j}$ in Eq. (1.8.1), it is easy to verify, if we take $(i,j)=(1,2)$ as an example and set $(x_1,x_2)=(s\cos\phi, s\sin\phi)$, that

$$D_{1,2}^r f(x) = \left(-\frac{\partial}{\partial\phi}\right)^r f(s\cos\phi, s\sin\phi, x_3,\ldots,x_d). \tag{4.5.1}$$

Definition 4.5.2. For $r\in\mathbb{N}_0$ and $t\ge 0$,

$$K_r(f,t)_p := \inf_{g\in\mathscr{W}_p^r(\mathbb{S}^{d-1})} \left\{ \|f-g\|_p + t^r \max_{1\le i<j\le d} \|D_{i,j}^r g\|_p \right\}. \tag{4.5.2}$$

Theorem 4.5.3. *Let $r\in\mathbb{N}$ and let $f\in L^p(\mathbb{S}^{d-1})$ if $1\le p<\infty$ and $f\in C(\mathbb{S}^{d-1})$ if $p=\infty$. For $0<t<1$,*

$$\omega_r(f;t)_p \sim K_r(f,t)_p, \qquad 1\le p\le\infty.$$

Proof. By (ii) of Lemma 4.2.2 and the triangle inequality,

$$\|\triangle_{i,j,\theta}^r f\|_p \le \|\triangle_{i,j,\theta}^r (f-g)\|_p + \|\triangle_{i,j,\theta}^r g\|_p \le c\|f-g\|_p + c\theta^r \|D_{i,j}^r g\|_p,$$

from which $\omega_r(f;t)_p \le cK_r(f,t)_p$ follows. On the other hand, for $t>0$, set $n=\lfloor \frac{1}{t}\rfloor$. Then by Lemma 2.6.3, Eq. (4.4.1), and (iii) of Lemma 4.2.2,

$$\begin{aligned} K_r(f,t)_p &\le \|f-L_n f\| + t^r \max_{1\le i<j\le d} \|D_{i,j}^r L_n f\|_p \\ &\le c\omega_r(f;n^{-1})_p + ct^r n^r \max_{1\le i<j\le d} \|\triangle_{i,j,n^{-1}}^r L_n f\|_p \\ &\le c\omega_r(f;n^{-1})_p \le c\omega_r(f;t)_p, \end{aligned}$$

where the last step follows from (i) of Lemma 4.2.2. □

The proof of the above theorem, together with Lemma 4.4.1 and (iii) of Lemma 4.2.2, yields a realization of the K-functional.

Corollary 4.5.4. *Under the assumption of Theorem 4.5.3,*

$$K_r(f,n^{-1})_p \sim \|f-L_n f\|_p + n^{-r} \max_{1\le i<j\le d} \|D_{i,j}^r L_n f\|_p.$$

In particular, this shows that the best approximation $E_n(f)_p$ can be characterized by the K-functional. Furthermore, we can now consider approximation in the Sobolev space.

Theorem 4.5.5. *If $r \in \mathbb{N}$ and $f \in \mathscr{W}_p^r(\mathbb{S}^{d-1})$, $1 \le p \le \infty$, then*

$$E_{2n}(f)_p \le cn^{-r} \max_{1\le i<j\le d} E_n(D_{i,j}^r f)_p. \tag{4.5.3}$$

Furthermore, $L_n f$, defined by Eq. (2.6.2)*, provides the near best simultaneous approximation for all $D_{i,j}^r f$, $1 \le i < j \le d$, in the sense that*

$$\|D_{i,j}^r(f - L_n f)\|_p \le cE_n(D_{i,j}^r f)_p, \quad 1 \le i < j \le d. \tag{4.5.4}$$

Proof. By Proposition 2.6.4, $L_n D_{i,j}^r = D_{i,j}^r L_n$. Thus, using Theorems 4.4.2 and 4.5.3, we obtain

$$\begin{aligned} E_{2n}(f)_p &= E_{2n}(f - L_n f)_p \le cK_r(f - L_n f, n^{-1})_p \\ &\le cn^{-r} \max_{1\le j<j\le d} \|D_{i,j}^r(f - L_n f)\|_p \\ &= cn^{-r} \max_{1\le j<j\le d} \|D_{i,j}^r f - L_n(D_{i,j}^r f)\|_p \\ &\le cn^{-r} \max_{1\le i<j\le d} E_n(D_{i,j}^r f)_p, \end{aligned}$$

where we used Eq. (4.5.2) in the third step, the fact that $L_n D_{i,j}^r = D_{i,j}^r L_n$ in the fourth step, and Eq. (2.6.4) in the last step. This proves Eq. (4.5.3). The inequality (4.5.4) follows immediately from the above proof. □

Corollary 4.5.6. *If $r \in \mathbb{N}$, $f \in \mathscr{W}_p^r(\mathbb{S}^{d-1})$, and $1 \le p \le \infty$, then*

$$E_n(f)_p \le cn^{-r}\|f\|_{\mathscr{W}_p^r}. \tag{4.5.5}$$

4.6 Computational Examples

In this section, we give several examples of functions whose modulus of smoothness can be determined. Since $\omega_r(f;t)_p$ is defined in terms of the forward difference of one variable, it is not difficult to derive its upper bound. The difficulty lies in proving the lower bound.

Example 4.6.1. For $x \in \mathbb{S}^{d-1}$ and $d \ge 3$, let $f_\alpha(x) = x^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_d) \ne 0$. If $0 \le \alpha_i < 1$ for $1 \le i \le d$, then for $r \ge 2$ and $1 \le p \le \infty$,

$$\omega_r(f,t)_{L^p(\mathbb{S}^{d-1})} \sim t^{\delta+1/p}, \qquad \delta = \min_{\alpha_i \ne 0}\{\alpha_1, \ldots, \alpha_d\}. \tag{4.6.1}$$

Consequently,

$$E_n(f_\alpha)_p \sim n^{-\delta-1/p}, \qquad 1 \le p \le \infty.$$

We need to consider only $\triangle^2_{1,2,\theta} f$, which, by Eq. (4.2.3), can be expressed as a forward difference

$$\triangle^r_{1,2,\theta} f_\alpha(x) = x_3^{\alpha_3} \cdots x_d^{\alpha_d} s^{\alpha_1+\alpha_2} \overrightarrow{\triangle}^r_\theta \left[(\cos\phi)^{\alpha_1}(\sin\phi)^{\alpha_2}\right],$$

where $(x_1, x_2) = (s\cos\phi, s\sin\phi)$. Hence by Eq. (4.2.5), we obtain

$$\|\triangle^r_{1,2,\theta} f_\alpha\|_p = c\left(\int_0^{2\pi} \left|\overrightarrow{\triangle}^r_\theta \left[(\cos\phi)^{\alpha_1}(\sin\phi)^{\alpha_2}\right]\right|^p \mathrm{d}\phi\right)^{1/p}.$$

Furthermore, using the well-known relation

$$\overrightarrow{\triangle}^r_\theta (fg)(\phi) = \sum_{k=0}^r \binom{n}{k} \overrightarrow{\triangle}^k_\theta f(\phi) \overrightarrow{\triangle}^{r-k}_\theta g(\phi + k\theta),$$

we can consider the differences for $\cos(\phi+\cdot)$ and $\sin(\phi+\cdot)$ separately. Since the sine and cosine functions cannot be both large or both small, we can divide the integral domain accordingly and estimate the integral in the L^p norm. Furthermore, in our range of α_i, we need to consider only the second difference ($r = 2$). Equation (4.6.1) also holds for $r = 1$ and $p = \infty$.

Example 4.6.2. For $d \ge 3$ and $\alpha \ne 0$, let $g_\alpha(x) = (1-x_1)^\alpha$, $x = (x_1, \ldots, x_d) \in \mathbb{S}^{d-1}$. Then for $1 \le p \le \infty$,

$$\omega_2(g_\alpha, t)_{L^p(\mathbb{S}^{d-1})} \sim \begin{cases} t^{2\alpha+\frac{d-1}{p}}, & -\frac{d-1}{2p} < \alpha < 1 - \frac{d-1}{2p}, \\ t^2|\log t|^{1/p}, & \alpha = 1 - \frac{d-1}{2p}, \quad p \ne \infty, \\ t^2, & \alpha > 1 - \frac{d-1}{2p}. \end{cases} \tag{4.6.2}$$

For $\alpha = 0$, $\omega_2(g_\alpha, t)_p = 0$. Consequently, for $-\frac{d-1}{2p} < \alpha < 1 - \frac{d-1}{2p}$ and $\alpha \ne 0$,

$$E_n(g_\alpha)_p \sim n^{-2\alpha-\frac{d-1}{p}}, \qquad 1 \le p \le \infty.$$

If neither i nor j equals 1, then $\triangle^2_{i,j,\theta} g_\alpha(x) = 0$. Thus, we need to consider only $\triangle^2_{1,j,\theta} g_\alpha$, and we can assume that $j = 2$. Since $x \in \mathbb{S}^{d-1}$ and $d \ge 3$ imply that $(x_1, x_2) \in \mathbb{B}^2$, it follows that by Eq. (4.2.3),

$$\begin{aligned}\|\triangle^2_{1,2,\theta} g_\alpha\|_p^p &= c\int_{\mathbb{B}^2} |\triangle^2_{1,2,\theta} g_\alpha(x_1,x_2)|^p (1-x_1^2-x_2^2)^{\mu-1}\mathrm{d}x \\ &= c\int_0^1 s\int_0^{2\pi} \left|\overrightarrow{\triangle}^2_\theta (1-s\cos\phi)^\alpha\right|^p \mathrm{d}\phi (1-s^2)^{\mu-1}\mathrm{d}s,\end{aligned}$$

where $\mu = \frac{d-2}{2}$ and the forward difference acts on ϕ; for $p = \infty$, the integral is replaced by the maximum taken over $0 \le s \le 1$ and $0 \le \phi \le 2\pi$. This last integral can be shown to give the order in Eq. (4.6.2). The proof is elementary but rather involved; see [50].

It is of interest to compare the two examples. As functions defined on $\mathbb{R}^d$, the functions x_1^α and $(1-x_1)^\alpha$ have the same smoothness, and a reasonable modulus of smoothness would confirm that. As functions on the sphere $\mathbb{S}^{d-1}$, however, they have different orders of smoothness, as seen in Examples 4.6.1 and 4.6.2, and their errors of best approximation are also different, as seen in these examples.

The phenomenon indicated in the previous paragraph can also be seen in the following example, in which the asymptotic order for $\|y_0\| < 1$ is different from that of $\|y_0\| = 1$.

Example 4.6.3. Let y_0 be a fixed point in $\mathbb{B}^d$, let $0 \ne \alpha > -\frac{d-1}{2p}$, and let $f_\alpha : \mathbb{S}^{d-1} \to \mathbb{R}$ be given by $f_\alpha(x) := \|x - y_0\|^{2\alpha}$. If $\alpha \ne 1 - \frac{d-1}{2p}$, then

$$\omega_2(f_\alpha, t)_{L^p(\mathbb{S}^{d-1})} \sim \|y_0\| t^2 (t + 1 - \|y_0\|)^{2(\alpha-1)+\frac{d-1}{p}} + \|y_0\| t^2, \tag{4.6.3}$$

where the constants of equivalence are independent of t and y_0. Moreover, if $\alpha = 1 - \frac{d-1}{2p}$, then

$$c^{-1}\|y_0\| t^2 \le \omega_2(f_\alpha, t)_{L^p(\mathbb{S}^{d-1})} \le c\|y_0\| t^2 |\log(t + 1 - \|y_0\|)|^{\frac{1}{p}}, \tag{4.6.4}$$

where c is a positive constant independent of y_0 and t.

In particular, if $-\frac{d-1}{2p} < \alpha < 1 - \frac{d-1}{2p}$, then

$$E_n(f)_{L^p(\mathbb{S}^{d-1})} \sim n^{-2}\|y_0\|(n^{-1} + 1 - \|y_0\|)^{2(\alpha-1)+\frac{d-1}{p}}.$$

The proof of Eqs. (4.6.3) and (4.6.4) is rather involved, and we again refer the reader to [50].

4.7 Other Moduli of Smoothness

In this section, we discuss briefly two other moduli of smoothness on the sphere.

Historically, the first modulus of smoothness is defined in terms of the spherical means, or the translation operator $T_\theta f$ in Definition 2.1.4, which we recall as

$$T_\theta f(x) = \frac{1}{\omega_{d-1}} \int_{\mathbb{S}_x^\perp} f(x\cos\theta + u\sin\theta) \mathrm{d}\sigma(u).$$

We denote this modulus of smoothness by $\omega_r^*(f,t)_p$. It is defined by, for $r = 1, 2, \ldots$,

$$\omega_r^*(f;t)_p := \sup_{|\theta|\le t} \|(I - T_\theta)^{r/2} f\|_p, \tag{4.7.1}$$

where $(I - T_\theta)^{r/2}$ is defined in terms of infinite series when $r/2$ is not an integer. A characterization of the best polynomial approximation, both direct and inverse theorems, can be established in terms of $\omega_r^*(f;t)_p$. Furthermore, this modulus of smoothness is equivalent to the K-functional defined by

$$K_r^*(f;t)_p := \inf_g \left\{ \|f - g\|_p + t^r \left\|(-\Delta_0)^{r/2} g\right\|_p \right\}, \tag{4.7.2}$$

where Δ_0 is the Laplace–Beltrami operator on the sphere and the infimum is taken over all g for which $(-\Delta_0)^{r/2} g \in L^p(\mathbb{S}^{d-1})$. More precisely, we have the following result.

Theorem 4.7.1. *Both Theorems 4.4.2 and 4.5.3 hold with $\omega_r(f;t)_p$ and $K_r(f;t)_p$ replaced by $\omega_r^*(f;t)_p$ and $K_r^*(f;t)_p$.*

These results turn out to hold in the more general setting of approximation in weighted spaces. The latter will be discussed in Chap. 10 with a complete proof, which includes a proof of Theorem 4.7.1 as a special case.

The fact that both moduli of smoothness $\omega_r(f;t)_p$ and $\omega_r^*(f;t)_p$ characterize the best approximation by polynomials does not imply that the two are equivalent, since the inverse theorem of the characterization is of weak type. Only a partial result is known in this regard.

Theorem 4.7.2. *Let $f \in L^p(\mathbb{S}^{d-1})$ with $1 < p < \infty$. For $0 < t < 1$, $\omega_r(f;t)_p \le c\,\omega_r^*(f;t)_p$ if $r \in \mathbb{N}$, and $\omega_r(f;t)_p \sim \omega_r^*(f;t)_p$ if $r = 1, 2$.*

Proof. By the equivalence between the moduli of smoothness and the K-functionals, it suffices to consider the K-functionals. For $r = 1$, the equivalence follows immediately form Theorem 3.5.3, where $f \in C^1(\mathbb{S}^{d-1})$ can clearly be weakened to the derivatives of f belonging to $L^p(\mathbb{S}^{d-1})$. Furthermore, by the commutativity of $D_{i,j}$ and $(-\Delta_0)^{r/2}$, we obtain

$$\left\|D_{i,j}^r f\right\|_p \le c \left\|(-\Delta_0)^{\frac{1}{2}} D_{i,j}^{r-1} f\right\|_p = \left\|D_{i,j}^{r-1} (-\Delta_0)^{\frac{1}{2}} f\right\|_p \le c \left\|(-\Delta_0)^{\frac{r}{2}} f\right\|_p$$

by iteration and Eq. (3.5.3), which implies that $K_r(f;t)_p \le c K_r^*(f;t)_p$. Finally, the equivalence of $r = 2$ follows from Eq. (1.8.3). □

The second modulus of smoothness on the sphere is due to Ditzian. Recall that $T(Q)f(x) := f(Q^{-1}x)$ for $Q \in SO(d)$. For $t > 0$, define

$$O_t := \left\{ Q \in SO(d) : \max_{x\in\mathbb{S}^{d-1}} \mathtt{d}(x, Qx) \le t \right\},$$

where $\mathtt{d}(x,y)$ is the geodesic distance on $\mathbb{S}^{d-1}$. For $r>0$ and $t>0$, define

$$\tilde{\omega}_r(f;t)_p := \sup_{Q\in O_t} \|\triangle_Q^r f\|_p, \qquad \text{where} \quad \triangle_Q^r := (I-T_Q)^r. \tag{4.7.3}$$

The main results on this modulus of smoothness are summarized as follows.

Theorem 4.7.3. *For $1 \le p \le \infty$, Theorem 4.4.2 holds with $\omega_r(f;t)_p$ replaced by $\tilde{\omega}_r(f;t)_p$. For $1<p<\infty$,*

$$\tilde{\omega}_r(f;t)_p \sim K_r^*(f;t)_p, \qquad 1<p<\infty, \tag{4.7.4}$$

whereas equivalence fails for $p=1$ and $p=\infty$

We will not prove this theorem. See the section below for its history and references. From Eq. (4.7.4) and the equivalence of $K_r^*(f;t)_p$ and $\omega_r^*(f,t)_p$, it follows that $\tilde{\omega}_r(f;t)$ and $\omega^*(f;t)_p$ are equivalent for $1<p<\infty$, and Theorem 4.7.2 shows that they are equivalent to $\omega_r(f;t)$ for $1<p<\infty$ and $r=1,2$. On the other hand, we have the following result.

Proposition 4.7.4. *For $f\in L^p(\mathbb{S}^{d-1})$ if $1\le p<\infty$ and $f\in C(\mathbb{S}^{d-1})$ if $p=\infty$,*

$$\omega_r(f,t)_p \le \tilde{\omega}_r(f,t)_p, \qquad 1\le p\le\infty, \quad r\in\mathbb{N}.$$

Proof. For $x\in\mathbb{S}^{d-1}$, a quick computation shows that

$$\langle Q_{i,j,\theta}x,x\rangle = (x_i^2+x_j^2)\cos\theta + \sum_{k\ne i,j} x_k^2 = \cos\theta + \sum_{k\ne i,j} x_k^2(1-\cos\theta) \ge \cos\theta.$$

Consequently, since $\cos d(x,y) = \langle x,y\rangle$, we obtain

$$\mathtt{d}(Q_{i,j,\theta}x,x) = \arccos\langle Q_{i,j,\theta}x,x\rangle \le \theta,$$

which shows that $Q_{i,j,\theta}\in O_t$ for $0<\theta\le t$. This completes the proof. □

According to these comparisons, $\omega_r(f;t)_p$ is at least no worse than the other two moduli. The examples in Sect. 4.6 show that it is computable, and a moment's reflection indicates that it is the easiest in regard to computability among the three moduli.

4.8 Notes and Further Results

Trigonometric approximation is at the root of approximation theory, and it is a treasure trove that covers a wide range of topics. Here are a few books that we consulted with when working on Sect. 1: [54, 137, 167, 197].

Approximation theory on the sphere has a long history. Much of the early work considered the convergence of spherical harmonics expansions, often called the Laplace series, see [71, Chap. 12]. We are interested mainly in the quantitative part, where most of the work in the literature is centered on the modulus of smoothness $\omega_r^*(f;t)_p$, which can be traced back to [145], while its early study in approximation theory appeared in [14, 135]. The main result, Theorem 4.7.1, was finally formulated and proved by Rustamov in [148], after various stages of earlier results and studies by several authors; see [133, 174] for references.

For $r = 1$ and $p = 1$, the modulus of smoothness $\tilde{\omega}_r(f;t)_p$ was introduced and used in [29] and further studied in [102]. For other spaces, including $L^p(\mathbb{S}^{d-1})$, $p > 0$, these moduli were introduced and investigated by Ditzian in [56]. The direct and weak converse theorems for $L^p(\mathbb{S}^{d-1})$, $1 \le p \le \infty$, were given in [57, p. 23] and [56, p. 197], respectively. The equivalence (4.7.4) was established in [44, (9.1)] and that the equivalence fails for $p = 1$ and ∞ was shown in [59].

The modulus of smoothness $\omega_r(f;t)_p$ and its equivalent K-functional were introduced in [50]. Since this modulus of smoothness is closely related to the classical modulus of smoothness of one variable, we believe that it gives the most satisfactory solution of the characterization of best approximation on the sphere. The main results of this chapter were established in [50], except approximation in Sobolev space, which was established in [51]. The proof incorporated numerous ideas from various earlier results.

There are other moduli of smoothness on the sphere in the literature. Several of them and their comparison to $\omega_r^*(f;t)_p$ appeared in [147].

The operator $L_n f$ was used in [148], but the use of such an operator appeared already in [95]. The fast decay of the kernel makes it an ideal building block for constructive approximation.

The K-functional $K_n^*(f;t)_p$ in Eq. (4.7.2) is defined in terms of the Sobolev space

$$W_p^r := \left\{ f \in L^p(\mathbb{S}^{d-1}) : \|f\|_{W_p^r} := \|f\|_p + \|(-\Delta_0)^{r/2} f\|_p < \infty \right\}. \tag{4.8.1}$$

In comparison with the Sobolev space in Definition 4.5.2, we have by the proof of Theorem 4.7.2 that for $1 < p < \infty$,

$$W_p^r \subset \mathscr{W}_p^r \quad \text{and} \quad \|f\|_{\mathscr{W}_p^r} \le c \|f\|_{W_p^r}.$$

Furthermore, for $r = 1$ or 2, or $p = 2$ and $r = 3, 4, \ldots$,

$$W_p^r = \mathscr{W}_p^r \quad \text{and} \quad \|f\|_{\mathscr{W}_p^r} \sim \|f\|_{W_p^r}.$$

For $r \in \mathbb{N}$, $1 \le p \le \infty$, and $\alpha \in [0,1)$, we can define a Lipschitz space $\mathscr{W}_p^{r,\alpha}$ as the space of all functions $f \in \mathscr{W}_p^r$ with

$$\|f\|_{\mathscr{W}_p^{r,\alpha}} := \|f\|_p + \max_{1 \le i < j \le d} \sup_{0 < |\theta| \le 1} \frac{\|\Delta_{i,j,\theta}(D_{i,j}^r f)\|_p}{|\theta|^\alpha} < \infty.$$

For $\alpha \in (0,1)$, it can be shown [51] that $\mathscr{W}_p^{r,\alpha}$ is equivalent to $H_p^{r+\alpha}$ defined by

$$H_p^{r+\alpha} := \left\{ f \in L^p(\mathbb{S}^{d-1}) : \|f\|_{H_p^{r+\alpha}} := \|f\|_p + \sup_{0<t\le 1} \frac{\omega_{r+1}(f,t)_p}{t^{r+\alpha}} < \infty \right\}.$$

More precisely, the following theorem holds.

Theorem 4.8.1. *If $r \in \mathbb{N}$, $1 \le p \le \infty$, and $\alpha \in (0,1)$, then $\mathscr{W}_p^{r,\alpha} = H_p^{r+\alpha}$ and*

$$\|f\|_{\mathscr{W}_p^{r,\alpha}} \sim \|f\|_{H_p^{\alpha+r}} \sim \|f\|_p + \sup_{n\ge 1} n^{r+\alpha} E_n(f)_p.$$

As an immediate corollary, this shows that for $f \in \mathscr{W}_p^{r,\alpha}$, $1 \le p \le \infty$,

$$E_n(f)_p \le c n^{-r-\alpha} \|f\|_{\mathscr{W}_p^{r,\alpha}}.$$

Chapter 5
Weighted Polynomial Inequalities

Polynomial inequalities have been playing crucial roles in approximation theory and related fields. Several such inequalities on the unit sphere will be established in this chapter. Since some of them will be needed in weighted approximation theory and harmonic analysis in later chapters, we prove them in the weighted L^p norm. We will work in the context of doubling weights, defined and discussed in the first section. A fundamental tool in our approach to polynomial inequalities is a maximal function for spherical polynomials, introduced and studied in the second section, which can be controlled pointwise by the Hardy–Littlewood maximal function and which possess several other useful properties. In the third section, we establish the Marcinkiewics–Zygmund inequality, which, in its most useful form, states that the norm of a polynomial with respect to a finite discrete measure, in fact a sum with well-separated points, is bounded by its L^p norm on the sphere. This inequality will play an important role in the next chapter. In the fourth section, we establish the Bernstein and Nikolskii inequalities.

For readers who are interested only in unweighted inequalities on the sphere, the first section can be omitted, and all weight functions in subsequent sections can be taken to be the constant 1. The proof in the unweighted case, however, does not simplify much.

5.1 Doubling Weights on the Sphere

A weight function on $\mathbb{S}^{d-1}$ is a nonnegative integrable function. For a given weight function w and a subset $E \subset \mathbb{S}^{d-1}$, we define

$$w(E) := \int_E w(x)\,\mathrm{d}\sigma(x).$$

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Given a spherical cap $B := \mathsf{c}(x,r)$ and a constant $c > 0$, we denote by cB the spherical cap $\mathsf{c}(x,cr)$, which has the same center as B but c times the radius.

Definition 5.1.1. A weight function w on $\mathbb{S}^{d-1}$ is said to satisfy the doubling condition if there exists a constant $L > 0$ such that

$$w(2B) \le Lw(B), \qquad \forall B = \mathsf{c}(x,r), \quad x \in \mathbb{S}^{d-1},\ r \in (0,\pi]. \tag{5.1.1}$$

The least constant L for which Eq. (5.1.1) is satisfied is called the doubling constant of w and is denoted by L_w.

Iterating Eq. (5.1.1) shows that $w(2^m B) \le L_w^m w(B) = 2^{m\log_2 L_w} w(B)$. We will use the symbol s_w to denote a number in $[0, \log_2 L_w]$ such that

$$\sup_B \frac{w(2^m B)}{w(B)} \le C_{L_w} 2^{m s_w}, \qquad m = 1,2,\ldots, \tag{5.1.2}$$

where C_{L_w} is a constant depending only on L_w, and the supremum is taken over all spherical caps $B \subset \mathbb{S}^{d-1}$. Evidently, s_w satisfies

$$\overline{\lim_{m\to\infty}} \frac{1}{m} \log_2 \left(\sup_B \frac{w(2^m B)}{w(B)} \right) \le s_w \le \log_2 L_w. \tag{5.1.3}$$

If $w(x) \equiv 1$, then for each spherical cap $\mathsf{c}(x,\theta)$, we have

$$w(\mathsf{c}(x,\theta)) = \int_{\mathsf{c}(x,\theta)} \mathrm{d}\sigma = \omega_{d-2} \int_0^\theta (\sin\phi)^{d-2} \mathrm{d}\phi \sim \theta^{d-1},$$

from which it follows readily that $s_w = d-1$ for the constant weight function.

Lemma 5.1.2. *Let w be a doubling weight on $\mathbb{S}^{d-1}$.*

(*i*) *If $0 < r < t$ and $x \in \mathbb{S}^{d-1}$, then*

$$w(\mathsf{c}(x,t)) \le C_{L_w} \left(\frac{t}{r}\right)^{s_w} w(\mathsf{c}(x,r)). \tag{5.1.4}$$

(*ii*) *For $x,y \in \mathbb{S}^{d-1}$ and $n = 1,2,\ldots$,*

$$w\left(\mathsf{c}\left(x, \frac{1}{n}\right)\right) \le C_{L_w} (1 + n\mathsf{d}(x,y))^{s_w} w\left(\mathsf{c}\left(y, \frac{1}{n}\right)\right). \tag{5.1.5}$$

Proof. (i) Let us assume that $2^{m-1} \le t/r \le 2^m$. Then by Eq. (5.1.2),

$$w(\mathsf{c}(x,t)) \le w(\mathsf{c}(x,2^m r)) \le C_{L_w} 2^{(m-1)s_w} w(\mathsf{c}(x,r)) \le C_{L_w} \left(\frac{t}{r}\right)^{s_w} w(\mathsf{c}(x,r)).$$

(ii) For $x,y \in \mathbb{S}^{d-1}$, $\mathtt{c}(x,n^{-1}) \subset \mathtt{c}\left(y,n^{-1}+\mathtt{d}(x,y)\right)$ by the triangle inequality; hence by Eq. (5.1.4), we obtain

$$w\left(\mathtt{c}\left(y,\frac{1}{n}\right)\right) \le w\left(\mathtt{c}\left(y,\frac{1}{n}+\mathtt{d}(x,y)\right)\right) \le C_{L_w}(1+n\mathtt{d}(x,y))^{s_w} w\left(\mathtt{c}\left(y,\frac{1}{n}\right)\right).$$

Multiplying by n^{d-1} proves the stated result. □

Definition 5.1.3. A weight function w on $\mathbb{S}^{d-1}$ is called an A_∞ weight if there exists a constant $\beta \ge 1$ such that for every spherical cap $B \subset \mathbb{S}^{d-1}$ and every measurable subset E of B,

$$\frac{w(B)}{w(E)} \le c\left(\frac{\operatorname{meas} B}{\operatorname{meas} E}\right)^{\beta}. \tag{5.1.6}$$

The least constant β in Eq. (5.1.6) is called the A_∞ constant of w.

Directly from the definition, an A_∞ weight must be a doubling weight. Below, we give two important examples of A_∞-weights on $\mathbb{S}^{d-1}$ that we will encounter in Chap. 7.

Example 5.1.4. Let $\kappa = (\kappa_1,\ldots,\kappa_d) \in \mathbb{R}_+^d$ and $|\kappa| = \kappa_1+\cdots+\kappa_d$. The weight function

$$w_\kappa(x) = \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1,\ldots,x_d) \in \mathbb{S}^{d-1}, \tag{5.1.7}$$

satisfies the A_∞ condition. Furthermore, the least constant s_{w_κ} for which Eq. (5.1.2) is satisfied is given by

$$s_{w_\kappa} = d-1+|\kappa| - \min_{1\le j\le d} \kappa_j. \tag{5.1.8}$$

Moreover, if $x \in \mathbb{S}^{d-1}$ and $\theta \in (0,\pi)$, then

$$w_\kappa(\mathtt{c}(x,\theta)) = \int_{c(x,\theta)} w_\kappa(y)\mathrm{d}\sigma \sim \theta^{d-1}\prod_{j=1}^{d}(|x_j|+\theta)^{\kappa_j}. \tag{5.1.9}$$

Proof. Let E be a measurable subset of a spherical cap $B = \mathtt{c}(x,\theta) \subset \mathbb{S}^{d-1}$. Let $\gamma := \operatorname{meas} E/\operatorname{meas} B$ and $\beta = 1+|\kappa|-\min_{1\le j\le d}\kappa_j$. We claim that

$$c_1\gamma^{\beta}\theta^{d-1}\prod_{j=1}^{d}(|x_j|+\theta)^{\kappa_j} \le w_\kappa(E) \le c_2\gamma\theta^{d-1}\prod_{j=1}^{d}(|x_j|+\theta)^{\kappa_j}. \tag{5.1.10}$$

For the moment, we take Eq. (5.1.10) for granted and proceed with our proof. Evidently, Eq. (5.1.9) follows from Eq. (5.1.10) with $E = B$. By Eqs. (5.1.2) and (5.1.3), the proof of Eq. (5.1.8) will follow from the following two inequalities:

$$\tfrac{w_\kappa(2^m B)}{w_\kappa(B)} \le c 2^{m s_{w_\kappa}}, \quad \forall B = \mathtt{c}(x,\theta), \quad x \in \mathbb{S}^{d-1}, \tag{5.1.11}$$

$$\overline{\lim}_{m\to\infty} \tfrac{1}{m} \log_2 \Big(\sup_B \tfrac{w_\kappa(2^m B)}{w_\kappa(B)} \Big) \ge s_{w_\kappa}, \tag{5.1.12}$$

where s_{w_κ} denotes the right-hand side of Eq. (5.1.8). To prove Eq. (5.1.11), we may assume, without loss of generality, that $|x_1| = \max_{1\le j\le d} |x_j|$. By Eq. (5.1.9), we then obtain, for $0 < \theta \le 2^{-m}$,

$$\frac{w_\kappa(2^m B)}{w_\kappa(B)} \sim \frac{2^{m(d-1)} \prod_{j=2}^d (|x_j| + 2^m\theta)^{\kappa_j}}{\prod_{j=2}^d (|x_j|+\theta)^{\kappa_j}} \le 2^{m(d-1)} 2^{m(|\kappa|-\kappa_1)} \le 2^{m s_{w_\kappa}},$$

while for $2^{-m} < \theta \le \pi$, the radius of $2^m B$ is greater than or equal to 1, so that $w_\kappa(2^m B) \sim 1$ and

$$\frac{w_\kappa(2^m B)}{w_\kappa(B)} \sim \frac{1}{\theta^{d-1} \prod_{j=2}^d (|x_j|+\theta)^{\kappa_j}} \le \left(\frac{1}{\theta}\right)^{d-1+|\kappa|-\kappa_1} \le 2^{m s_{w_\kappa}}.$$

Together, these two inequalities give Eq. (5.1.11). To prove Eq. (5.1.12), we may assume, without loss of generality, that $\kappa_1 = \min_{1\le j\le d} \kappa_j$. Setting $x = e_1 := (1,0,\dots,0) \in \mathbb{S}^{d-1}$ and using Eq. (5.1.9), we deduce then

$$\begin{aligned}\overline{\lim_{m\to\infty}} \frac{1}{m} \log_2 \left(\sup_B \frac{w_\kappa(2^m B)}{w_\kappa(B)} \right) &\ge \overline{\lim_{m\to\infty}} \frac{1}{m} \log_2 \left(\sup_{0<\theta<2^{-m}} \frac{w_\kappa(\mathtt{c}(e_1, 2^m\theta))}{w_\kappa(\mathtt{c}(e_1,\theta))} \right) \\ &= \overline{\lim_{m\to\infty}} \frac{1}{m} \log_2 \left(\sup_{0<\theta<2^{-m}} \frac{2^{m(d-1)} \prod_{j=2}^d 2^{m\kappa_j} \theta^{\kappa_j}}{\prod_{j=2}^d \theta^{\kappa_j}} \right) = s_{w_\kappa},\end{aligned}$$

which establishes Eq. (5.1.12). Finally, by Eqs. (5.1.9) and (5.1.10), we obtain

$$c_1 \gamma^\beta = c_1 \left(\frac{\operatorname{meas} E}{\operatorname{meas} B} \right)^\beta \le \frac{w_\kappa(E)}{w_\kappa(B)} \le c_2 \frac{\operatorname{meas} E}{\operatorname{meas} B} = c_2 \gamma, \tag{5.1.13}$$

from which the A_∞ property of w_κ follows by the definition.

It remains to prove the claim (5.1.10). We may assume, without loss of generality, that $0 < \theta \le 1/(4\sqrt{d})$, since Eq. (5.1.10) holds trivially if $\theta \ge 1/(4\sqrt{d})$. Let $\varepsilon \in (0, \frac{1}{2})$ be a sufficiently small absolute constant to be determined later, and let $K_j := \{y \in \mathtt{c}(x,\theta) : \quad |y_j| \le \varepsilon\gamma\theta\}$ for $j = 1,2,\dots,d$. We first assert that

$$\operatorname{meas} K_j \le c_d\, \varepsilon \operatorname{meas} E, \qquad 1 \le j \le d. \tag{5.1.14}$$

It is enough to consider $j = 1$. Write $x = (x_1,\dots,x_d) = (\cos s, \xi \sin s)$ for some $s \in [0,\pi]$ and $\xi \in \mathbb{S}^{d-2}$. If $|x_1| = |\cos s| > 2\theta$, then $|y_1| \ge |x_1| - \mathtt{d}(x,y) \ge \theta$ for all $y \in \mathtt{c}(x,\theta)$, which implies that $K_1 = \emptyset$. Hence, we can assume $|x_1| = |\cos s| \le 2\theta$. Since

$0 < \theta < 1/(4\sqrt{d})$, we have $\sin s \ge \sqrt{1-1/(4d)} := \delta_d$. Let $y = (\cos t, \eta \sin t) \in \mathsf{c}(x,\theta)$ with $t \in [0,\pi]$ and $\eta \in \mathbb{S}^{d-2}$. Then by

$$\|\xi \sin s - \eta \sin t\|^2 = (\sin s - \sin t)^2 + \sin s \sin t \|\xi - \eta\|^2$$

and elementary trigonometric identities, we see that

$$|s-t| + (\sin s \sin t)^{1/2}\|\xi - \eta\| \sim \|x-y\| \sim \mathsf{d}(x,y) \le \theta.$$

The term $(\sin s \sin t)^{1/2}$ can be dropped, which is obvious if $\sin t \ge \delta_d/2$, since $\sin s \ge \delta_d$, whereas if $\sin\theta \le \delta_d/2$, then $|s-t| \ge \delta_d/2$. In particular, we conclude that $\mathsf{d}(\xi,\eta) \sim \|\xi - \eta\| \le c\theta$. Consequently,

$$\begin{aligned}
\operatorname{meas} K_1 &= \int_0^\pi (\sin t)^{d-2}\left(\int_{\mathbb{S}^{d-2}} \chi_{K_1}(\cos t, \eta \sin t)\, \mathrm{d}\sigma(\eta)\right) \mathrm{d}t \\
&\le \int_{\pi/2-\arcsin(\varepsilon\gamma\theta)}^{\pi/2+\arcsin(\varepsilon\gamma\theta)} \left(\int_{\{\eta\in\mathbb{S}^{d-2}:\ \mathsf{d}(\eta,\xi)\le c\theta\}} \mathrm{d}\sigma(\eta)\right) \mathrm{d}t \\
&\le c_d \arcsin(\varepsilon\gamma\theta)\theta^{d-2} \le c\varepsilon \operatorname{meas} E,
\end{aligned}$$

since $\gamma = \operatorname{meas} E/\operatorname{meas} B$ and $\operatorname{meas} B \sim \theta^{d-1}$. This proves Eq. (5.1.14).

Let $I_1 := \{j : 1 \le j \le d,\ |x_j| \ge 4\theta\}$ and $I_2 := \{j : 1 \le j \le d,\ |x_j| < 4\theta\}$. It follows from the triangle inequality that if $j \in I_1$, then $|x_j| + \theta \sim |x_j|$, and if $j \in I_2$, then $|x_j| + \theta \sim \theta$; furthermore, if $y \in \mathsf{c}(x,\theta)$ and $j \in I_1$, then $|x_j| \sim |y_j|$. Consequently, we deduce that

$$\begin{aligned}
w_\kappa(E) &= \int_E \prod_{j=1}^d |y_j|^{\kappa_j}\, \mathrm{d}\sigma(y) \sim \prod_{j\in I_1} |x_j|^{\kappa_j} \int_E \prod_{j\in I_2} |y_j|^{\kappa_j}\, \mathrm{d}\sigma(y) \\
&\ge \prod_{j\in I_1} |x_j|^{\kappa_j} \int_{E\setminus \cup_{j=1}^d K_j} \prod_{j\in I_2} (\varepsilon\gamma\theta)^{\kappa_j}\, \mathrm{d}\sigma(y).
\end{aligned} \tag{5.1.15}$$

Choosing $\varepsilon = \frac{1}{2dC_d}$ in Eq. (5.1.14) gives $\sum_{j=1}^d \operatorname{meas} K_j \le dc_d \varepsilon \operatorname{meas} E \le \frac{1}{2} \operatorname{meas} E$. Hence the above inequality leads to

$$w_\kappa(E) \ge c \operatorname{meas} E \gamma^{\beta-1} \prod_{j=1}^d (|x_j|+\theta)^{\kappa_j} \ge c\gamma^\beta \theta^{d-1} \prod_{j=1}^d (|x_j|+\theta)^{\kappa_j}$$

by the definition of γ. Moreover, using Eq. (5.1.15) directly, we have the following upper estimates:

$$w_\kappa(E) \le c \prod_{j\in I_1} |x_j|^{\kappa_j} \int_E \prod_{j\in I_2} \theta^{\kappa_j}\, \mathrm{d}\sigma(y) \sim \gamma\theta^{d-1} \prod_{j=1}^d (|x_j|+\theta)^{\kappa_j}.$$

This completes the proof of the claim (5.1.10). □

Example 5.1.5. Let $\kappa = (\kappa_1, \dots, \kappa_m) \in \mathbb{R}_+^m$, $v = (v_1, \dots, v_m)$ with $v_j \in \mathbb{S}^{d-1}$, $1 \le j \le m$. Then the weight function

$$w_{\kappa,v}(x) = \prod_{j=1}^{m} |\langle x, v_j \rangle|^{\kappa_j}$$

is an A_∞ weight on $\mathbb{S}^{d-1}$. Furthermore, if $x \in \mathbb{S}^{d-1}$ and $\theta \in (0, \pi)$, then

$$w_{\kappa,v}(\mathsf{c}(x,\theta)) \sim \theta^{d-1} \prod_{j=1}^{m} (|\langle x, v_j \rangle| + \theta)^{\kappa_j}. \tag{5.1.16}$$

Proof. Following the proof in the above example, we can show that if E is a measurable subset of a spherical cap $B \subset \mathbb{S}^{d-1}$, then

$$c_1 \left(\frac{\operatorname{meas} E}{\operatorname{meas} B} \right)^{1+|\kappa|} \le \frac{w_\alpha(E)}{w_\alpha(B)} \le c_2 \frac{\operatorname{meas} E}{\operatorname{meas} B}$$

for some positive constants c_1, c_2 depending only on d, m, and κ, from which it will follow that $w_{\kappa,v}$ is an A_∞ weight. Since only Eq. (5.1.16) will be needed in later chapters, we give a detailed proof of Eq. (5.1.16) below.

Without loss of generality, we may assume that $v_i \ne v_j$ if $i \ne j$. Set

$$\mathscr{A} = \{i : \ 1 \le i \le m, \ |\langle x, v_i \rangle| < 4\theta\}, \quad \mathscr{B} = \{i : \ 1 \le i \le m, \ |\langle x, v_i \rangle| \ge 4\theta\}.$$

Since $|\langle y, v_j \rangle - \langle x, v_j \rangle| \le \|x - y\| \le d(x,y)$, the triangle inequality implies that for $y \in \mathsf{c}(x, 2\theta)$, if $j \in \mathscr{A}$ then $|\langle y, v_j \rangle| \le 6\theta$ and if $j \in \mathscr{B}$, then $|\langle y, v_j \rangle| \sim |\langle x, v_j \rangle|$. Consequently,

$$w_{\kappa,v}(y) \sim \prod_{i \in \mathscr{A}} |\langle y, v_i \rangle|^{\kappa_i} \prod_{j \in \mathscr{B}} |\langle x, v_j \rangle|^{\kappa_j} \le \prod_{i \in \mathscr{A}} (6\theta)^{\kappa_i} \prod_{j \in \mathscr{B}} |\langle x, v_j \rangle|^{\kappa_j}, \tag{5.1.17}$$

which is the upper estimate of Eq. (5.1.16). To prove the lower estimate, let

$$E_j = \left\{ y \in \mathsf{c}\left(x, \frac{\theta}{4}\right) : \ \left| \mathsf{d}(y, v_j) - \frac{\pi}{2} \right| \le \varepsilon_{d,m} \theta \right\}, \quad 1 \le j \le m,$$

where $\varepsilon_{d,m}$ is a sufficiently small positive constant depending only on d and m. A straightforward calculation shows that

$$\sum_{j=1}^{m} \operatorname{meas}(E_j) \le C_d \varepsilon_{d,m} m \theta^{d-1} \le \frac{1}{2} \operatorname{meas}\left(\mathsf{c}\left(x, \frac{\theta}{4}\right) \right),$$

provided that $\varepsilon_{d,m}$ is small enough. Thus, there must exist a point $y_0 \in \mathsf{c}(x, \frac{\theta}{4})$ such that $y_0 \notin \cup_{j=1}^m E_j$. By the definition of E_j, we then have

$$|\langle y_0, v_j\rangle| = \left|\sin\left(\frac{\pi}{2} - d(y_0, v_j)\right)\right| = \sin\left|\frac{\pi}{2} - d(y_0, v_j)\right| \geq \sin(\varepsilon_{d,m}\theta)$$

for all j, $1 \leq j \leq m$. It then follows that $\mathsf{c}(y_0, \frac{\varepsilon_{d,m}\theta}{2}) \subset \mathsf{c}(x,\theta)$ and that for every $y \in \mathsf{c}(y_0, \varepsilon_{d,m}\theta/2)$ and $i \in \mathscr{A}$,

$$5\theta > |\langle y, v_i\rangle| \geq \sin(\varepsilon_{d,m}\theta) - \frac{1}{2}\varepsilon_{d,m}\theta \geq \left(\frac{2}{\pi} - \frac{1}{2}\right)\varepsilon_{d,m}\theta.$$

Consequently, by the equivalence in Eq. (5.1.17), we obtain

$$w_{\kappa,v}(y) \sim \prod_{i\in\mathscr{A}} \theta^{\kappa_i} \prod_{j\in\mathscr{B}} |\langle x, v_j\rangle|^{\kappa_j}, \quad \forall y \in \mathsf{c}(y_0, \varepsilon_{d,m}\theta/2).$$

Integrating over $\mathsf{c}(y_0, \varepsilon_{d,m}\theta/2) \subset \mathsf{c}(x,\theta)$ then gives the desired lower estimate of Eq. (5.1.16). This completes the proof. □

5.2 A Maximal Function for Spherical Polynomials

Let w be a doubling weight on $\mathbb{S}^{d-1}$ and let s_w denote the positive number defined in Eq. (5.1.2). Since $\mathrm{d}\mu = w(x)\mathrm{d}\sigma$ is clearly a doubling measure, the space $(\mathbb{S}^{d-1}, \mathrm{d}\mu)$ is a homogeneous space. Denote the Hardy–Littlewood maximal function associated with w by M_w,

$$M_w g(x) = \sup_{0<r\leq\pi} \frac{1}{w(\mathsf{c}(x,r))} \int_{\mathsf{c}(x,r)} |g(y)| w(y)\,\mathrm{d}\sigma(y).$$

According to Theorem 3.1.2, for every $f \in L^p(\mathbb{S}^{d-1}, w\mathrm{d}\sigma)$,

$$\|M_w g\|_{p,w} \leq c_p \|g\|_{p,w}, \quad 1 < p \leq \infty, \tag{5.2.1}$$

where $\|\cdot\|_{p,w}$ denotes the weighted L^p norm with respect to the measure $w\mathrm{d}\sigma$. The main goal of this section is to introduce another maximal function $f^*_{\beta,n}$ that will serve as our main tool in proving polynomial inequalities.

Definition 5.2.1. For $\beta > 0$, $f \in C(\mathbb{S}^{d-1})$, and $n \in \mathbb{N}$, define

$$f^*_{\beta,n}(x) := \max_{y\in\mathbb{S}^{d-1}} |f(y)|(1 + n\mathsf{d}(x,y))^{-\beta}, \qquad x \in \mathbb{S}^{d-1}. \tag{5.2.2}$$

Since $\mathtt{d}(x,x)=0$, it follows directly from the definition that $|f(x)| \le f^*_{\beta,n}(x)$. More importantly, however, $f^*_{\beta,n}$ is controlled pointwise by the maximal function M_w whenever f is a spherical polynomial of degree at most n.

Theorem 5.2.2. *For $f \in \Pi_n(\mathbb{S}^{d-1})$, $\beta > 0$, and $\gamma := s_w/\beta$,*

$$f^*_{\beta,n}(x) \le c_{\beta,L_w}\left(M_w\left(|f|^\gamma\right)(x)\right)^{1/\gamma}, \qquad x \in \mathbb{S}^{d-1}. \tag{5.2.3}$$

Proof. Let L_n be the kernel defined via a cutoff function as in Eq. (2.6.3), and for $\delta > 0$ and $y,u \in \mathbb{S}^{d-1}$, set

$$A_{n,\delta}(u,y) := \sup_{z \in \mathtt{c}(y,\frac{\delta}{n})} |L_n(\langle u,y\rangle) - L_n(\langle u,z\rangle)|. \tag{5.2.4}$$

Using the fact that $f = f * L_n$ for $f \in \Pi_n(\mathbb{S}^{d-1})$, we obtain, for $x,y \in \mathbb{S}^{d-1}$,

$$\begin{aligned}
\max_{z \in \mathtt{c}(y,\frac{\delta}{n})} \frac{|f(y)-f(z)|}{(1+n\mathtt{d}(x,y))^\beta} &\le f^*_{\beta,n}(x)\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}}\left(\frac{1+n\mathtt{d}(x,u)}{1+n\mathtt{d}(x,y)}\right)^\beta A_{n,\delta}(u,y)\,\mathrm{d}\sigma(u)\\
&\le f^*_{\beta,n}(x)\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}}(1+n\mathtt{d}(u,y))^\beta A_{n,\delta}(u,y)\,\mathrm{d}\sigma(u)\\
&\le c_\beta \delta f^*_{\beta,n}(x),
\end{aligned} \tag{5.2.5}$$

where the last integral is bounded by $c_\beta\delta$, since by Eq. (2.6.5), $A_{n,\delta}(u,y) \le cn^{d-1}$, and by Eq. (2.6.6), $A_{n\delta}(u,y) \le c\delta n^{d-1}(1+n\mathtt{d}(u,y))^{-\ell}$ if $4\delta/n \le \mathtt{d}(u,y) \le \pi$ for every positive integer $\ell > \beta + d$. This implies, in particular, that

$$|f(y)| \le \min_{z \in \mathtt{c}(y,\frac{\delta}{n})} |f(z)| + c_\beta \delta f^*_{\beta,n}(x)(1+n\mathtt{d}(x,y))^\beta, \quad x,y \in \mathbb{S}^{d-1}.$$

Since $\gamma = s_w/\beta$, we then obtain, for $x,y \in \mathbb{S}^{d-1}$ and $\delta = 1/(4c_\beta)$, that

$$\frac{|f(y)|^\gamma}{(1+n\mathtt{d}(x,y))^{s_w}} \le 2^\gamma(1+n\mathtt{d}(x,y))^{-s_w}\min_{z \in \mathtt{c}(y,\frac{\delta}{n})} |f(z)|^\gamma + \frac{1}{2^\gamma}\left(f^*_{\beta,n}(x)\right)^\gamma,$$

which implies, on taking the maximum over $x \in \mathbb{S}^{d-1}$, the inequality

$$\left(f^*_{\beta,n}(x)\right)^\gamma \le c(1+n\mathtt{d}(x,y))^{-s_w}\min_{z \in \mathtt{c}(y,\frac{\delta}{n})} |f(z)|^\gamma, \tag{5.2.6}$$

where $c = 2^\gamma/(1-2^{-\gamma})$. It remains to estimate $\min_{z \in \mathtt{c}(y,\frac{\delta}{n})} |f(z)|^\gamma$.

If $\mathtt{d}(x,y) \le \frac{\delta}{n}$, then $\mathtt{c}(y,\frac{\delta}{n}) \subset \mathtt{c}(x,\frac{2\delta}{n}) \subset \mathtt{c}(y,\frac{3\delta}{n})$. By Eqs. (5.1.1) and (5.1.5),

$$\begin{aligned}\min_{z\in \mathtt{c}(y,\frac{\delta}{n})} |f(z)|^\gamma &\le \frac{1}{w(\mathtt{c}(y,\frac{\delta}{n}))}\int_{\mathtt{c}(y,\frac{\delta}{n})} |f(z)|^\gamma w(z)\,\mathrm{d}\sigma(z)\\ &\le C_{L_w}L_w(1+n\mathtt{d}(x,y))^{s_w}\frac{1}{w(\mathtt{c}(x,\frac{2\delta}{n}))}\int_{\mathtt{c}(x,\frac{2\delta}{n})}|f(z)|^\gamma w(z)\,\mathrm{d}\sigma(z)\\ &\le C_{L_w}L_w(1+n\mathtt{d}(x,y))^{s_w}M_w(|f|^\gamma)(x).\end{aligned}$$

If $\frac{\delta}{n} \le \theta := \mathtt{d}(x,y) \le \pi$, then $\mathtt{c}(y,\frac{\delta}{n}) \subset \mathtt{c}(x,2\theta) \subset \mathtt{c}(y,3\theta)$, and by Eq. (5.1.4),

$$w\left(\mathtt{c}\left(y,\frac{\delta}{n}\right)\right) \ge \frac{1}{C_{L_w}}\left(\frac{3\theta n}{\delta}\right)^{-s_w} w(\mathtt{c}(y,3\theta)) \ge \frac{1}{C_{L_w}}\left(\frac{3\theta n}{\delta}\right)^{-s_w} w(\mathtt{c}(x,2\theta)),$$

from which we obtain that

$$\begin{aligned}\min_{z\in \mathtt{c}(y,\frac{\delta}{n})} |f(z)|^\gamma &\le C_{L_w}\left(\frac{3\theta n}{\delta}\right)^{s_w}\frac{1}{w(\mathtt{c}(x,2\theta))}\int_{\mathtt{c}(x,2\theta)}|f(z)|^\gamma w(z)\,\mathrm{d}\sigma(z)\\ &\le C_{L_w}\left(\frac{3}{\delta}\right)^{s_w}(1+n\mathtt{d}(x,y))^{s_w}M_w\left(|f|^\gamma\right)(x).\end{aligned}$$

Substituting these estimates into Eq. (5.2.6) completes the proof. □

In the discussion below, we can often consider $\|\cdot\|_{p,w}$ with $0<p<\infty$, even though $\|\cdot\|_{w,p}$ is no longer a norm when $0<p<1$. A consequence of Theorem 5.2.2 is the following useful corollary.

Corollary 5.2.3. *If $0<p\le\infty$, $f\in\Pi_n(\mathbb{S}^{d-1})$, and $\beta>\frac{s_w}{p}$, then*

$$\|f\|_{p,w} \le \|f^*_{\beta,n}\|_{p,w} \le C\|f\|_{p,w},$$

where C depends also on L_w and β when β is either large or close to $\frac{s_w}{p}$.

Proof. The first inequality follows from $f(x)\le f^*_{\beta,n}(x)$. The second follows from Eq. (5.2.3) and the boundedness of $\|(M(|f|^\gamma))^{1/\gamma}\|_p$, which requires that $p/\gamma>1$ or equivalently, $\beta>\frac{s_w}{p}$. □

Definition 5.2.4. For $f\in C(\mathbb{S}^{d-1})$ and $r>0$, we define

$$\operatorname{osc}(f)(x,r) := \sup_{y,z\in\mathtt{c}(x,r)} |f(y)-f(z)|, \qquad x\in\mathbb{S}^{d-1}. \tag{5.2.7}$$

Lemma 5.2.5. *If $f\in\Pi_n(\mathbb{S}^{d-1})$ and $\delta\in(0,1]$, then for every $\beta>0$,*

$$\operatorname{osc}(f)\left(x,\frac{\delta}{n}\right) \le c_\beta\,\delta f^*_{\beta,n}(x), \qquad x\in\mathbb{S}^{d-1},$$

where the constant c_β depends only on d and β when β is large.

Proof. From $f * L_n = f$ for $f \in \Pi_n(\mathbb{S}^{d-1})$ and Eq. (5.2.4), we have

$$\begin{aligned}\operatorname{osc}(f)\left(x, \frac{\delta}{n}\right) &\le \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} |f(u)| A_{n,\delta}(x,u)\, \mathrm{d}\sigma(u) \\ &\le f^*_{\beta,n}(x) \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} (1 + n\mathtt{d}(u,x))^{\beta} A_{n,\delta}(x,u)\, \mathrm{d}\sigma(u) \\ &\le c_\beta \delta f^*_{\beta,n}(x),\end{aligned}$$

where the last step follows from Eq. (5.2.5). □

5.3 The Marcinkiewicz–Zygmund Inequalities

In many applications, we need to deal with finite sums of function evaluations instead of integrals. The Marcinkiewicz–Zygmund inequality shows that these sums can often be bounded by integrals if the points on which the function evaluation takes place are well separated. We start with a definition that quantifies the separation of points.

Definition 5.3.1. Let $\varepsilon > 0$. A subset Λ of $\mathbb{S}^{d-1}$ is called ε-separated if $\mathtt{d}(\xi, \eta) \ge \varepsilon$ for every two distinct points $\xi, \eta \in \Lambda$. An ε-separated subset Λ of $\mathbb{S}^{d-1}$ is called maximal if $\mathbb{S}^{d-1} = \bigcup_{\eta\in\Lambda} \mathtt{c}(\eta, \varepsilon)$.

In the following, we denote by $\#\Lambda$ the cardinality of the set Λ.

Lemma 5.3.2. *(i) If $\Lambda \subset \mathbb{S}^{d-1}$ is ε-separated, then $\#\Lambda \le c_d \varepsilon^{-d+1}$; if in addition, Λ is maximal, then $c'_d \varepsilon^{-d+1} \le \#\Lambda \le c_d \varepsilon^{-d+1}$.*
(ii) If $\Lambda \subset \mathbb{S}^{d-1}$ is ε-separated and $\beta \ge 1$, then

$$\sum_{\eta\in\Lambda} \chi_{\mathtt{c}(\eta,\beta\varepsilon)}(x) \le c_d \beta^{d-1} \qquad \forall x \in \mathbb{S}^{d-1}; \tag{5.3.1}$$

if in addition, Λ is maximal, then the sum in Eq. (5.3.1) is greater than or equal to 1 for $x \in \mathbb{S}^{d-1}$.

Proof. If $\Lambda \subset \mathbb{S}^{d-1}$ is ε-separated, then the spherical caps $\mathtt{c}(\eta, \frac{\varepsilon}{2})$, $\eta \in \Lambda$, are mutually disjoint, whence

$$\sum_{\eta\in\Lambda} \operatorname{meas} \mathtt{c}\left(\eta, \frac{\varepsilon}{2}\right) = \operatorname{meas} \bigcup_{\eta\in\Lambda} \mathtt{c}\left(\eta, \frac{\varepsilon}{2}\right) \le \operatorname{meas}(\mathbb{S}^{d-1}),$$

which implies, since $\operatorname{meas} \mathtt{c}(\eta, \varepsilon) \sim \varepsilon^{d-1}$, that $\#\Lambda \le c_d \varepsilon^{-d+1}$. If, in addition, Λ is maximal, then $\operatorname{meas}(\mathbb{S}^{d-1}) \le \sum_{\eta\in\Lambda} \operatorname{meas} \mathtt{c}(\eta, \varepsilon)$, which gives the lower estimate $c'_d \varepsilon^{-d+1} \le \#\Lambda$. This proves (i). Assertion (ii) can be proved by a similar argument of volume comparison. Indeed, let $A_x := \{\eta \in \Lambda : x \in \mathsf{c}(\eta, \beta\varepsilon)\}$ for $x \in \mathbb{S}^{d-1}$; then $\bigcup_{\eta \in A_x} \mathsf{c}(\eta, \frac{1}{2}\varepsilon) \subset \mathsf{c}(x, (\beta + \frac{1}{2})\varepsilon)$ by the triangle inequality, and, hence, by the disjointness of $\mathsf{c}(\eta, \frac{\varepsilon}{2})$,

$$\sum_{\eta \in A_x} \operatorname{meas} \mathsf{c}\left(\eta, \frac{1}{2}\varepsilon\right) \le \operatorname{meas} \mathsf{c}\left(x, \left(\beta + \frac{1}{2}\right)\varepsilon\right) \le c_d \beta^{d-1} \varepsilon^{d-1},$$

from which Eq. (5.3.1) follows from $\# A_x = \sum_{\eta \in \Lambda} \chi_{\mathsf{c}(\eta, \beta\varepsilon)}(x)$. When Λ is maximal, then the fact that the sum in Eq. (5.3.1) is bounded below by 1 follows trivially. □

Remark 5.3.3. For a maximal ε-separated set, what we need is essentially

$$1 \le \sum_{\eta \in \Lambda} \chi_{\mathsf{c}(\eta, \varepsilon)}(x) \le c_d, \qquad \forall x \in \mathbb{S}^{d-1}. \tag{5.3.2}$$

We shall call a subset that satisfies Eq. (5.3.2) an extended maximal set.

Theorem 5.3.4. *Let $\varepsilon = \frac{\delta}{n}$ for $n \in \mathbb{N}_0$ and $\delta \in (0,1)$. If Λ is an ε-separated subset of $\mathbb{S}^{d-1}$, then for all $f \in \Pi_n(\mathbb{S}^{d-1})$ and $0 < p < \infty$,*

$$\left(\sum_{\eta \in \Lambda} \left| \operatorname{osc}(f)\left(\eta, \frac{\delta}{n}\right) \right|^p w\left(\mathsf{c}\left(\eta, \frac{\delta}{n}\right)\right) \right)^{\frac{1}{p}} \le A_p \delta \|f\|_{p,w}, \tag{5.3.3}$$

where A_p depends on p when p is close to 0, and on d and L_w.

Proof. By Lemma 5.2.5, we have, for $f \in \Pi_n(\mathbb{S}^{d-1})$ and $\eta \in \Lambda$,

$$\operatorname{osc}(f)\left(\eta, \frac{\delta}{n}\right) \le c_p \delta f^*_{2s_w/p,n}(\eta),$$

where c_p depends only on L_w and p when p is small. Since

$$f^*_{2s_w/p,n}(y) \sim f^*_{2s_w/p,n}(\eta), \quad \text{if } y \in \mathsf{c}\left(\eta, \frac{\delta}{n}\right),$$

it then follows, by Lemma 5.3.2 (ii), that

$$\begin{aligned}
\sum_{\eta \in \Lambda} \left| \operatorname{osc}(f)\left(\eta, \frac{\delta}{n}\right) \right|^p w\left(\mathsf{c}\left(\eta, \frac{\delta}{n}\right)\right) &\le (c_p\delta)^p \sum_{\eta \in \Lambda} \int_{\mathsf{c}(\eta, \frac{\delta}{n})} \left(f^*_{2s_w/p,n}(y)\right)^p w(y) \mathrm{d}\sigma(y) \\
&\le (c_p\delta)^p \int_{\mathbb{S}^{d-1}} \left(f^*_{2s_w/p,n}(y)\right)^p w(y) \mathrm{d}\sigma(y) \\
&\le c (c_p\delta)^p \int_{\mathbb{S}^{d-1}} |f(y)|^p w(y) \, \mathrm{d}\sigma(y),
\end{aligned}$$

where the last step follows from Corollary 5.2.3. □

Lemma 5.3.5. *If μ is a finite nonnegative measure on $\mathbb{S}^{d-1}$ satisfying*

$$\mu\left(\mathsf{c}\left(x,\frac{1}{n}\right)\right) \le K w\left(\mathsf{c}\left(x,\frac{1}{n}\right)\right), \quad x \in \mathbb{S}^{d-1}, \tag{5.3.4}$$

for some positive integer n, then for all $0<p<\infty$ and $f \in \Pi_m(\mathbb{S}^{d-1})$, $m \ge n$,

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p \, \mathrm{d}\mu(x) \le cK \left(\frac{m}{n}\right)^{s_w} \|f\|_{p,w}^p,$$

where c depends only on L_w and p when p is close to 0.

Proof. Let Λ be a maximal $\frac{1}{m}$-separated subset of $\mathbb{S}^{d-1}$ and set $\beta = s_w/p+1$. For $f \in \Pi_m(\mathbb{S}^{d-1})$, using $f(x) \le c f_{\beta,m}^*(x) \sim f_{\beta,m}^*(\eta)$ for $\eta \in \mathsf{c}(x,\frac{1}{m})$, we have

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} |f(x)|^p \mathrm{d}\mu(x) &\le c \sum_{\eta\in\Lambda} \int_{\mathsf{c}(\eta,\frac{1}{m})} |f(x)|^p \mathrm{d}\mu(x)\\ &\le c \sum_{\eta\in\Lambda} \left(f_{\beta,m}^*(\eta)\right)^p \int_{\mathsf{c}(\eta,\frac{1}{n})} \mathrm{d}\mu(x).\end{aligned}$$

Applying Eqs. (5.3.4) and (5.1.4), we then obtain

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} |f(x)|^p \mathrm{d}\mu(x) &\le cK \left(\frac{m}{n}\right)^{s_w} \sum_{\eta\in\Lambda} (f_{\beta,m}^*(\eta))^p w\big(\mathsf{c}(\eta,m^{-1})\big)\\ &\le cK \left(\frac{m}{n}\right)^{s_w} \int_{\mathbb{S}^{d-1}} (f_{\beta,m}^*(y))^p w(y) \mathrm{d}\sigma(y) \le cK \left(\frac{m}{n}\right)^{s_w} \|f\|_{p,w}^p,\end{aligned}$$

where the last step follows from Corollary 5.2.3. This completes the proof. □

We are now ready to prove the Marcinkiewicz–Zygmund inequality for spherical polynomials.

Theorem 5.3.6. *Let Λ be a $\frac{\delta}{n}$-separated subset of $\mathbb{S}^{d-1}$ and $\delta \in (0,1]$.*

(i) *For all $0<p<\infty$ and $f \in \Pi_m(\mathbb{S}^{d-1})$ with $m \ge n$,*

$$\sum_{\eta\in\Lambda} \left(\max_{x\in\mathsf{c}(\eta,\frac{\delta}{n})} |f(x)|^p\right) w\left(\mathsf{c}\left(\eta,\frac{\delta}{n}\right)\right) \le c_{w,p} \left(\frac{m}{n}\right)^{s_w} \|f\|_{p,w}^p, \tag{5.3.5}$$

where $c_{w,p}$ depends on p when p is close to 0, and on L_w.

(ii) *If, in addition, Λ is maximal and $\delta \in (0,1/(4A_r))$ with A_r the constant in Eq. (5.3.3) for some $r \in (0,1)$, then for $f \in \Pi_n(\mathbb{S}^{d-1})$, $\|f\|_\infty \sim \max_{\eta\in\Lambda} |f(\eta)|$, and for $r \le p < \infty$,*

$$\|f\|_{p,w} \sim \left(\sum_{\eta\in\Lambda} \big(\min_{x\in \mathsf{c}\left(\eta,\frac{\delta}{n}\right)} |f(x)|^p \big) w\left(\mathsf{c}\left(\eta, \frac{\delta}{n}\right)\right)\right)^{1/p} \tag{5.3.6}$$

$$\sim \left(\sum_{\eta\in\Lambda} \big(\max_{x\in \mathsf{c}\left(\eta,\frac{\delta}{n}\right)} |f(x)|^p \big) w\left(\mathsf{c}\left(\eta, \frac{\delta}{n}\right)\right)\right)^{1/p}, \tag{5.3.7}$$

where the constants of equivalence depend only on r when r is close to 0*, and on* L_w.

Proof. (i) For convenience, we let n_1 be an integer such that $\frac{n}{2\delta} < n_1 \le \frac{n}{\delta}$. For every $\eta \in \Lambda$, choose $\xi_\eta \in \mathsf{c}(\eta, \frac{\delta}{n})$ such that $f(\xi_\eta) = \max_{x\in\mathsf{c}(\eta,\frac{\delta}{n})} |f(x)|^p$. Let μ be a nonnegative measure supported in the set $\{\xi_\eta : \eta \in \Lambda\}$ defined by $\mu(\xi_\eta) = w(\mathsf{c}(\eta, \frac{\delta}{n}))$ for $\eta \in \Lambda$. Then

$$\sum_{\eta\in\Lambda} \left(\max_{x\in\mathsf{c}(\eta,\frac{\delta}{n})} |f(x)|^p \right) w\left(\mathsf{c}(\eta, \frac{\delta}{n})\right) = \int_{\mathbb{S}^{d-1}} |f(x)|^p \, \mathrm{d}\mu(x). \tag{5.3.8}$$

For every $x \in \mathbb{S}^{d-1}$, the doubling property of w shows that

$$\begin{aligned} \mu\left(\mathsf{c}\left(x, \frac{1}{n_1}\right)\right) &\le \sum_{\xi_\eta\in\mathsf{c}(x,\frac{1}{n_1}),\eta\in\Lambda} w\left(\mathsf{c}\left(\eta, \frac{1}{n_1}\right)\right) \\ &\le L_w^2 \sum_{\eta\in\Lambda\cap\mathsf{c}(x,\frac{2}{n_1})} w\left(\mathsf{c}\left(\eta, \frac{1}{4n_1}\right)\right) \\ &\le L_w^2 \int_{\mathsf{c}(x,\frac{3}{n_1})} w(y)\,\mathrm{d}\sigma(y) \le L_w^4 w\left(\mathsf{c}\left(x, \frac{1}{n_1}\right)\right), \end{aligned}$$

which allows us to use Lemma 5.3.5 to conclude that for $f \in \Pi_m(\mathbb{S}^{d-1})$,

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p \,\mathrm{d}\mu(x) \le cL_w^4 \left(\frac{m}{n_1}\right)^{s_w} \|f\|_{p,w}^p \le cL_w^4 \left(\frac{m}{n}\right)^{s_w} \|f\|_{p,w}^p.$$

Together with Eq. (5.3.8), this completes the proof of (i).

(ii) Let $\delta_r := (4A_r)^{-1}$. We claim that if $r \le p < \infty$, $\delta \in (0, \delta_r)$ and Λ is a maximal $\frac{\delta}{n}$-separated subset, then for $f \in \Pi_n(\mathbb{S}^{d-1})$,

$$\|f\|_{p,w} \le c_* \left(\sum_{\eta\in\Lambda} \left(\min_{x\in\mathsf{c}(\eta,\frac{\delta}{n})} |f(x)|^p \right) w\left(\mathsf{c}\left(\eta, \frac{\delta}{n}\right)\right)\right)^{1/p}, \tag{5.3.9}$$

where c_* depends only on r and L_w. Assume this inequality. Then together with Eq. (5.3.5), we have Eqs. (5.3.6) and (5.3.7) immediately. Furthermore, since the constant c in Eq. (5.3.9) is independent of p, the equivalence for the case of $p = \infty$ can be deduced from Eq. (5.3.9) as well.

It remains to prove Eq. (5.3.9). We observe that

$$\begin{aligned}\|f\|_{p,w}^p &\le \sum_{\eta\in\Lambda} \int_{\mathsf{c}(\eta,\frac{\delta}{n})} |f(x)|^p w(x)\,\mathrm{d}\sigma(x) \\ &\le 2^p \sum_{\eta\in\Lambda} \left|\mathrm{osc}(f)\left(\eta,\frac{\delta}{n}\right)\right|^p w\left(\mathsf{c}\left(\eta,\frac{\delta}{n}\right)\right) \\ &\quad + 2^p \sum_{\eta\in\Lambda} \left(\min_{y\in\mathsf{c}(\eta,\frac{\delta}{n})} |f(y)|^p\right) w\left(\mathsf{c}\left(\eta,\frac{\delta}{n}\right)\right).\end{aligned}$$

Using Theorem 5.3.4, we then obtain that for $r < p < \infty$,

$$\|f\|_{p,w}^p \le (2A_r\delta)^p \|f\|_{p,w}^p + 2^p \sum_{\eta\in\Lambda} \left(\min_{y\in\mathsf{c}(\eta,\frac{\delta}{n})} |f(y)|^p\right) w\left(\mathsf{c}\left(\eta,\frac{\delta}{n}\right)\right).$$

Since $\delta \le \delta_r = 1/(4A_r)$, this proves inequality (5.3.9) and thus (ii). □

5.4 Further Inequalities Between Sums and Integrals

In this section we prove several other inequalities that will be useful in the next chapter, in which we study cubature rules on the sphere.

For $n = 1, 2, \ldots$, it is often convenient to work with an approximation w_n of the weight function w, which is defined by

$$w_n(x) := n^{d-1} \int_{\mathsf{c}(x,\frac{1}{n})} w(y)\,\mathrm{d}\sigma(y) = n^{d-1} w\left(\mathsf{c}\left(x,\frac{1}{n}\right)\right), \qquad (5.4.1)$$

and for convenience, we also let $w_0(x) := w_1(x)$.

Theorem 5.4.1. *For $f \in \Pi_n(\mathbb{S}^{d-1})$ and $0 < p < \infty$,*

$$c^{-1}\|f\|_{p,w_n} \le \|f\|_{p,w} \le c\|f\|_{p,w_n},$$

where c depends only on d, L_w, and on p when p is small.

Proof. Each w_n is again a doubling weight, and $L_{w_n} \sim L_w$ with equivalence constants independent of n. According to Corollary 5.2.3, it suffices to prove that

$$\left\|f^*_{2s/p,n}\right\|_{p,w} \sim \left\|f^*_{2s/p,n}\right\|_{p,w_n} \quad \text{with} \quad s := \max\{s_w, s_{w_n}\}. \tag{5.4.2}$$

To this end, let $\Lambda \subset \mathbb{S}^{d-1}$ be a maximal $\frac{1}{n}$-separated subset and observe that for $x \in \mathsf{c}(\eta, \frac{1}{n})$, $f^*_{2s/p,n}(x) \sim f^*_{2s/p,n}(\eta)$ and $w_n(x) \sim w_n(\eta)$. It follows by Lemma 5.3.2 that

$$\begin{aligned}\left\|f^*_{2s/p,n}\right\|^p_{p,w} &\sim \sum_{\eta\in\Lambda} \int_{\mathsf{c}(\eta,\frac{1}{n})} \left(f^*_{2s/p,n}(x)\right)^p w(x)\,\mathrm{d}\sigma(x) \\ &\sim \sum_{\eta\in\Lambda} \int_{\mathsf{c}(\eta,\frac{1}{n})} \left(f^*_{2s/p,n}(x)\right)^p w_n(x)\,\mathrm{d}\sigma(x) \sim \left\|f^*_{2s/p,n}\right\|^p_{p,w_n},\end{aligned}$$

which proves Eq. (5.4.2) and hence the theorem. □

Next we give a partial converse of the Marcinkiewicz–Zygmund inequality.

Lemma 5.4.2. *If Λ is a maximal $\frac{\delta}{n}$-separated subset of $\mathbb{S}^{d-1}$ for some $\delta \in (0,1]$ and $f \in \Pi_n(\mathbb{S}^{d-1})$ is nonnegative on Λ, i.e., $f(\eta) \ge 0$ for all $\eta \in \Lambda$, then*

$$\int_{\mathbb{S}^{d-1}} f(x)w(x)\,\mathrm{d}\sigma(x) \ge c \sum_{\eta\in\Lambda} f(\eta) w\left(\mathsf{c}\left(\eta, \frac{\delta}{n}\right)\right),$$

where the constant c is equal to $c_d^{-1} - c_* A_1 \delta$, with c_d the constant in Eq. (5.3.1), c_* the constant in Eq. (5.3.9), and A_1 being the constant in Eq. (5.3.3) with $p = 1$.

Proof. Let $N(x) := \sum_{\eta\in\Lambda} \chi_{\mathsf{c}(\eta,\frac{\delta}{n})}(x)$. By Lemma 5.3.2 (ii),

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} f(x)w(x)\mathrm{d}\sigma(x) &= \sum_{\eta\in\Lambda} \int_{\mathsf{c}(\eta,\frac{\delta}{n})} f(x)\frac{w(x)}{N(x)}\mathrm{d}\sigma(x) \\ \ge c_d^{-1} \sum_{\eta\in\Lambda} f(\eta) \int_{\mathsf{c}(\eta,\frac{\delta}{n})} w(x)\mathrm{d}\sigma(x) &- \sum_{\eta\in\Lambda} \int_{\mathsf{c}(\eta,\frac{\delta}{n})} |f(x) - f(\eta)| w(x)\mathrm{d}\sigma(x).\end{aligned}$$

Applying Eq. (5.3.3) with $p = 1$ followed by Eq. (5.3.9), we then obtain

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} f(x)w(x)\,\mathrm{d}\sigma(x) &\ge c_d^{-1} \sum_{\eta\in\Lambda} f(\eta) w\left(\mathsf{c}\left(\eta, \frac{\delta}{n}\right)\right) - A_1\delta\|f\|_{1,w} \\ &\ge (c_d^{-1} - c_* A_1 \delta) \sum_{\eta\in\Lambda} f(\eta) w\left(\mathsf{c}\left(\eta, \frac{\delta}{n}\right)\right),\end{aligned}$$

which proves the desired inequality. □

Our next lemma is a partial converse of Lemma 5.3.5.

Lemma 5.4.3. *Let μ be a nonnegative measure on $\mathbb{S}^{d-1}$ such that*

$$\int_{\mathbb{S}^{d-1}} f(x)w(x)\mathrm{d}\sigma(x) = \int_{\mathbb{S}^{d-1}} f(x)\mathrm{d}\mu(x), \quad \forall f \in \Pi_n(\mathbb{S}^{d-1}), \tag{5.4.3}$$

for some positive integer n. Then for all $x \in \mathbb{S}^{d-1}$,

$$\mu\left(\mathsf{c}\left(x, \frac{2}{n}\right)\right) \le cw\left(\mathsf{c}\left(x, \frac{2}{n}\right)\right), \tag{5.4.4}$$

where the constant c depends only on L_w and d. Moreover,

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p \,\mathrm{d}\mu(x) \le c \int_{\mathbb{S}^{d-1}} |f(x)|^p w(x)\mathrm{d}\sigma(x), \quad \forall f \in \Pi_{2n}(\mathbb{S}^{d-1}). \tag{5.4.5}$$

Proof. Let $m := \lfloor \frac{d-1}{2} + \frac{s_w}{2} \rfloor + 1$ and $n_1 := \lfloor \frac{n}{2m} \rfloor$. Define

$$S_n(\cos\theta) := \gamma_n \left(\frac{\sin(n_1 + \frac{1}{2})\theta}{\sin\frac{\theta}{2}}\right)^{2m}, \tag{5.4.6}$$

where γ_n is chosen such that $\frac{\omega_{d-1}}{\omega_d} \int_0^\pi S_n(\cos\theta)(\sin\theta)^{d-2}\mathrm{d}\theta = 1$. Then S_n is an even nonnegative trigonometric polynomial of degree at most n. From $\sin\frac{\theta}{2} \sim \theta$ for $\theta \in [0, \pi]$, a change of variable $t \mapsto (n_1 + \frac{1}{2})\theta$ in the integral then shows that $\gamma_n \sim n^{d-1-2m}$, which in turn implies, considering $n_1\theta \le 1$ and $n_1\theta \ge 1$ separately if necessary, that

$$0 \le S_n(\cos\theta) \le cn^{d-1}(1 + n\theta)^{-2m}, \qquad \theta \in [0, \pi], \tag{5.4.7}$$

and furthermore,

$$S_n(\cos\theta) \ge cn^{d-1}, \quad \theta \in \left[0, \frac{2}{n}\right]. \tag{5.4.8}$$

Since S_n is even, $S_n(\cos\theta)$ is an algebraic polynomial in $\cos\theta$ of degree at most n. It follows that $S_n(\langle x, \cdot\rangle)$ is a spherical polynomial of degree at most n for each fixed $x \in \mathbb{S}^{d-1}$. Thus by Eq. (5.4.3), for each $x \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} S_n(\langle x, y\rangle)\mathrm{d}\mu(y) = \int_{\mathbb{S}^{d-1}} S_n(\langle x, y\rangle)w(y)\mathrm{d}\sigma(y). \tag{5.4.9}$$

On the left-hand side of Eq. (5.4.9), by Eq. (5.4.8) and the positivity of S_n,

$$\int_{\mathbb{S}^{d-1}} S_n(\langle x, y\rangle)\,\mathrm{d}\mu(y) \ge cn^{d-1}\mu\left(\mathsf{c}\left(x, \frac{2}{n}\right)\right),$$

while on the right-hand side, using Theorem 5.4.1, Eqs. (5.4.4), and (5.1.5), we have

$$\int_{\mathbb{S}^{d-1}} S_n(\langle x,y\rangle)w(y)\mathrm{d}\sigma(y) \le c\int_{\mathbb{S}^{d-1}} S_n(\langle x,y\rangle)w_n(y)\mathrm{d}\sigma(y)$$
$$\le cn^{d-1}w_n(x)\int_{\mathbb{S}^{d-1}}(1+n\mathsf{d}(x,y))^{s_w-2m}\,\mathrm{d}\sigma(y) \le cw_n(x).$$

Substituting the last two inequalities into Eq. (5.4.9) yields the estimate (5.4.4) on using Eq. (5.4.1) and the doubling property of the weight w. Finally, by Eq. (5.4.4), the inequality (5.4.5) is an immediate consequence of Lemma 5.3.5. □

Lemma 5.4.4. *Let $f: \mathbb{S}^{d-1} \to [0,\infty)$ be a nonnegative function satisfying*

$$f(y) \le c_f(1+n\mathsf{d}(x,y))^{\alpha} f(x), \quad \forall x,y \in \mathbb{S}^{d-1}, \tag{5.4.10}$$

for a positive integer n and a nonnegative number α. Then for each p, $0<p<\infty$, there exists a nonnegative spherical polynomial $g \in \Pi_n(\mathbb{S}^{d-1})$ such that

$$c^{-1}f(x)^{\frac{1}{p}} \le g(x) \le cf(x)^{\frac{1}{p}}, \qquad \forall x \in \mathbb{S}^{d-1}, \tag{5.4.11}$$

where the constant c depends only on c_f, α, and p when p is close to zero. Furthermore, if $f(x) = F(\langle x,e\rangle)$ for a fixed $e \in \mathbb{S}^{d-1}$, then g in Eq. (5.4.11) can be chosen as a zonal polynomial $g(x) = G(\langle x,e\rangle)$.

Proof. We define a function $S_n(\cos\theta)$ as in Eq. (5.4.6) but with $m = \lfloor \alpha/p\rfloor + d + 1$, and $n_1 = \lfloor \frac{n}{2m}\rfloor$. Then $S_n(\cos\theta)$ is a polynomial in $\cos\theta$ of degree at most n that satisfies Eqs. (5.4.7) and (5.4.8). We then define

$$g(x) = \int_{\mathbb{S}^{d-1}} f(y)^{\frac{1}{p}} S_n(\langle x,y\rangle)\,\mathrm{d}\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{5.4.12}$$

and show that g has the desired properties. Since $S_n(\langle x,y\rangle)$ is a polynomial of degree at most n in x, g is a spherical polynomial of degree at most n. Furthermore, a moment's reflection shows, where we expand f in zonal spherical harmonics if necessary, that if f is a zonal function, then so is g. Since both f and S_n are nonnegative, g is nonnegative. Now, by Eqs. (5.4.4) and (5.4.10),

$$g(x) \le c_f^{\frac{1}{p}} f(x)^{\frac{1}{p}} \int_{\mathbb{S}^{d-1}} (1+n\mathsf{d}(x,y))^{\frac{\alpha}{p}} S_n(\langle x,y\rangle)\mathrm{d}\sigma(y) \le cf(x)^{\frac{1}{p}},$$

whereas by Eq. (5.4.8), we deduce

$$g(x) \ge \int_{\mathsf{c}(x,\frac{1}{2n})} f(y)^{\frac{1}{p}} S_n(\langle x,y\rangle)\mathrm{d}\sigma(y) \ge cf(x)^{\frac{1}{p}} \int_0^{\frac{1}{2n}} n^{d-1}\theta^{d-2}\,\mathrm{d}\theta \ge cf(x)^{\frac{1}{p}}.$$

Hence g satisfies Eq. (5.4.11), and the proof is complete. □

Our next result is a refinement of the Marcinkiewicz–Zygmund inequality.

Theorem 5.4.5. *Let μ be a nonnegative finite measure on $\mathbb{S}^{d-1}$ for which*

$$\int_{\mathbb{S}^{d-1}} f(x)\,w(x)\mathrm{d}\sigma(x) = \int_{\mathbb{S}^{d-1}} f(x)\mathrm{d}\mu(x), \quad \forall f \in \Pi_{3n}(\mathbb{S}^{d-1}), \tag{5.4.13}$$

holds for a positive integer n. Then for all p, $0 < p < \infty$, and $f \in \Pi_n(\mathbb{S}^{d-1})$,

$$\|f\|_{p,w} \sim \left(\int_{\mathbb{S}^{d-1}} |f(x)|^p \,\mathrm{d}\mu(x)\right)^{\frac{1}{p}} =: \|f\|_{p,\mathrm{d}\mu}, \tag{5.4.14}$$

where the constants of equivalence depend only on L_w and p when p is close to 0.

Proof. Because of Eq. (5.4.5), we need to prove only the inequality

$$\|f\|_{p,w}^p = \int_{\mathbb{S}^{d-1}} |f(x)|^p w(x)\mathrm{d}\sigma(x) \le c\int_{\mathbb{S}^{d-1}} |f(x)|^p \mathrm{d}\mu(x) = c\|f\|_{p,\mathrm{d}\mu}^p. \tag{5.4.15}$$

Let L_n be the kernel defined by the cutoff function as defined in Eq. (2.6.3). Since $f = f * L_n$ for $f \in \Pi_n(\mathbb{S}^{d-1})$ and the integral of $|L_n(\langle x,y\rangle)|$ over the sphere is uniformly bounded, by Hölder's inequality we obtain that for $x \in \mathbb{S}^{d-1}$,

$$|f(x)| \le c\left(\int_{\mathbb{S}^{d-1}} |f(y)|^2 |L_n(\langle x,y\rangle)|\,\mathrm{d}\sigma(y)\right)^{\frac{1}{2}},$$

which implies, by Eqs. (5.1.5), (5.4.1) and the estimate of L_n in Theorem 2.6.5, that

$$|f(x)|^p w_n(x) \le c\left(n^{d-1}\int_{\mathbb{S}^{d-1}} [f(y)]^2 \frac{(w_n(y))^{\frac{2}{p}}}{(1+n\mathrm{d}(x,y))^{\ell-\frac{2s_w}{p}}}\mathrm{d}\sigma(y)\right)^{\frac{p}{2}}.$$

The reason for passing to $|f(y)|^2$ lies in Eq. (5.4.13), for which we need to get rid of the absolute value of f. By Lemma 5.4.4 and Eq. (5.1.5), there exist a nonnegative spherical polynomial $Q_1 \in \Pi_{[n/2]}^d$ and a nonnegative zonal spherical polynomial $Q_2(\langle x,y\rangle) \in \Pi_{[n/2]}^d$ such that

$$Q_1(y) \sim (w_n(y))^{\frac{2}{p}-1} \quad \text{and} \quad Q_2(\langle x,y\rangle) \sim n^{d-1}(1+n\mathrm{d}(y,x))^{-\ell+\frac{2s_w}{p}}, \tag{5.4.16}$$

where ℓ is a fixed integer such that $\ell > (d-1)\max\{\frac{2}{p},1\}$. Hence,

$$\begin{aligned} |f(x)|^p w_n(x) &\le c\left(\int_{\mathbb{S}^{d-1}} [f(y)]^2 Q_2(\langle x,y\rangle)Q_1(y)w_n(y)\mathrm{d}\sigma(y)\right)^{\frac{p}{2}} \\ &\le c\left(\int_{\mathbb{S}^{d-1}} [f(y)]^2 Q_2(\langle x,y\rangle)Q_1(y)w(y)\mathrm{d}\sigma(y)\right)^{\frac{p}{2}} \\ &= c\left(\int_{\mathbb{S}^{d-1}} [f(y)]^2 Q_2(\langle x,y\rangle)Q_1(y)\,\mathrm{d}\mu(y)\right)^{\frac{p}{2}}, \end{aligned} \tag{5.4.17}$$

where the second step uses Theorem 5.4.1 and the last step uses Eq. (5.4.13).

We now prove Eq. (5.4.15) for the case $0 < p \le 2$. Let Λ be a maximal $\frac{1}{n}$-separated subset of $\mathbb{S}^{d-1}$. Since $|f(y)| \le f^*_{2/p,n}(y) \sim f^*_{2/p,n}(\eta)$ for $\eta \in \mathsf{c}(y,\frac{1}{n})$, and by Eq. (5.4.16), $Q_2(\langle x,y\rangle) \sim Q_2(\langle x,\eta\rangle)$ for $\eta \in \mathsf{c}(y,\frac{1}{n})$, we deduce from Eq. (5.4.17) that

$$\begin{aligned}|f(x)|^p w_n(x) &\le c \sum_{\eta\in\Lambda} \left| \int_{\mathsf{c}(\eta,\frac{1}{n})} |f(y)|^2 Q_2(\langle x,y\rangle) Q_1(y) \mathrm{d}\mu(y) \right|^{\frac{p}{2}} \\ &\le c \sum_{\eta\in\Lambda} \left[\left(f^*_{2/p,n}(\eta)\right)^{2-p} Q_2(\langle x,\eta\rangle)\,(w_n(\eta))^{\frac{2}{p}-1} \int_{\mathsf{c}(\omega,\frac{1}{n})} |f(y)|^p \,\mathrm{d}\mu \right]^{\frac{p}{2}}.\end{aligned}$$

By Eq. (5.4.16), $\int_{\mathbb{S}^{d-1}} Q_2(\langle x,y\rangle)^{p/2} \mathrm{d}\sigma(x) \le c n^{(d-1)(\frac{p}{2}-1)}$ for all $y \in \mathbb{S}^{d-1}$. Hence integrating over $x \in \mathbb{S}^{d-1}$ gives

$$\|f\|^p_{p,w_n} \le c n^{(d-1)(\frac{p}{2}-1)} \sum_{\eta\in\Lambda} \left[\left(f^*_{2/p,n}(\eta)\right)^{2-p} (w_n(\eta))^{\frac{2}{p}-1} \left\| f\chi_{\mathsf{c}(\eta,\frac{1}{n})} \right\|^p_{p,\mathrm{d}\mu} \right]^{\frac{p}{2}}.$$

Applying Hölder's inequality to the sum and using Eq. (5.3.1), we obtain

$$\begin{aligned}\|f\|^p_{p,w_n} &\le c \left(\|f\|^p_{p,\mathrm{d}\mu}\right)^{\frac{p}{2}} \left(\frac{1}{n^{d-1}} \sum_{\eta\in\Lambda} |f^*_{2/p,n}(\eta)|^p w_n(\eta) \right)^{1-\frac{p}{2}} \\ &\le c \left(\|f\|^p_{p,\mathrm{d}\mu}\right)^{\frac{p}{2}} \left(\sum_{\eta\in\Lambda} \int_{\mathsf{c}(\eta,\frac{1}{n})} \left(f^*_{2/p,n}(y)\right)^p w_n(y) \mathrm{d}\sigma(y) \right)^{1-\frac{p}{2}} \\ &\le c \left(\|f\|^p_{p,\mathrm{d}\mu}\right)^{\frac{p}{2}} \|f^*_{2/p,n}\|^{p(1-\frac{p}{2})}_{p,w_n} \le c \left(\|f\|^p_{p,\mathrm{d}\mu}\right)^{\frac{p}{2}} \|f\|^{p(1-\frac{p}{2})}_{p,w},\end{aligned}$$

where the last step uses Corollary 5.2.3 and Theorem 5.4.1, from which Eq. (5.4.15) follows. This completes the proof in the case $0 < p \le 2$.

Next, we consider the case $2 < p < \infty$. Since $p/2 > 1$, using (5.4.17) and Hölder's inequality, we obtain

$$\begin{aligned}|f(x)|^p w_n(x) &\le c \left(\int_{\mathbb{S}^{d-1}} |f(y)|^p Q_2(\langle x,y\rangle)\,\mathrm{d}\mu(y) \right) \\ &\quad \times \left(\int_{\mathbb{S}^{d-1}} Q_2(\langle x,y\rangle) |Q_1(y)|^{\frac{p}{p-2}} \,\mathrm{d}\mu(y) \right)^{\frac{p}{2}-1}. \qquad (5.4.18)\end{aligned}$$

By Lemma 5.4.4 and Eq. (5.4.16), there exists a nonnegative spherical polynomial $Q_3 \in \Pi_n(\mathbb{S}^{d-1})$ such that

$$Q_3(y) \sim Q_1(y)^{\frac{p}{p-2}} \sim w_n(y)^{-1}, \qquad \forall y \in \mathbb{S}^{d-1}.$$

Hence, using Eqs. (5.4.5) and (5.4.16), we see that the second integral in (5.4.18) is bounded by

$$c\int_{\mathbb{S}^{d-1}} Q_2(\langle x,y\rangle)Q_3(y)\mathrm{d}\mu(y) \le c\int_{\mathbb{S}^{d-1}} Q_2(\langle x,y\rangle)Q_3(y)w_n(y)\mathrm{d}\sigma(y) \le c\int_{\mathbb{S}^{d-1}} Q_2(\langle x,y\rangle)\,\mathrm{d}\sigma(y) \le c.$$

Using this inequality in Eq. (5.4.18) and integrating over $x\mathbb{S}^{d-1}$, we conclude that

$$\|f\|_{p,w}^p \le c\|f\|_{p,w_n}^p \le c\int_{\mathbb{S}^{d-1}} |f(y)|^p\,\mathrm{d}\mu(y),$$

where we have used again the fact that the integral of Q_2 is bounded. This proves Eq. (5.4.15) for $2 < p < \infty$ and completes the proof. □

5.5 Nikolskii and Bernstein Inequalities

In this section, w denotes a doubling weight on $\mathbb{S}^{d-1}$ normalized so that its integral over $\mathbb{S}^{d-1}$ is equal to 1, and s_w is the constant defined in Eq. (5.1.2). Recall that $s_w = d-1$ if w is a constant weight.

Theorem 5.5.1 (Nikolskii's inequality). *If $0 < p < q \le \infty$ and $f \in \Pi_n(\mathbb{S}^{d-1})$, then*

$$\|f\|_{q,w} \le cn^{(\frac{1}{p}-\frac{1}{q})s_w}\|f\|_{p,w}, \tag{5.5.1}$$

where c depends only on d, p, q, and L_w.

Proof. We first consider the case $0 < p < q = \infty$. Let $\delta = \frac{1}{12c_{w,p}n}$ with $c_{w,p}$ the constant in Eq. (5.3.5). Let Λ be a maximal δ-separated subset of $\mathbb{S}^{d-1}$. By (ii) of Theorem 5.3.6, we obtain that for $f \in \Pi_n(\mathbb{S}^{d-1})$,

$$\|f\|_\infty \le c\max_{\eta\in\Lambda}|f(\eta)| \le c\left(\min_{\eta\in\Lambda} w\left(\mathsf{c}\left(\eta,\frac{\delta}{n}\right)\right)\right)^{-\frac{1}{p}}\left(\sum_{\eta\in\Lambda} w\left(\mathsf{c}\left(\eta,\frac{\delta}{n}\right)\right)|f(\eta)|^p\right)^{\frac{1}{p}} \le c\|f\|_{p,w}\max_{\eta\in\Lambda}\left(w\left(\mathsf{c}\left(\eta,\frac{1}{n}\right)\right)\right)^{-\frac{1}{p}}. \tag{5.5.2}$$

Let m be a positive integer such that $2^{m-1} \le n\pi \le 2^m$. By Eq. (5.1.5), for every $\eta \in \Lambda$,

$$1=w(\mathbb{S}^{d-1})=w(\mathsf{c}(\eta,\pi))\le C_{L_w}2^{ms_w}w\left(\mathsf{c}\left(\eta,\frac{1}{n}\right)\right) \le c_{L_w}(2\pi n)^{s_w}w\left(\mathsf{c}\left(\eta,\frac{1}{n}\right)\right),$$

which implies that $(w(\mathtt{c}(\eta, \frac{1}{n})))^{-\frac{1}{p}} \le cn^{s_w/p}$, and by Eq. (5.5.2), it implies the inequality Eq. (5.5.1) for the case $0 < p < q = \infty$.

The case $0 < p < q < \infty$ follows from that of $q = \infty$. Indeed, one has

$$\|f\|_{q,w}^q \le \|f\|_\infty^{q-p} \|f\|_{p,w}^p \le cn^{s_w(q-p)/p} \|f\|_{p,w}^q,$$

using the inequality for $q = \infty$, which proves Eq. (5.5.1) for $q < \infty$. □

Next we establish a weighted Bernstein inequality, which extends Lemma 4.2.4. Recall that the differential operators $D_{i,j}$ are defined by $D_{i,j} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$.

Theorem 5.5.2. *If $f \in \Pi_n(\mathbb{S}^{d-1})$, $\ell \in \mathbb{N}$, and $0 < p < \infty$, then*

$$\max_{1 \le i < j \le d} \|D_{i,j}^\ell f\|_{p,w} \le cn^\ell \|f\|_{p,w},$$

where c depends on L_w but is independent of f, n, and p when p is bounded away from zero.

Proof. Let $L_n(t)$ be the kernel defined via a cutoff function η in Eq. (2.6.3). We first need the following estimate of this kernel: For $x, y \in \mathbb{S}^{d-1}$ and $m \in \mathbb{N}$,

$$\left| D_{i,j} \left[L_n(\langle \cdot, y \rangle) \right](x) \right| \le c_m n^d (1 + n\mathtt{d}(x,y))^{-m}, \quad 1 \le i \ne j \le d, \tag{5.5.3}$$

with c_m depending only on m, η, and d. This follows directly from Theorem 2.6.5. Indeed, since $|x_i y_j - x_j y_i| \le |x_i - y_i| + |y_j - x_j| \le 2d(x,y)$,

$$\begin{aligned} \left| D_{i,j} \left[L_n(\langle \cdot, y \rangle) \right](x) \right| &= \left| L_n'(\langle x, y \rangle) \right| |x_i y_j - x_j y_i| \\ &\le cn^{d+1} (1 + n\mathtt{d}(x,y))^{-m-1} \mathtt{d}(x,y) \le cn^d (1 + n\mathtt{d}(x,y))^{-m}. \end{aligned}$$

Using the fact that $f * L_n = f$ for all $f \in \Pi_n(\mathbb{S}^{d-1})$, we obtain

$$|D_{i,j} f(x)| = \left| \int_{\mathbb{S}^{d-1}} f(y) D_{i,j} \left[L_n(\langle \cdot, y \rangle) \right](x) \, \mathrm{d}\sigma(y) \right|,$$

which, using Eq. (5.5.3) with $m > \frac{2}{p} s_w + d - 1$, is controlled by

$$\begin{aligned} & cn^d \int_{\mathbb{S}^{d-1}} |f(y)| (1 + n\mathtt{d}(x,y))^{-m} \mathrm{d}\sigma(y) \\ & \le cn^d f_{2s_w/p,n}^*(x) \int_{\mathbb{S}^{d-1}} (1 + n\mathtt{d}(x,y))^{-m + \frac{2}{p} s_w} \mathrm{d}\sigma(y) \\ & \le Cn f_{2s_w/p,n}^*(x). \end{aligned}$$

Thus, using Corollary 5.2.3, we deduce the desired Bernstein's inequality for $\ell = 1$. Iteration over ℓ completes the proof. □

Recall that the Laplace–Beltrami operator Δ_0 has a decomposition in terms of $D_{i,j}$, $\Delta_0 = \sum_{1\le i<j\le d} D_{i,j}^2$. The following weighted Bernstein inequality for Δ_0 is an immediate consequence of Theorem 5.5.2.

Corollary 5.5.3. *If $\ell \in \mathbb{N}$, $0 < p < \infty$, and $f \in \Pi_n(\mathbb{S}^{d-1})$, then*

$$\|\Delta_0^\ell f\|_{p,w} \le Cn^{2\ell}\|f\|_{p,w},$$

where c depends only on ℓ, L_w, and p when p is close to zero.

5.6 Notes and Further Results

A good reference for polynomial inequalities of one variable is [20]. Doubling weights arrived on the scene relatively late. The study of weighted polynomial inequalities in this chapter follows the approach of Mastroianni and Totik [116], who first proved that a number of important weighted polynomial inequalities—such as the Bernstein, Marcinkiewicz–Zygmund, Nikolskii, and Remez inequalities—hold under the doubling condition or the slightly stronger A_∞-condition on the weights. Most of the inequalities of [116] hold for $0 < p < 1$ as well, as observed by Erdélyi [69]. For polynomial approximation with doubling weights in one variable, we refer to [115, 117].

The L^p-Markov–Bernstein-type inequalities for trigonometric polynomials on arcs of a circle were established by Lubinsky [112] and by Kobindarajah and Lubinsky [96]. It was shown in [70] that weighted versions of these inequalities with doubling weights can be deduced using the results of [112] and the techniques of [116]. These weighted Markov–Bernstein-type inequalities were applied in [46, 119] to deduce the Marcinkiewicz–Zygmund-type inequalities on spherical caps. More general results on Marcinkiewicz–Zygmund-type inequalities for all arcs of the circle were established earlier in [97].

The Marcinkiewiecz–Zygmund inequality on the sphere, without weights, was established in [23, 122, 129]. In the unweighted case, the maximal function in the second section was introduced and studied in [39]. Most of the results on the weighted inequalities in this chapter were proved in [38], which also includes the following Remez inequality:

Theorem 5.6.1. *Let w be an A_∞ weight on $\mathbb{S}^{d-1}$. Let E be a subset of $\mathbb{S}^{d-1}$ and assume that* $\operatorname{meas} E = t^{d-1}$ *with $0 \le t \le 2^{-1/(d-1)}$. Then for $0 < p < \infty$ and $f \in \Pi_n(\mathbb{S}^{d-1})$,*

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p w(x)\,\mathrm{d}\sigma(x) \le c^{nt+1} \int_{\mathbb{S}^{d-1}\setminus E} |f(x)|^p w(x)\,\mathrm{d}\sigma(x),$$

where c depends only on d, p, and the A_∞ constant of w.

For weighted polynomial approximation on the sphere, we refer to [37, 50].

Chapter 6
Cubature Formulas on Spheres

In problems that deal with data, as frequently encountered in applied mathematics, it is often necessary to discretize integrals to obtain discrete processes of approximation. Cubature formulas, a synonym for numerical integration formulas, are essential tools for discretizing integrals. In contrast to the one-variable case, fundamental problems of cubature formulas in several variables are still open, including those on the sphere. In this chapter, we discuss several aspects of cubature formulas on the sphere.

After a brief introduction in the first section, the classical cubature formulas on the sphere in spherical coordinates are given in the second section; despite the problem of points accumulating around the poles, they are among the few formulas that are explicitly available. For a given set of discrete points, the existence of a positive cubature rule that preserves polynomials of degrees up to the order of the square root of the number of nodes is proven in the third section. Such formulas are most useful when the points are well separated. An example of such points, taken from an equal-area partition of the sphere's surface, is discussed in the fourth section. Finally, in the fifth section, we consider cubature rules on the sphere with equal coefficients, a synonym for spherical design, and give the recent affirmative proof of a conjecture on the optimal asymptotic bounds of the number of nodes.

6.1 Cubature Formulas

A cubature formula is a finite linear sum of function evaluations that approximates an integral. The strength of a cubature formula is often measured by the number of polynomials that it preserves.

Definition 6.1.1. Let w be a weight function on $\mathbb{S}^{d-1}$. A cubature formula

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$$Q_n(f) := \sum_{k=1}^{N} \lambda_k f(x_k), \quad \lambda_k \in \mathbb{R}, \quad x_k \in \mathbb{S}^{d-1},$$

is of degree n for the measure $w(x)\mathrm{d}\sigma$ on $\mathbb{S}^{d-1}$ if

$$\int_{\mathbb{S}^{d-1}} f(x)w(x)\mathrm{d}\sigma = Q_n(f), \qquad \forall f \in \Pi_n(\mathbb{S}^{d-1}), \tag{6.1.1}$$

and there is at least one f in $\Pi_{n+1}(\mathbb{S}^{d-1})$ for which equality fails to hold. A cubature formula is positive if $\lambda_k > 0$ for $1 \le k \le N$.

The points x_k in $Q_n(f)$ are called nodes and the coefficients λ_k in $Q_n(f)$ are called weights of the cubature formula. We assume that w is normalized so that $\int_{\mathbb{S}^{d-1}} w(x)\mathrm{d}\sigma = 1$, which implies, in particular, that $\sum_{k=1}^{N} \lambda_k = 1$, since $Q_n(1) = 1$. We are particularly interested in positive cubature formulas, since they are numerically stable and there will not be wild oscillation in their weights, since $0 \le \lambda_k \le 1$ if $Q_n(f)$ is positive.

The strength of a cubature formula is measured by its degree. For a fixed number of nodes N, the greater the value of n, the stronger the $Q_n(f)$. The formula of highest degree is called the Gaussian type, a tribute to the Gaussian quadrature formula of one variable. This correlation between the number of points N and the degree of precision n is often considered by asking how many points are needed for a fixed degree. A classical result is the following.

Theorem 6.1.2. *If a cubature formula on the sphere is of degree n, then its number of nodes N satisfies*

$$N \ge \dim \Pi_{\lfloor \frac{n}{2} \rfloor}(\mathbb{S}^{d-1}) = \binom{m+d-1}{m} + \binom{m+d-2}{m-1}, \quad m = \left\lfloor \frac{n}{2} \right\rfloor. \tag{6.1.2}$$

Proof. Assume that $Q_n(f)$ is a cubature formula of degree n on the sphere with N nodes and $N < M := \dim \Pi_m(\mathbb{S}^{d-1})$. Consider $P(x) = \sum_\alpha a_\alpha x^\alpha$, where the sum is over a set of M linearly independent monomials that form a basis of $\Pi_m(\mathbb{S}^{d-1})$. The linear system of equations $P(x_j) = 0$, $1 \le j \le N$, has N equations and M variables a_α, and it has, since $M > N$, a nontrivial solution, which gives a polynomial P of degree at most m that vanishes on all nodes. Consequently, $Q_n(P^2) = 0$. On the other hand, $Q_n(P^2) = \int_{\mathbb{S}^{d-1}} [P(x)]^2 w(x)\mathrm{d}\sigma > 0$, which is a contradiction. □

For the integral with respect to $\mathrm{d}\sigma$, that is, $w(x) = 1$, there is an improved lower bound for cubature formulas of odd degree, which is based on the following lemma.

Lemma 6.1.3. *For* $\lambda = \frac{d-2}{2}$, $d \ge 3$, *let* F *be a nonnegative function defined on* $[-1,1]$ *with Gegenbauer expansion*

$$F(x) = \sum_{k=0}^{\infty} \hat{F}_k C_k^\lambda(x), \quad \hat{F}_k = (h_k^\lambda)^{-1} c_\lambda \int_{-1}^{1} F(t) C_k^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}}\mathrm{d}t,$$

where $h_k^\lambda := \int_{-1}^1 |C_k^\lambda(t)|^2(1-t^2)^{\lambda-\frac{1}{2}}\,\mathrm{d}t$*, which satisfies* $\hat{F}_0 > 0$ *and* $\hat{F}_k \le 0$ *for* $k > n$ *for a positive integer* n*. Then the number* N *of nodes of a positive cubature formula of degree* n *for the integral* $\int_{\mathbb{S}^{d-1}} f(x)\mathrm{d}\sigma$ *satisfies*

$$N \ge F(1)/\hat{F}_0. \tag{6.1.3}$$

Proof. Let $Q_n(f)$ be a cubature formula for $\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(x)\mathrm{d}\sigma$, as in Eq. (6.1.1) with $w(x) = 1/\omega_d$. Then $\sum_{k=1}^N \lambda_k = 1$ by setting $f(x) = 1$. By Eq. (1.2.7), the reproducing kernel of $\mathscr{H}_k^d$ is given by $Z_k(x,y) = \frac{k+\lambda}{\lambda} C_k^\lambda(\langle x,y\rangle)$. Furthermore, by Eq. (1.2.3), for $k \ge 1$,

$$\sum_{i=1}^N \sum_{j=1}^N \lambda_i \lambda_j Z_k(x_i,x_j) = \sum_{\ell=1}^{\dim \mathscr{H}_k^d} \left| \sum_{i=1}^N \lambda_i Y_\ell(x_i) \right|^2 \ge 0,$$

where $\{Y_\ell : 1 \le \ell \le \dim \mathscr{H}_k^d\}$ is an orthonormal basis of $\mathscr{H}_k^d$. Furthermore, the above sum is equal to zero for $1 \le k \le n$ by the cubature formula and the fact that the integral of Z_k over $\mathbb{S}^{d-1}$ is zero for $k \ge 1$. Hence, by the assumption on $\hat{F}_k$,

$$\begin{aligned} I := \sum_{i=1}^N \sum_{j=1}^N \lambda_i \lambda_j F(\langle x_i,x_j\rangle) &= \sum_{k=0}^\infty \hat{F}_k \sum_{i=1}^N \sum_{j=1}^N \lambda_i \lambda_j \frac{\lambda}{k+\lambda} Z_k(x_i,x_j) \\ &\le \hat{F}_0 \sum_{i=1}^N \sum_{j=1}^N \lambda_i \lambda_j = \hat{F}_0. \end{aligned}$$

On the other hand, since F is nonnegative and $\lambda_j \ge 0$, by the Cauchy–Schwarz inequality, we have

$$I \ge \sum_{j=1}^N \lambda_j^2 F(\langle x_j,x_j\rangle) = F(1)\sum_{j=1}^N \lambda_j^2 \ge F(1)\frac{1}{N}\left(\sum_{j=1}^N \lambda_j\right)^2 = \frac{F(1)}{N}.$$

Putting together these two inequalities gives the desired lower bound of N. □

Theorem 6.1.4. *If a positive cubature formula for the integral* $\int_{\mathbb{S}^{d-1}} f(x)\mathrm{d}\sigma$ *is of degree* $2m+1$*, then its number* N *of nodes satisfies*

$$N \ge 2\binom{m+d-1}{m}. \tag{6.1.4}$$

Proof. We apply the lemma with the function F defined by

$$F(x) := (1+x)Q^2(x) \quad \text{with} \quad Q(x) = \sum_{0 \le 2k \le m} \frac{m-2k+\lambda}{\lambda} C_{m-2k}^\lambda(x).$$

Evidently, $\hat{F}_0 = c_\lambda \int_{-1}^{1} F(x)(1-x^2)^{\lambda-1/2}\mathrm{d}x > 0$ and $\hat{F}_k = 0$ for $k > 2m+1$, since F is of degree $2m+1$. Since Q^2 is even, we can drop $1+x$ and use Eq. (A.5.1) to write

$$\hat{F}_0 = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} Q^2(\langle x,y\rangle)\mathrm{d}\sigma(y) = \sum_{0\le 2k\le m} \dim \mathscr{H}^d_{m-2k} = \dim \mathscr{P}^d_m = \binom{m+d-1}{m},$$

where the second equality follows from Eqs. (1.2.5) and (1.2.9). Furthermore,

$$Q(1) = \sum_{0\le 2k\le m} \dim \mathscr{H}^d_{m-2k} = \dim \mathscr{P}^d_m,$$

so that $F(1) = 2(\dim \mathscr{P}^d_m)^2$, from which Eq. (6.1.4) follows. □

A given lower bound on N is called sharp if there exists a cubature formula of specified degree with the number of nodes equal to the lower bound. The lower bounds in Eqs. (6.1.2) and (6.1.4), however, are known to be sharp only for a few special values of n and d, and they are known to be not sharp in general. In fact, even the asymptotic order

$$N \ge \frac{2}{(d-1)!}\left(\frac{n}{2}\right)^{d-1} + \mathscr{O}\left(n^{d-2}\right), \qquad n\to\infty, \tag{6.1.5}$$

of Eq. (6.1.2) is not sharp for $d > 2$ in the sense that all cubature formulas of degree n need $N = \mu n^{d-1} + \mathscr{O}(1)n^{d-2}$ nodes for a larger μ; see Sect. 6.6.

If $d = 2$, the space $\Pi_n(\mathbb{S}^1)$ is the space of trigonometric polynomials of degree n on $\mathbb{S}^1 \equiv [0,2\pi)$. The lower bound (6.1.2) becomes $N \ge n$ when n is odd, and it is attainable for all odd n. A cubature formula on a circle or on an interval is usually called a quadrature formula.

Proposition 6.1.5. *Let a_n be fixed elements in $[0,2\pi)$. For $n = 0,1\ldots$, the quadrature formula*

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{d}\theta = \frac{1}{n}\sum_{j=0}^{n-1} f\left(a_n + \frac{2\pi j}{n}\right) \tag{6.1.6}$$

is exact for all $f \in \Pi_n(\mathbb{S}^1) = \mathrm{span}\{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta\}$.

Proof. By the periodicity of the trigonometric functions, the integral is unchanged under a change of variable, so that we need to consider only the case $a_n = 0$. Since $e^{\mathrm{i}k\theta} = \cos k\theta + i \sin k\theta$, we can work with the basis $\{e^{\mathrm{i}k\theta} : -n \le k \le n\}$ of $\Pi_n(\mathbb{S}^1)$. An elementary computation shows that for $1 \le k \le n-1$,

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{\mathrm{i}k\frac{2\pi j}{n}} = \frac{1}{n}\frac{1-e^{2\pi\mathrm{i}k}}{1-e^{\mathrm{i}k\frac{2\pi}{n}}} = 0 = \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}k\theta}\mathrm{d}\theta,$$

whereas for $k = 0$, both sides are obviously 1. Equality fails for $k = n$. □

For $d \ge 3$, constructing positive cubature formulas of high degree is a difficult problem. An obvious approach is to solve the system of equations

$$\sum_{k=1}^{N} \lambda_k f_j(x_k) = \int_{\mathbb{S}^{d-1}} f_j(x) w(x) \mathrm{d}\sigma(x), \quad j = 1, \ldots, M := \dim \Pi_n(\mathbb{S}^{d-1}), \quad (6.1.7)$$

where $\{f_j : 1 \le j \le M\}$ is a basis of $\Pi_n(\mathbb{S}^{d-1})$. If the nodes x_k in $Q_n(f)$ are preassigned, then Eq. (6.1.7) is a system of linear equations in the variables $\lambda_1, \ldots, \lambda_N$, which, however, may not have a positive solution even if it is solvable, and the cubature formula of degree n obtained in this way uses at least $\dim \Pi_n(\mathbb{S}^{d-1})$ points, 2^{d-1} times the order in Eq. (6.1.5). To keep the number of points small, one can assume that both x_k and λ_k are variables, in which case Eq. (6.1.7) is a nonlinear system of equations and can be solved, if a solution exists, only by numerical methods. In this regard, a theorem of Sobolev on cubature formulas invariant under a finite group can often be used to simplify the matter.

Let G be a finite subgroup of the $d \times d$ orthogonal group $\mathrm{O}(d)$. For a function f defined on $\mathbb{R}^d$ and $\tau \in G$, define τf by $\tau f(x) := f(x\tau)$, $\forall x \in \mathbb{S}^{d-1}$. The function f is said to be invariant under G if $\tau f = f$ for all $\tau \in G$. Define

$$\Pi_n^G(\mathbb{S}^{d-1}) := \{f \in \Pi_n(\mathbb{S}^{d-1}) : f \text{ invariant under } G\}.$$

Definition 6.1.6. A cubature formula $Q_n(f)$ is invariant under a finite subgroup G of $O(d)$ if $Q_n(\tau f) = Q_n(f)$ for all $\tau \in G$.

Theorem 6.1.7. *Assume that w is invariant under a finite subgroup G of $O(d)$. If the cubature formula $Q_n(f)$ is invariant under G, then $Q_n(f)$ is of degree n if and only if Eq. (6.1.1) holds for all polynomials in $\Pi_n^G(\mathbb{S}^{d-1})$.*

Proof. Since $\Pi_n^G(\mathbb{S}^{d-1}) \subset \Pi_n(\mathbb{S}^{d-1})$, we need to consider only one direction. Let $Q_n(f)$ be invariant and suppose that Eq. (6.1.1) holds for all polynomials in $\Pi_n^G(\mathbb{S}^{d-1})$. For $f \in \Pi_n(\mathbb{S}^{d-1})$, let $f_G = \frac{1}{\#G}\sum_{\tau\in G} \tau f$, where $\#G$ denotes the cardinality of G. Then

$$\begin{aligned} Q_n(f) &= \frac{1}{\#G}\sum_{\tau\in G} Q_n(\tau f) = Q_n(f_G) = \int_{\mathbb{S}^{d-1}} f_G(x) w(x) \mathrm{d}\sigma \\ &= \frac{1}{\#G}\sum_{\tau\in G} \int_{\mathbb{S}^{d-1}} f(x) w(x\tau^{-1}) \mathrm{d}\sigma = \int_{\mathbb{S}^{d-1}} f(x) w(x) \mathrm{d}\sigma, \end{aligned}$$

by the invariance of $Q_n(f)$ and w. □

It should be noted that since the dimension of $\Pi_n^G(\mathbb{S}^{d-1})$ is far smaller than that of $\Pi_n(\mathbb{S}^{d-1})$, the size of the system of Eqs. (6.1.7) for an invariant cubature formula can be drastically reduced to facilitate computation.

Example 6.1.8. The vertices of the icosahedron are nodes of a spherical 5-design on $\mathbb{S}^2$. More precisely, let $X := \{(0, \pm 1, \pm\tau)/\gamma, (\pm 1, \pm\tau, 0)/\gamma, (\pm\tau, 0, \pm 1)/\gamma\}$, where $\tau = (1+\sqrt{5})/2$ and $\gamma = \sqrt{1+\tau^2}$; then X is a subset of $\mathbb{S}^2$ and

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} f(x)\mathrm{d}\sigma(x) = \frac{1}{12}\sum_{x\in X} f(x), \qquad f \in \Pi_5(\mathbb{S}^2). \tag{6.1.8}$$

Proof. The set of vertices of the icosahedron is invariant under the icosahedral group. It is known that the first two nontrivial polynomials invariant under the icosahedral group are $p_1(x) = x_1^2 + x_2^2 + x_3^2$, of degree 2, and $p_2(x) = (x_1^2 - \tau^2 x_2^2)(x_2^2 - \tau^2 x_3^2)(x_3^2 - \tau^2 x_1^2)$, of degree 6. If $f = p_1$, then both sides of Eq. (6.1.8) are equal to 1, so that the theorem holds for $f = p_1$. If $f = p_2$, it is easy to verify that Eq. (6.1.8) fails to hold. Accordingly, Eq. (6.1.8) holds for all $f \in \Pi_5(\mathbb{S}^2)$ by Theorem 6.1.7 but not for all f in $\Pi_6(\mathbb{S}^2)$. □

We note that the number of nodes, 12, of this cubature formula of degree 5 attains the lower bound given in Eq. (6.1.4), one of the rare examples.

6.2 Product-Type Cubature Formulas on the Sphere

Product-type cubature formulas on the sphere are constructed by parameterizing the integral over the sphere in polar coordinates. The number of nodes of such a formula is far more than that of Eq. (6.1.5), and the nodes cluster around the north and south poles of the sphere. Despite these defects, these cubature formulas are useful and are essentially the only family of formulas on the sphere that are positive and explicitly constructed.

We will need the Gaussian quadrature rules for the Gegenbauer weight function $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$. It is well known that such quadrature formulas exist and that they are based on the zeros of the Gegenbauer polynomials. The polynomial $C_n^\lambda(t)$ has n distinct real zeros in $[-1,1]$, which we denote by $t_{k,n}^{(\lambda)}$ with

$$-1 < t_{1,n}^{(\lambda)} < t_{2,n}^{(\lambda)} < \cdots < t_{n,n}^{(\lambda)} < 1.$$

Since $C_n^\lambda(-t) = (-1)^n C_n^\lambda(t)$, the zeros are symmetric with respect to the origin, that is, $t_{k,n}^{(\lambda)} = -t_{n-k+1,n}^{(\lambda)}$. Furthermore, we define $\theta_{k,n}^{(\lambda)}$ by

$$t_{k,n}^{(\lambda)} := \cos\theta_{k,n}^{(\lambda)}, \quad \theta_{k,n}^{(\lambda)} \in (0,\pi), \quad 1 \le k \le n. \tag{6.2.1}$$

Proposition 6.2.1. *Let $\lambda > -1/2$ and $t_{k,n} = t_{k,n}^{(\lambda)}$. For each $n \in \mathbb{N}$, the Gaussian quadrature of degree $2n-1$ for w_λ is given by*

$$\int_{-1}^{1} f(x) w_\lambda(x)\mathrm{d}x = \sum_{k=1}^{n} \mu_{k,n}^{(\lambda)} f(t_{k,n}), \qquad \forall f \in \Pi_{2n-1}, \tag{6.2.2}$$

where the quadrature weights $\mu_{k,n}^{(\lambda)} > 0$ *are given by*

$$\mu_{k,n}^{(\lambda)} = \frac{\pi}{2^{2\lambda}[\Gamma(\lambda+1)]^2} \frac{\Gamma(n+2\lambda)}{(1-t_{k,n}^2)[C_{n-1}^{\lambda+1}(t_{k,n})]^2}. \tag{6.2.3}$$

This proposition is classical. The formula for the weights are given in [162, p. 352], which we have simplified by applying (4.7.27) of [162]. Through a change of variables $x = \cos\theta$, Eq. (6.2.2) becomes, with $\theta_{k,n} = \theta_{k,n}^{(\lambda)}$,

$$\int_0^\pi f(\cos\theta)(\sin\theta)^{2\lambda}\mathrm{d}\theta = \sum_{k=1}^{n} \mu_{k,n}^{(\lambda)} f(\cos\theta_{k,n}), \qquad \forall f \in \Pi_{2n-1}. \tag{6.2.4}$$

Gaussian quadrature is known to have the highest degree of precision among all quadratures with the same number of nodes.

We are now ready to construct product-type cubature formulas on the sphere. First we consider $\mathbb{S}^2$ in $\mathbb{R}^3$. In spherical coordinates (1.6.3), set

$$g(\phi,\theta) := f(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta), \quad 0 \le \phi \le 2\pi, 0 \le \theta \le \pi.$$

Recall that the Gegenbauer polynomial $C_n^{\frac{1}{2}}$ is equal to P_n, the Legendre polynomial.

Theorem 6.2.2. *For* $n \in \mathbb{N}$, *let* $\phi_{k,n} = \pi k/n$, $0 \le k \le 2n-1$. *Let* $\theta_{j,n} = \theta_{j,n}^{(\frac{1}{2})}$ *be associated with the zeros of the Legendre polynomial as in Eq.* (6.2.1) *and* $\mu_{j,n} = \mu_{j,n}^{(\frac{1}{2})}$. *Then the cubature formula*

$$\int_{\mathbb{S}^2} f(x)\mathrm{d}\sigma(x) = \frac{\pi}{n} \sum_{k=0}^{2n-1} \sum_{j=1}^{n} \mu_{j,n} g(\phi_{k,n}, \theta_{j,n}) \tag{6.2.5}$$

is of degree $2n-1$, *that is, Eq.* (6.2.5) *holds for all* $f \in \Pi_{2n-1}(\mathbb{S}^2)$.

Proof. In spherical coordinates, we have Eq. (1.6.4)

$$\int_{\mathbb{S}^2} f(x)\mathrm{d}\sigma = \int_0^\pi \left(\int_0^{2\pi} g(\phi,\theta)\mathrm{d}\phi \right) \sin\theta\mathrm{d}\theta.$$

We need to verify that Eq. (6.2.5) holds for all polynomials in $\Pi_n(\mathbb{S}^2)$. We choose to work with the orthogonal basis in Eq. (1.6.5), which means that we need to establish Eq. (6.2.5) for $(\sin\theta)^k C_{m-k}^{k+\frac{1}{2}}(\cos\theta)(a_k\cos k\phi + b_k\sin k\phi)$ with $0 \le k \le m \le$

$2n-1$. If $0 < k \le 2n-1$, then both sides of Eq. (6.2.5) equal zero according to the trigonometric quadrature Eq. (6.1.6) with n replaced by $2n-1$. The remaining case $k = 0$ amounts to showing that

$$\frac{1}{2}\int_0^\pi C_m^{\frac{1}{2}}(\cos\theta)\sin\theta\,\mathrm{d}\theta = \sum_{j=1}^n \mu_{j,n} C_m^{\frac{1}{2}}(\cos\theta_{j,n}), \quad 0 \le m \le 2n-1,$$

which, however, follows immediately from Eq. (6.2.4) with $\lambda = 1/2$. □

This construction shows why the cubature formula (6.2.5) is said to be of product type. The number of nodes of Eq. (6.2.5) is $2n^2$, whereas the order of the lower bound in Eq. (6.1.5) is $n^2 + \mathscr{O}(n)$ for a cubature formula of degree $2n-1$. Geometrically, the nodes of the cubature formula (6.2.5) are distributed on n parallel circles, each of which contains $2n$ equally spaced points. This distribution, however, means that the nodes are heavily clustered at the north and south poles $(0,0,\pm 1)$, instead of being more evenly distributed over the sphere.

Using different quadrature rules for $\mathrm{d}x$ on $[-1,1]$, we can derive other cubature rules of similar type with different positions of parallel circles. In particular, if we use the Gauss–Lobatto quadrature rule for $\mathrm{d}x$ on $[-1,1]$, which includes points at the two endpoints, we obtain a product cubature formula similar to the one in Eq. (6.2.5) but with two additional nodes located at the north and the south poles.

The product cubature rule on $\mathbb{S}^{d-1}$ has more or less the same structure and can be constructed by induction. In spherical coordinates (1.5.1), let

$$g(\theta_1,\ldots,\theta_{d-1}) := f(\sin\theta_{d-1}\ldots\sin\theta_2\sin\theta_1, \sin\theta_{d-1}\ldots\sin\theta_2\cos\theta_1,\ldots,\cos\theta_{d-1}).$$

Theorem 6.2.3. *For $n \in \mathbb{N}$, let $\phi_{k,n} = \pi k/n$, $0 \le k \le 2n-1$. Then the cubature formula*

$$\int_{\mathbb{S}^{d-1}} f(x)\mathrm{d}\sigma = \frac{\pi}{n}\sum_{k=0}^{2n-1}\sum_{j_2=1}^{n}\cdots\sum_{j_{d-1}=1}^{n}\prod_{i=2}^{d-1}\mu_{i,n}^{(\frac{i-1}{2})} g\left(\phi_{k,n}, \theta_{j_2,n}^{(\frac{1}{2})},\ldots,\theta_{j_{d-1},n}^{(\frac{d-2}{2})}\right) \qquad (6.2.6)$$

is of degree $2n-1$, that is, Eq. (6.2.6) holds for all $f \in \Pi_{2n-1}(\mathbb{S}^{d-1})$.

Proof. Writing in the spherical coordinates

$$\int_{\mathbb{S}^{d-1}} f(x)\mathrm{d}\sigma = \int_0^\pi\int_0^{2\pi}\cdots\int_0^{2\pi} g(\theta_1,\theta_2,\ldots,\theta_{d-1})\prod_{j=1}^{d-2}(\sin\theta_{d-j})^{d-j-1}\mathrm{d}\theta_{d-1}\cdots\mathrm{d}\theta_1,$$

we work with the orthogonal basis in Eq. (1.5.6), for which we need to establish Eq. (6.2.6) for $\prod_{j=1}^{d-2}(\sin\theta_{d-j})^{|\alpha^{j+1}|}C_{\alpha_j}^{\lambda_j}(\cos\theta_{d-j})(a_k\cos\alpha_{d-1}\theta_1 + b_k\sin\alpha_{d-1}\theta_1)$ with $0 \le |\alpha| \le 2n-1$. If $0 < \alpha_{d-1} \le 2n-1$, then both sides of Eq. (6.2.6) equal zero according to Eq. (6.1.6) with n replaced by $2n-1$. For $\alpha_{d-1}=0$, the integral over θ_2 becomes

$$\int_0^{\pi} C_{\alpha_{d-2}}^{\frac{1}{2}}(\cos\theta)\sin\theta \mathrm{d}\theta = \sum_{j=1}^{n} \mu_{j,n}^{\frac{1}{2}} C_{\alpha_{d-2}}^{\frac{1}{2}}\left(\cos\theta_{j,n}^{\frac{1}{2}}\right), \quad 0 \le \alpha_{d-2} \le 2n-1,$$

which follows immediately from Eq. (6.2.4) with $\lambda = 1/2$. Continuing this process, for $\alpha_{d-1} = \cdots = \alpha_{d-k+1} = 0$, the integral over θ_k becomes

$$\int_0^{\pi} C_{\alpha_{d-k}}^{\frac{k-1}{2}}(\cos\theta)(\sin\theta)^{k-1}\mathrm{d}\theta = \sum_{j=1}^{n} \mu_{j,n}^{(\frac{k-1}{2})} C_{\alpha_{d-k}}^{\frac{k-1}{2}}\left(\cos\theta_{j,n}^{(\frac{k-1}{2})}\right), \quad 0 \le \alpha_{d-k} \le 2n-1,$$

which follows from Eq. (6.2.4) with $\lambda = (k-1)/2$ for $2 \le k \le d-1$. The proof is complete. □

It is worth mentioning that the cubature formula Eq. (6.2.6) can also be deduced from a product of an integral over $[0,\pi]$ and an integral over $\mathbb{S}^{d-2}$ by Eq. (1.5.4). The number of nodes of the cubature formula Eq. (6.2.6) is $2n^d$.

6.3 Positive Cubature Formulas

Recall the ε-separated subset defined in Definition 5.3.1. Each $\frac{1}{n}$-separated subset, say Λ, of $\mathbb{S}^{d-1}$ contains $\mathscr{O}(n^{d-1})$ points, and almost all such subsets admit an interpolation operator $I_n f$ in an appropriate polynomial space Π_Λ that includes $\Pi_{cn}(\mathbb{S}^{d-1})$ as a subspace and has $\dim \Pi_\Lambda = \#\Lambda$, so that $I_n f(\xi) = f(\xi)$, $\forall \xi \in \Lambda$, where f is a generic function. The interpolation operator is a projection operator in the sense that $I_n f = f$ for all $f \in \Pi_\Lambda$, and it is linear and can be written as

$$I_n f(x) = \sum_{\xi \in \Lambda} f(\xi)\ell_\xi(x),$$

where $\ell_\xi \in \Pi_\Lambda$ are determined by $\ell_\xi(\eta) = \delta_{\xi,\eta}$, $\forall \xi,\eta \in \Lambda$, usually called fundamental interpolation polynomials. Integrating $I_n(x)$ with respect to $w(x)\mathrm{d}\sigma$ over $\mathbb{S}^{d-1}$ then leads to a cubature formula of degree cn with $\xi \in \Lambda$ as nodes and $\lambda_\xi = \int_{\mathbb{S}^{d-1}} \ell_\xi(x) w(x)\mathrm{d}\sigma$ as weights. This cubature formula, however, is most likely not a positive one, that is, some of λ_ξ will be negative, and there could be wild fluctuation among the values of λ_ξ.

There is, however, another possibility: a positive cubature formula can exist if we lower the degree of precision somewhat, or equivalently, keep the same degree but increase the number of nodes. Indeed, the main result of this section is to show that each maximal $\frac{\delta}{n}$-separated subset in $\mathbb{S}^{d-1}$, for a sufficiently small $\delta > 0$, admits a positive cubature formula of degree n, and furthermore, the weights of this formula can be chosen to have more or less equal values. For the proof of this result, we shall need two lemmas from convex optimization.

The first one, Gordan's lemma, states, geometrically, that the origin does not lie in the convex hull of a set of vectors $\{a^0, a^1, \ldots, a^m\}$ in a Euclidean space if and only if there is an open half-space $\{y : \langle y, x\rangle > 0\}$ that contains $\{a^0, a^1, \ldots, a^m\}$. The precise statement is given as follows.

Lemma 6.3.1 (Gordan's lemma). *Let V be a finite-dimensional real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then for any elements $a^0, a^1, \ldots, a^m$ of V, exactly one of the following two systems has a solution:*

(1) $\sum_{i=0}^{m} \lambda_i a^i = 0, \quad \sum_{i=0}^{m} \lambda_i = 1, \quad 0 \le \lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{R};$

(2) $\langle a^i, x\rangle > 0,\ i = 0, 1, \ldots, m,$ *for some* $x \in V$.

Proof. If (1) is solvable, it is clear that (2) has no solution. Conversely, if (2) has no solution, we need to show that (1) is solvable. Without loss of generality, we can assume $V = \mathbb{R}^N$. Define

$$f(x) := \log\left(\sum_{i=0}^{m} \exp(\langle a^i, x\rangle)\right), \quad x \in \mathbb{R}^N.$$

A simple computation shows then

$$\nabla f(x) = \sum_{j=0}^{m} \lambda_j(x) a^j, \quad \text{with} \quad \lambda_j(x) = \frac{\exp(\langle a^j, x\rangle)}{\sum_{i=0}^{m} \exp(\langle a^i, x\rangle)} \ge 0. \tag{6.3.1}$$

It is evident that $\sum_{j=1}^{m} \lambda_j(x) = 1$. Let $\varphi \in C^\infty(\mathbb{R}^N)$ satisfy $\varphi(x) = 0$ for $\|x\| \le \frac{1}{2}$, and $\varphi(x) = 1$ for $\|x\| \ge 1$. Define

$$F_k(x) := f(x) + k^{-1}\varphi(kx)\|x\|, \quad k = 1, 2, \ldots.$$

Since f is nonnegative, because $f(x) \ge \log(\max_{0\le i\le m} \exp(\langle a^i, x\rangle)) \ge 0$, it follows that $\lim_{\|x\|\to\infty} F_k(x) = \infty$. Consequently, F_k attains a global minimum at some $x_k \in \mathbb{R}^N$, and therefore,

$$0 = \nabla F_k(x_k) = \nabla f(x_k) + \|x_k\|\nabla\varphi(kx_k) + k^{-1}\varphi(kx_k)x_k/\|x_k\|.$$

Since $\nabla\varphi$ is supported in $\{x : 1/2 \le \|x\| \le 1\}$, it follows that $\left|\nabla f(x_k)\right| \le ck^{-1}$, which goes to 0 as $k \to \infty$. Let $\lambda_j^k = \lambda_j(x_k)$. Since the sequence $\{(\lambda_0^k, \lambda_1^k, \ldots, \lambda_m^k) : k = 1, 2, \ldots\}$ is bounded in $\mathbb{R}^{m+1}$, it has a convergent subsequence by Weierstrass's theorem. Taking the limit of this convergent subsequence shows, by Eq. (6.3.1) and the fact that $|\nabla f(x_k)| \to 0$ as $k \to \infty$, that the system (1) has a solution. □

Our second lemma is Farkas's lemma, which states, geometrically, that every point ζ not lying in the finitely generated cone $\{\sum_{j=1}^{m} \mu_j a^j : 0 \le \mu_1, \mu_2, \ldots, \mu_m \in \mathbb{R}\}$ can be separated from the cone by a hyperplane.

Lemma 6.3.2 (Farkas's lemma). *Let V be a finite-dimensional real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then for any points $a^1, a^2, \ldots, a^m$ and ζ in V, exactly one of the following two systems has a solution:*

(1) $\sum_{j=1}^{m} \mu_j a^j = \zeta \quad 0 \le \mu_1, \mu_2, \ldots, \mu_m \in \mathbb{R}$;

(2) $\langle a^j, x \rangle \ge 0,\ j = 1, 2, \ldots, m,$ *and* $\langle \zeta, x \rangle < 0$ *for some* $x \in V$.

Proof. If (1) has a solution, then taking inner product with x of (1) shows immediately that (2) has no solution. Conversely, assume that (2) has no solution. We shall show that (1) has a solution by induction on m. For $m = 1$, the assumption implies that $\langle a^1, x \rangle$ and $\langle \zeta, x \rangle$ have the same sign for all $x \in V$, which implies that $\zeta = \mu a^1$ for some positive μ, whence (1). Suppose now that the assertion has been proved for a set of $m-1$ vectors in some finite-dimensional real Hilbert space. Define $a^0 = -\zeta$. The unsolvability of (2) then implies the unsolvability of (2) in Lemma 6.3.1, which implies that there are nonnegative scalars $\lambda_0, \ldots, \lambda_m$, not all zero, such that $\lambda_0 \zeta = \sum_{j=1}^{m} \lambda_j a^j$. If $\lambda_0 > 0$, the proof is complete. So suppose $\lambda_0 = 0$ and, without loss of generality, $\lambda_m > 0$. We then have

$$a^m = -\lambda_m^{-1} \sum_{j=1}^{m-1} \lambda_j a^j. \tag{6.3.2}$$

Define now $Y = \{y \in V : \langle y, a^m \rangle = 0\}$, and let $P_Y : V \to Y$ denote the orthogonal projection onto Y. By the induction hypothesis, the system

$$\langle a^j, y \rangle \ge 0,\ j = 1, 2, \ldots, m-1,\ \langle \zeta, y \rangle < 0 \text{ for some } y \in Y$$

has no solution, or equivalently,

$$\langle P_Y a^j, y \rangle \ge 0\ j = 1, 2, \ldots, m-1,\ \langle P_Y \zeta, y \rangle < 0 \text{ for some } y \in Y$$

has no solution. Applying the induction hypothesis to the subspace Y shows that there are nonnegative real numbers $\mu_1, \ldots, \mu_{m-1}$ such that $\sum_{j=1}^{m-1} \mu_j P_Y a^j = P_Y \zeta$. This means that $\zeta - \sum_{j=1}^{m-1} \mu_j a^j$ is orthogonal to the space Y. Since Y is the orthogonal complement of span$\{a^m\}$ in V, there is a $\mu_m \in \mathbb{R}$ such that

$$\mu_m a^m = \zeta - \sum_{j=1}^{m-1} \mu_j a^j. \tag{6.3.3}$$

If $\mu_m \geq 0$, we immediately obtain a solution of (1), whereas if $\mu_m < 0$, we can substitute Eq. (6.3.2) into Eq. (6.3.3) to obtain

$$(-\mu_m)\lambda_m^{-1} \sum_{j=1}^{m-1} \lambda_j a^j = \zeta - \sum_{j=1}^{m-1} \mu_j a^j,$$

which again gives a solution of (1). □

We are now ready to prove the existence of positive cubature formulas that we can establish for a doubling weight. In the following theorem, δ_0 denotes a sufficiently small positive constant depending only on the doubling constant of w.

Theorem 6.3.3. *Let w be a doubling weight on $\mathbb{S}^{d-1}$. Given a maximal $\frac{\delta}{n}$-separated subset $\Lambda \subset \mathbb{S}^{d-1}$ with $\delta \in (0, \delta_0)$, there exist positive numbers λ_η, $\eta \in \Lambda$ such that $\lambda_\eta \sim w\big(\mathsf{c}(\eta, \frac{\delta}{n})\big)$ for all $\eta \in \Lambda$ and*

$$\int_{\mathbb{S}^{d-1}} f(x)w(x)\,\mathrm{d}\sigma(x) = \sum_{\eta\in\Lambda} \lambda_\eta f(\eta), \quad f \in \Pi_n(\mathbb{S}^{d-1}). \tag{6.3.4}$$

Proof. We shall use Lemma 6.3.2 with $V = \Pi_n(\mathbb{S}^{d-1})$ endowed with the inner product

$$\langle f, g\rangle := \int_{\mathbb{S}^{d-1}} f(x)g(x)\,\mathrm{d}\sigma(x).$$

The reproducing kernel of $\Pi_n(\mathbb{S}^{d-1})$ under $\langle\cdot,\cdot\rangle$ is $K_n(x,y) = \frac{1}{\omega_d}\sum_{k=0}^n Z_k(x,y)$, where Z_k is defined in Eq. (1.2.2). By definition,

$$\langle f, K_n(x,\cdot)\rangle = f(x), \quad x \in \mathbb{S}^{d-1}, \quad f \in V. \tag{6.3.5}$$

Let $\{\eta_1, \ldots, \eta_N\}$ be an enumeration of Λ. We define the functions ζ and a^j in the space V as follows:

$$a^j(x) := K_n(x, \eta_j), \quad j = 1, 2, \ldots, N,$$

$$\zeta(x) := \int_{\mathbb{S}^{d-1}} K_n(x,y)w(y)\mathrm{d}\sigma(y) - \frac{1}{2c_d}\sum_{j=1}^N K_n(x,\eta_j)w\left(\mathsf{c}\left(\eta_j, \frac{\delta}{n}\right)\right),$$

where c_d is the constant in Eq. (5.3.1). By Eq. (6.3.5), for all $f \in V$,

$$\langle f, a^j\rangle = f(\eta_j), \quad j = 1, 2, \ldots, N, \tag{6.3.6}$$

$$\langle f, \zeta\rangle = \int_{\mathbb{S}^{d-1}} f(y)w(y)\mathrm{d}\sigma(y) - \frac{1}{2c_d}\sum_{j=1}^N f(\eta_j)w\left(\mathsf{c}\left(\eta_j, \frac{\delta}{n}\right)\right). \tag{6.3.7}$$

If $\langle f, a^j \rangle \geq 0$ for all j, then f is nonnegative on the set Λ, and we can apply Lemma 5.4.2 to conclude that

$$\int_{\mathbb{S}^{d-1}} f(x)w(x)\mathrm{d}\sigma \geq \left(\frac{1}{c_d} - c'\delta\right) \sum_{j=1}^{N} f(\eta_j) w\left(\mathsf{c}\left(\eta_j, \frac{\delta}{n}\right)\right),$$

where $c' = c_* A_1$ is a constant independent of δ, which implies that

$$\langle f, \zeta \rangle \geq \left(\frac{1}{2c_d} - c'\delta\right) \sum_{j=1}^{N} f(\eta_j) w(\mathsf{c}(\eta_j, n^{-1}\delta)) \geq 0$$

if $0 < \delta \leq 1/(2c_d c')$. Consequently, the corresponding system (2) of Lemma 6.3.2 is not solvable, and as a result, the system (1) of Lemma 6.3.2 has a solution; that is, there exist $\mu_1, \ldots, \mu_N \geq 0$ such that $\zeta = \sum_{j=1}^{N} \mu_j a^j$. By Eqs. (6.3.6) and (6.3.7), taking the inner product with f, this implies that for all $f \in \Pi_n(\mathbb{S}^{d-1})$,

$$\int_{\mathbb{S}^{d-1}} f(y)w(y)\,\mathrm{d}\sigma(y) - \frac{1}{2c_d} \sum_{j=1}^{N} f(\eta_j) w\left(\mathsf{c}\left(\eta_j, \frac{\delta}{n}\right)\right) = \sum_{j=1}^{N} \mu_j f(\eta_j),$$

which gives the desired cubature formula

$$\int_{\mathbb{S}^{d-1}} f(y)w(y)\,\mathrm{d}\sigma(y) = \sum_{j=1}^{N} \lambda_j f(\eta_j) \tag{6.3.8}$$

with $\lambda_j := \mu_j + (2c_d)^{-1} w(\mathsf{c}(\eta_j, n^{-1}\delta))$ for $1 \leq j \leq N$. Since $\mu_j \geq 0$, we immediately have $\lambda_j \geq (2c_d)^{-1} w(\mathsf{c}(\eta_j, n^{-1}\delta))$. Furthermore, on defining a finite measure μ supported on the finite set Λ, by $\mu\{\eta_j\} := w(\mathsf{c}(\eta_j, \frac{\delta}{n}))$ for $1 \leq j \leq N$, we can write the right-hand side of Eq. (6.3.8) as $\int_{\mathbb{S}^{d-1}} f(y)\mathrm{d}\mu(y)$, which allows us to apply Lemma 5.4.3 to obtain the upper estimate $\lambda_j \leq c w(\mathsf{c}(\eta_j, n^{-1}\delta))$. The proof is complete. □

Let N be the number of nodes of Eq. (6.3.4). Then the degree of precision of the cubature formula is on the order of $N^{1/(d-1)}$. This is the same order as that given in Eq. (6.1.5), although the constant in front of the asymptotic order could be rather small. The desirable features of the cubature formula (6.3.4) make it an important tool for theoretical studies.

Since the nodes are given, constructing the cubature formula (6.3.4) amounts to determining the weights λ_η for each $\eta \in \Lambda$. However, since the weights satisfy an underdetermined linear system of more variables than equations, it is a difficult task to identify a positive solution as specified in the theorem among the infinitely many solutions of the system. In fact, at this moment, no practical method for constructing such a cubature formula is known when n is moderately large.

6.4 Area-Regular Partitions of $\mathbb{S}^{d-1}$

Except for $d = 2$, there are generally no equally spaced points on the sphere $\mathbb{S}^{d-1}$ for $d > 2$. Identifying a good collection of well-distributed points on the sphere is important for many problems. The main result in this section is to show that for a given positive integer N, the sphere $\mathbb{S}^{d-1}$ can be partitioned into N equal-area patches in an orderly manner. Once such a partition is established, we can take, say, the central point of each patch to get a collection of well-distributed points.

The existence of an equal-area partition is intuitively obvious, and our main task is to provide an algorithm that shows how a desired partition is arranged. To be more precise, we give a definition.

Definition 6.4.1. A finite collection $\mathscr{R} := \{R_1, \ldots, R_N\}$ of closed subsets of $\mathbb{S}^{d-1}$ is called an area-regular partition of $\mathbb{S}^{d-1}$ if

(1) $\mathbb{S}^{d-1} = \bigcup_{j=1}^{N} R_j$ and $R_i^\circ \cap R_j^\circ = \emptyset$ for $1 \le i \ne j \le N$, where R_i° denotes the interior of R_i;
(2) $\operatorname{meas}(R_j) = N^{-1} \operatorname{meas}(\mathbb{S}^{d-1})$ for all $1 \le j \le N$.

The partition norm of $\mathscr{R}$ is defined by

$$\|\mathscr{R}\| := \max_{R \in \mathscr{R}} \max_{x,y \in R} \|x - y\|.$$

The theorem below shows that there exists an area-regular partition of the sphere with the additional property that each R_i in the partition is a product domain in spherical coordinates.

Theorem 6.4.2. *For each positive integer N, there exists an area-regular partition $\mathscr{R} := \{R_1, \ldots, R_N\}$ of the sphere $\mathbb{S}^{d-1}$ with partition norm $\|\mathscr{R}\| \le c_d N^{-\frac{1}{d-1}}$.*

Proof. The assertion holds obviously if $N = 1, 2$ or $d = 2$. Hence, we assume that $d \ge 3$ and $N \ge 3$. To illustrate the idea, we start with the case $d = 3$.

Let $n = \lfloor \sqrt{N} - 3/5 \rfloor$ and let $\theta_k \in [0, \pi/2]$, $0 \le k \le n$, be defined by $\cos\theta_k = 1 - \frac{k(k+1)}{N}$. Then $0 = \theta_0 < \theta_1 < \cdots < \theta_n < \frac{\pi}{2}$. Furthermore, $(\sin\frac{\theta_k}{2})^2 = \frac{k(k+1)}{2N}$ and $\frac{2k}{N} = \cos\theta_{k-1} - \cos\theta_k = 2\sin\frac{\theta_k - \theta_{k-1}}{2} \sin\frac{\theta_k + \theta_{k-1}}{2}$. It follows that

$$\theta_k \sim kN^{-\frac{1}{2}}, \quad \theta_k - \theta_{k-1} \sim N^{-\frac{1}{2}}, \quad N - n(n+1) \sim N^{\frac{1}{2}}. \tag{6.4.1}$$

We first partition $\mathbb{S}^2$ into parallel spherical belts. Let $e = (0, 0, 1)$ and

$$A_k^+ := \{x \in \mathbb{S}^2 : \theta_{k-1} \le \mathtt{d}(x, e) \le \theta_k\}, \quad 1 \le k \le n,$$

$$B_n := \{x \in \mathbb{S}^2 : \theta_n \le \mathtt{d}(x, e) \le \pi - \theta_n\},$$

$$A_k^- := \{x \in \mathbb{S}^2 : \pi - \theta_k < \mathtt{d}(x, e) \le \pi - \theta_{k-1}\}, \quad 1 \le k \le n.$$

The sets A_1^+ and A_1^- are spherical caps centered at the north pole and the south pole, respectively. For $2 \le k \le n$, A_k^+ and A_k^- are spherical belts in the northern and the southern hemispheres, respectively, and B_n is a spherical belt around the equator. A straightforward computation shows that

$$\operatorname{meas}(A_k^+) = \operatorname{meas}(A_k^-) = 2\pi \int_{\theta_{k-1}}^{\theta_k} \sin\theta \,\mathrm{d}\theta = \frac{4\pi k}{N}, \quad 1 \le k \le n,$$
$$\operatorname{meas}(B_n) = 4\pi \cos\theta_n = \frac{4\pi(N - n(n+1))}{N}.$$

Next we partition each spherical belt into patches of equal area. Let

$$I_{k,j} := [\theta_{k-1}, \theta_k] \times \left[\frac{2(j-1)\pi}{k}, \frac{2j\pi}{k}\right], \quad 1 \le j \le k \le n,$$
$$J_{n,j} := [\theta_n, \pi - \theta_n] \times \left[\frac{2(j-1)\pi}{N - n(n+1)}, \frac{2j\pi}{N - n(n+1)}\right], \quad 1 \le j \le N - n(n+1).$$

Write $\xi_\varphi = (\cos\varphi, \sin\varphi)$ for $\varphi \in [0, 2\pi]$. Define

$$A_{k,j}^+ := \left\{(\xi_\varphi \sin\theta, \cos\theta) \in A_k^+ : (\theta, \varphi) \in I_{k,j}\right\}, \quad 1 \le j \le k \le n,$$
$$A_{k,j}^- := \left\{(\xi_\varphi \sin\theta, \cos\theta) \in A_k^- : (\pi - \theta, \varphi) \in I_{k,j}\right\}, \quad 1 \le j \le k \le n,$$
$$B_{n,j} := \left\{(\xi_\varphi \sin\theta, \cos\theta) \in B_n : (\theta, \varphi) \in J_{n,j}\right\}, \quad 1 \le j \le N - n(n+1).$$

Evidently, the sets $A_{k,j}^+, B_{k,j}, A_{k,n}^-$ have disjoint interiors, and their union is the whole sphere $\mathbb{S}^2$. Using spherical coordinates, it follows immediately form the definition that

$$\operatorname{meas}(A_{k,j}^+) = \operatorname{meas}(A_{k,j}^-) = \frac{\operatorname{meas}(A_k)}{k} = \frac{4\pi}{N}, \quad 1 \le j \le k \le n,$$
$$\operatorname{meas}(B_{n,j}) = \frac{\operatorname{meas}(A_{n+1})}{N - n(n+1)} = \frac{4\pi}{N}, \quad 1 \le j \le N - n(n+1).$$

Thus, all these patches have the same measure ω_3/N, which in turn implies that the number of these patches is N. Finally, since $\|\xi_\varphi \sin\theta - \xi_{\varphi'} \sin\theta'\|^2 = (\sin\theta - \sin\theta')^2 + 2\sin\theta\sin\theta' \|\xi_\varphi - \xi_{\varphi'}\|^2$, it follows from Eq. (6.4.1) that for $1 \le j \le k \le n \sim \sqrt{N}$,

$$\operatorname{diam}(A_{k,j}^\pm) \sim \sup_{(\theta,\varphi),(\theta',\varphi') \in I_{k,j}} \left[|\cos\theta - \cos\theta'| + \|\xi_\varphi \sin\theta - \xi_{\varphi'} \sin\theta'\|\right]$$
$$\sim (\cos\theta_{k-1} - \cos\theta_k) + (\theta_k - \theta_{k-1}) + \frac{\theta_k}{k} \sim N^{-\frac{1}{2}},$$

and that for $1 \le j \le N - n(n+1) \sim \sqrt{N}$,

$$\operatorname{diam}(B_{n,j}) \sim 2\cos\theta_n + (1 - \sin\theta_n) + \frac{1}{N - n(n+1)} \sim N^{-\frac{1}{2}}.$$

This completes the assertion for the case $\mathbb{S}^2$.

The partition of $\mathbb{S}^{d-1}$ can be carried out via induction. Let us assume now that the assertion is true for $\mathbb{S}^{d-2}$ and proceed with $\mathbb{S}^{d-1}$ along the same lines as the case $\mathbb{S}^2$. Let $n = \lfloor \sqrt{N} - 3/5 \rfloor$ and let θ_k, $0 = \theta_0 < \theta_1 < \cdots < \theta_n < \frac{\pi}{2}$, be defined by $\int_0^{\theta_k} (\sin\theta)^{d-2} \mathrm{d}\theta = \frac{\omega_d}{\omega_{d-1}} \frac{k(k+1)}{2N}$. The existence of such θ_k is obvious, since $g(t) := \int_0^t (\sin\theta)^{d-2} \mathrm{d}\theta$ is evidently an increasing function and $g(\frac{\pi}{2}) = \frac{1}{2}\frac{\omega_d}{\omega_{d-1}}$. For convenience, we identity a point $(\xi \sin\theta, \cos\theta)$ on $\mathbb{S}^{d-1}$ with $(\theta, \xi) \in [0,\pi] \times \mathbb{S}^{d-2}$, and denote by $\mathrm{d}\sigma_m$ the usual Lebesgue measure on $\mathbb{S}^{m-1}$. Then

$$\int_{\mathbb{S}^{d-1}} f(x)\,\mathrm{d}\sigma_d(x) = \int_{\mathbb{S}^{d-2}} \int_0^{\pi} f(\theta,\xi)(\sin\theta)^{d-2} \mathrm{d}\theta\,\mathrm{d}\sigma_{d-1}(\xi). \tag{6.4.2}$$

We can then partition $\mathbb{S}^{d-1}$ into parallel spherical belts $A_k^{\pm}$ and B_n exactly as in the case $d = 3$. With $e = (0, \ldots, 0, 1)$, we have then

$$\begin{aligned} A_k^+ &= \{(\theta,\xi) \in \mathbb{S}^{d-1} : \theta_{k-1} \le \theta \le \theta_k\}, \quad 1 \le k \le n, \\ B_n^+ &= \{(\theta,\xi) \in \mathbb{S}^{d-1} : \theta_n \le \theta \le \pi - \theta_n\}, \\ A_k^- &= \{(\theta,\xi) \in \mathbb{S}^{d-1} : \pi - \theta_k \le \theta \le \pi - \theta_{k-1}\}, \quad 1 \le k \le n. \end{aligned}$$

By the definition of θ_k, it follows readily that

$$\begin{aligned} \operatorname{meas}(A_k^+) = \operatorname{meas}(A_k^-) &= \omega_{d-1} \int_{\theta_{k-1}}^{\theta_k} (\sin\theta)^{d-2} \mathrm{d}\theta = \frac{k}{N}\omega_d, \quad 1 \le k \le n, \\ \operatorname{meas}(B_n) &= \omega_d - 2\sum_{k=1}^{n} \operatorname{meas}(A_k^+) = \frac{N - n(n+1)}{N}\omega_d. \end{aligned}$$

Next, we partition $A_k^{\pm}$ equally using the area-regular partition

$$\mathscr{R}_k := \{R_{k,1}, \ldots, R_{k,k}\}, \quad \text{with } \|\mathscr{R}_k\| \le c_d k^{-\frac{1}{d-2}},$$

of $\mathbb{S}^{d-2} = \cup_{j=1}^{k} R_{k,j}$, which exists by the induction hypothesis, and define

$$\begin{aligned} A_{k,j}^+ &:= \{(\theta,\xi) : \theta_{k-1} \le \theta \le \theta_k, \xi \in E_{k,j}\}, \quad 1 \le j \le k \le n, \\ A_{k,j}^- &:= \{(\theta,\xi) : \pi - \theta_k \le \theta \le \pi - \theta_{k-1}, \xi \in E_{k,j}\}, \quad 1 \le j \le k \le n; \end{aligned}$$

furthermore, we partition B_n equally using the area-regular partition

$$\mathscr{R}_0 := \{R_{0,1}, \dots, R_{0,M}\}, \quad \|\mathscr{R}_0\| \le c_d M^{-\frac{1}{d-2}}, \quad M := N - n(n+1),$$

of $\mathbb{S}^{d-2} = \bigcup_{j=1}^{M} R_{0,j}$, which again exists by the induction hypothesis, and define

$$B_{n,j} := \{(\theta, \xi) : \theta_n \le \theta \le \pi - \theta_n, \xi \in E_{0,j}\}, \quad 1 \le j \le N - n(n+1).$$

Using Eq. (6.4.2) and following the proof in the case $\mathbb{S}^2$, one can easily verify that the $A^{\pm}_{k,j}$ and $B_{n,j}$ constitute an area-regular partition of $\mathbb{S}^{d-1}$ with partition norm $\le C_d N^{-\frac{1}{d-1}}$. This completes the proof. □

A couple of remarks are in order. Clearly, the partition is not unique. For example, in the construction given in the proof, the result still holds if partitions in any spherical belt are rotated by an angle around the x_{d+1}-axis. In the case $N = n(n+1)$, the construction can be carried out with the belt B_n the empty set.

Since each element of the partitions in the proof of the theorem is a product region in a product domain in spherical coordinates, it has a center point. The collection of the central points is a well-distributed set of points in $\mathbb{S}^{d-1}$. Recall that the concept of extended maximal ε-separated set is defined in Remark 5.3.3.

Corollary 6.4.3. *Let $d = 3$ and let $\mathscr{R} = \{R_1, \dots, R_N\}$ be the area-regular partition of $\mathbb{S}^2$ as constructed in the proof of Theorem 6.4.2. Let x_i be the center point of R_i. Then $\Lambda = \{x_1, \dots, x_N\}$ is an extended maximal $\frac{c}{n}$-separated subset of $\mathbb{S}^2$ for some absolute constant $c > 0$, where $n = \lfloor \sqrt{N} \rfloor$.*

Proof. This is an immediate consequence of the proof of Theorem 6.4.2. Indeed, the distance between the center of two neighboring patches is proportional to the diameter of the patches, which is of order $\sqrt{N}$. □

As an immediate application of area-regular partition, we state a result on the collection of points pertinent to the partition in which $x_i \in R_i$ does not have to be the center point of R_i.

Theorem 6.4.4. *Let $\mathscr{R} = \{R_1, \dots, R_N\}$ be an area-regular partition of $\mathbb{S}^{d-1}$ and $\{x_i : x_i \in R_i, 1 \le i \le N\}$ a collection of points in $\mathbb{S}^{d-1}$. There exists a constant $r_d \in (0,1)$, depending only on d, such that if $\|\mathscr{R}\| \le r_d m^{-1}$ for some $m \in \mathbb{N}$, then for each $f \in \Pi_m(\mathbb{S}^{d-1})$,*

$$\frac{1}{2}\|f\|_{L^1(\mathbb{S}^{d-1})} \le \frac{1}{N}\sum_{j=1}^{N} |f(x_j)| \le \frac{3}{2}\|f\|_{L^1(\mathbb{S}^{d-1})}.$$

Proof. Recall that $\operatorname{osc}(f)(x,r) := \sup_{y,z \in \mathsf{c}(x,r)} |f(y) - f(z)|$. For $f \in \Pi_m(\mathbb{S}^{d-1})$,

$$\left| \frac{1}{N} \sum_{j=1}^{N} |f(x_j)| - \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} |f(x)| \, \mathrm{d}\sigma(x) \right| \le \frac{1}{\omega_d} \sum_{j=1}^{N} \int_{R_j} |f(x_j) - f(x)| \, \mathrm{d}\sigma(x)$$

$$\le \frac{1}{\omega_d} \sum_{j=1}^{N} \int_{R_j} \operatorname{osc}(f)(x, r_d m^{-1}) \, \mathrm{d}\sigma(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \operatorname{osc}(f)(x, r_d m^{-1}) \, \mathrm{d}\sigma(x),$$

which, by Lemma 5.2.5 and Corollary 5.2.3, is dominated by

$$c_d r_d \int_{\mathbb{S}^{d-1}} f^*_{d,m}(x) \, \mathrm{d}\sigma(x) \le c'_d r_d \|f\|_1.$$

Choosing $r_d = \frac{1}{2c'_d}$ proves the desired inequality. □

Corollary 6.4.5. *Under the assumption of Theorem 6.4.4, for each $f \in \Pi_m(\mathbb{S}^{d-1})$,*

$$\frac{1}{2\sqrt{d}} \|\nabla_0 f\|_{L^1(\mathbb{S}^{d-1})} \le \frac{1}{N} \sum_{j=1}^{N} \|\nabla_0 f(x_j)\| \le \frac{3}{2} \sqrt{d} \|\nabla_0 f\|_{L^1(\mathbb{S}^{d-1})}.$$

Proof. Write $\nabla_0 f = (f_1, \ldots, f_d)$ for $f \in \Pi_m(\mathbb{S}^{d-1})$. By definition, each f_j is a spherical polynomial of degree at most m on $\mathbb{S}^{d-1}$. Since

$$\|\nabla_0 f\| = \sqrt{f_1^2 + \cdots + f_d^2} \le |f_1| + \cdots + |f_d| \le \sqrt{d} \|\nabla_0 f\|,$$

the proof follows by applying Theorem 6.4.4 to each component f_j. □

6.5 Spherical Designs

A spherical design of degree n, or n-design, is a finite set of N points on $\mathbb{S}^{d-1}$ such that the average value of every polynomial $f \in \Pi_n(\mathbb{S}^{d-1})$ on the set equals the average value of f on $\mathbb{S}^{d-1}$. In other words, an n-design is a cubature formula of degree n on the sphere with equal cubature weights,

$$\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(x) \mathrm{d}\sigma(x) = \frac{1}{N} \sum_{k=1}^{N} f(x_k), \qquad f \in \Pi_n(\mathbb{S}^{d-1}), \tag{6.5.1}$$

where $x_k \in \mathbb{S}^{d-1}$. Spherical designs are intimately related to combinatorics, where the name n-design originated, as well as to isomorphic embedding of classical Banach spaces, statistics, and, naturally, approximation theory.

The additional constraint of equal weights means that only the nodes are at our disposal, which makes it harder to construct such cubature formulas. On the other hand, the equal-weights constraint allows us to apply tools in geometry and analysis to facilitate construction, as can be seen in Example 6.1.8.

The lower bound for the number of nodes in Eq. (6.1.2) clearly applies to equal-weights cubature formulas. The additional constraint suggests that number of nodes in such cubature formulas is in general greater than those in ordinary cubature formulas of the same degree. It has long been conjectured, however, that a spherical n-design containing $\mathscr{O}(n^{d-1})$ nodes on $\mathbb{S}^{d-1}$ exists. Recently, this conjecture was confirmed in [19]. The result is the following theorem.

Theorem 6.5.1. *There exists a positive constant K_d, depending only on d, such that for each positive integer $N \ge K_d n^{d-1}$, there exists a set of N points $x_1, \dots, x_N \in \mathbb{S}^{d-1}$ for which Eq. (6.5.1) holds for all $f \in \Pi_n(\mathbb{S}^{d-1})$.*

The rest of this section is devoted to the proof of this theorem. The proof is based on the following lemma in Brouwer degree theory, the proof of which can be found in [134, Theorems 1.2.6 and 1.2.9].

Lemma 6.5.2. *Let V be a real finite-dimensional space with inner product $\langle \cdot, \cdot \rangle_V$. Let Ω be a bounded open subset of V with boundary $\partial\Omega$ and such that $0 \in \Omega$. If $\mathscr{F} : V \mapsto V$ is a continuous mapping satisfying $\langle x, \mathscr{F}(x) \rangle_V > 0$ for all $x \in \partial\Omega$, then there exists a point $x \in \Omega$ such that $\mathscr{F}(x) = 0$.*

For the proof of Theorem 6.5.1, we will use the Brouwer degree theorem with V the space of polynomials in $\Pi_n(\mathbb{S}^{d-1})$ with zero mean and inner product of $L^2(\mathbb{S}^{d-1})$. To construct the mapping $\mathscr{F}$, we start with a set of points taken from an area-regular partition and change the points along paths in the direction of the gradient, which is permissible because of the lemma below on the initial value problem of a system of first-order differential equations. First we need a couple of definitions.

Every polynomial in $\Pi_n(\mathbb{S}^{d-1})$ can be expanded as a sum of spherical harmonics and hence has a unique extension to a solid harmonic polynomial on $\mathbb{R}^d$. Let $\mathscr{A}_n^d$ denote the space of harmonic polynomials of degree at most n on $\mathbb{R}^d$. Since each harmonic polynomial in $\mathscr{A}_n^d$ is uniquely determined by its restriction to the sphere $\mathbb{S}^{d-1}$, $\|\cdot\|_{L^2(\mathbb{S}^{d-1})}$ is a norm of the space $\mathscr{A}_n^d$. For $f \in \mathscr{A}_n^d$ and $y \in \mathbb{R}^d$, we define an operator $\mathscr{D}$ by

$$\mathscr{D}f(y) := \|y\|^2 \nabla f(y) - (y, \nabla f(y))y.$$

By Eq. (1.8.12), $\mathscr{D}f(\xi) = \nabla_0 f(\xi)$ for $\xi \in \mathbb{S}^{d-1}$. We need a normalization of $\mathscr{D}f$ that is bounded. Let $\varepsilon > 0$ be a fixed number depending only on d, and define the mapping $\Phi : \mathscr{A}_n^d \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\Phi(f, y) := \frac{\mathscr{D}f(y)}{h_\varepsilon(\|\mathscr{D}f(y)\|)} \quad \text{with} \quad h_\varepsilon(t) := \max\{t, \varepsilon\}.$$

Lemma 6.5.3. *For each fixed $P \in \mathscr{A}_n^d$ and $x \in \mathbb{S}^{d-1}$, the system of differential equations*

$$\begin{cases} y'(t) = \boldsymbol{\Phi}(P, y(t)), & t \ge 0, \\ y(0) = x, \end{cases} \tag{6.5.2}$$

has a unique solution $y = y(P,t)$ *for* $t \in [0,\infty)$. *Moreover,* $y(P,t) \in \mathbb{S}^{d-1}$*, and the mapping* $P \mapsto y(P,t)$ *is continuous on* $\mathscr{A}_n^d$ *for each fixed* $t \ge 0$.

Proof. It suffices to prove the assertion for $t \in [0,M]$ for an arbitrary $M > 2$. The proof uses the standard Picard successive approximation. The initial value problem (6.5.2) has a solution $y = \phi(P,t)$ if and only if

$$\phi(P,t) = x + \int_0^t \boldsymbol{\Phi}(P, \phi(P,s))\,\mathrm{d}s. \tag{6.5.3}$$

For a fixed $x \in \mathbb{S}^{d-1}$ and a fixed $P \in \mathscr{A}_n^d$, we define $\phi_0(P,t) = x \in \mathbb{S}^{d-1}$ and

$$\phi_k(P,t) := x + \int_0^t \boldsymbol{\Phi}(P, \phi_{k-1}(P,s))\,\mathrm{d}s, \quad k = 1, 2, \ldots, \tag{6.5.4}$$

for $t \in [0,M]$. Let $E := \{y \in \mathbb{R}^d : \|y - x\| \le M\}$. The definition of h_ε implies that $\|\boldsymbol{\Phi}(P,y)\| \le 1$ for all $y \in \mathbb{R}^d$, from which it follows that $\phi_k(P,t) \in E$ for all k and $t \in [0,M]$. Since each component of $\mathscr{D}P$ is a polynomial of degree at most $n+1$, and h_ε is a Lipschitz function on $[0,\infty)$ that is bounded below by ε, it is easily seen, by the triangle inequality, that

$$\|\boldsymbol{\Phi}(P,y) - \boldsymbol{\Phi}(P,z)\| \le L\|y - z\|, \quad y, z \in E, \quad P \in \mathscr{A}_n^d, \tag{6.5.5}$$

where the constant L is equal to $L_{P,E,\varepsilon}$. Consequently, by induction on k, it is easily seen that for all $t \in [0,M]$,

$$\|\phi_k(P,t) - \phi_{k-1}(P,t)\| \le \frac{L^{k-1} t^k}{k!}, \quad t \in [0,M],$$

from which we deduce that

$$\sum_{k=1}^{\infty} \sup_{t \in [0,M]} \|\phi_k(P,t) - \phi_{k-1}(P,t)\| \le L^{-1} e^{ML} < \infty.$$

Since $\phi_k = \phi_0 + (\phi_1 - \phi_0) + \cdots + (\phi_k - \phi_{k-1})$, it follows that the sequence $\{\phi_k\}_{k=1}^\infty$ converges uniformly to a function $\phi(t) \equiv \phi(P,t)$ on $[0,M]$. Letting $k \to \infty$ in Eq. (6.5.4) shows that $\phi(P,t)$ satisfies Eq. (6.5.3); hence it is a solution of the initial value problem (6.5.2). Furthermore, the definition of $\mathscr{D}P$ shows that $\langle y, \mathscr{D}P(y) \rangle = 0$, and hence $\langle y, \boldsymbol{\Phi}(P,y) \rangle = 0$. Thus, for all $y \in \mathbb{R}^d$, by Eq. (6.5.2),

$$\frac{\partial}{\partial s}(\|\phi(P,s)\|^2) = 2\Big\langle \phi(P,s), \frac{\partial \phi(P,s)}{\partial s} \Big\rangle = 2\big\langle \phi(P,s), \boldsymbol{\Phi}\big(P, \phi(P,s)\big) \big\rangle = 0.$$

Hence $\|\phi(P,s)\|$ is a constant, and moreover, $\|\phi(P,s)\| = \|\phi(P,0)\| = \|x\| = 1$, or $\phi(P,s) \in \mathbb{S}^{d-1}$, for all $s \in [0,M]$.

Next, we show that the solution is unique. Assume that $\phi_1(s)$ and $\phi_2(s)$ are two solutions of Eq. (6.5.2) on $[0,M]$. Then the above argument shows that $\phi_1(s), \phi_2(s) \in \mathbb{S}^{d-1} \subset E$, and using Eqs. (6.5.3) and (6.5.5),

$$\|\phi_1(t) - \phi_2(t)\| \le L \int_0^t \|\phi_1(s) - \phi_2(s)\| \mathrm{d}s.$$

Setting $g(t) := \int_0^t \|\phi_1(s) - \phi_2(s)\| \, \mathrm{d}s$, we can rewrite the last equation as $g'(t) \le Lg(t)$, or equivalently, $(g(t)e^{-Lt})' \le 0$. Thus, $0 \le g(t)e^{-Lt} \le g(0) = 0$. This shows that $g(t) = 0$ for all $t \in [0,M]$. Thus, $\phi_1 = \phi_2$.

Finally, we show that $\phi(P,s)$ is continuous in $P \in \mathscr{A}_n^d$ for each fixed $s \ge 0$. Since h_ε is a Lipschitz function, the triangle inequality shows that

$$\sup_{y \in E} \|\Phi(P,y) - \Phi(Q,y)\| \le C\varepsilon^{-1} \sup_{y \in E} \|\mathscr{D}P(y) - \mathscr{D}Q(y)\|.$$

Evidently, we can add an extra term $\|P - Q\|_{L^2(\mathbb{S}^{d-1})}$ on the right-hand side of the last inequality. Since $\sup_{y \in E} \|\mathscr{D}f(y)\| + \|f\|_{L^2(\mathbb{S}^{d-1})}$ is a norm of $\mathscr{A}_n^d$ and different norms on a finite-dimensional space are equivalent, we conclude that for $P, Q \in \mathscr{A}_n^d$,

$$\sup_{y \in E} \|\Phi(P,y) - \Phi(Q,y)\| \le C_{E,n,\varepsilon} \|P - Q\|_{L^2(\mathbb{S}^{d-1})}.$$

Consequently, by Eqs. (6.5.4) and (6.5.5), it is easy to see that

$$\|\phi_k(P,t) - \phi_k(Q,t)\| \le C_{E,n,\varepsilon} t \|P - Q\|_{L^2(\mathbb{S}^{d-1})} + L \int_0^t \|\phi_{k-1}(P,s) - \phi_{k-1}(Q,s)\| \mathrm{d}s,$$

so that by induction on k with $\|\phi_0(P,t) - \phi_0(Q,t)\| = 0$ as a starting point, for $P, Q \in \mathscr{A}_n^d$ and $t \in [0,M]$,

$$\begin{aligned} \|\phi_k(P,t) - \phi_k(Q,t)\| &\le C_{E,n,\varepsilon} \sum_{j=0}^{k-1} \frac{L^j t^{j+1}}{(j+1)!} \|P - Q\|_{L^2(\mathbb{S}^{d-1})} \\ &\le C_{E,n,\varepsilon} L^{-1} e^{LM} \|P - Q\|_{L^2(\mathbb{S}^{d-1})}, \end{aligned}$$

where L depends on P but is independent of Q. Letting $k \to \infty$, we conclude that $\phi(P,s)$ is continuous in $P \in \mathscr{A}_n^d$. □

The space V in our application of the Brouwer degree theorem is defined by

$$\Pi_{n,0}^d := \left\{ f \in \Pi_n(\mathbb{S}^{d-1}) : \int_{\mathbb{S}^{d-1}} f(y) \, \mathrm{d}\sigma(y) = 0 \right\}.$$

The following is the key lemma for the proof of Theorem 6.5.1.

Lemma 6.5.4. *For each positive integer $N \ge K_d n^{d-1}$ with K_d a sufficiently large positive constant, there exists a continuous mapping $P \mapsto (x_1(P),\dots,x_N(P))$ from $\Pi_{n,0}^d$ to $(\mathbb{S}^{d-1})^N$, the N-fold product of $\mathbb{S}^{d-1}$, such that*

$$\sum_{j=1}^{N} P(x_j(P)) > 0 \tag{6.5.6}$$

whenever $P \in \Pi_{n,0}^d$ and $\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} \|\nabla_0 P(x)\|\,\mathrm{d}\sigma(x) = 1$.

Proof. Let $\mathscr{R} = \{R_1,\dots,R_N\}$ be an area-regular partition of $\mathbb{S}^{d-1}$ with

$$\|\mathscr{R}\| \le c_d N^{-\frac{1}{d-1}} \le c_d K_d^{-\frac{1}{d-1}} n^{-1}.$$

For each i, $1 \le i \le N$, choose a point $x_i \in R_i$. For a fixed $P \in \Pi_{n,0}^d$ and a fixed x_i, consider the initial value problem

$$\begin{cases} y_i'(s) = \dfrac{\mathscr{D}P(y_i)}{h_\varepsilon(\|\mathscr{D}P(y_i)\|)}, & s \ge 0, \\ y_i(0) = x_i, \end{cases}$$

where $\varepsilon = \frac{1}{4\sqrt{d}}$. By Lemma 6.5.3, this initial value problem has a solution $y_i(s) = y_i(P,s)$ for $s \in [0,\infty)$, which is continuous in $P \in \Pi_{n,0}^d$ and $y_i(P,s) \in \mathbb{S}^{d-1}$. Since $\nabla_0 P(\xi) = \mathscr{D}P(\xi)$ for $\xi \in \mathbb{S}^{d-1}$, the solution $y_i(s)$ satisfies

$$y_i'(s) = \frac{\nabla_0 P(y_i(s))}{h_\varepsilon(\|\nabla_0 P(y_i(s))\|)}, \quad s \ge 0. \tag{6.5.7}$$

Let r_d be the constant in Theorem 6.4.4. Setting

$$x_j(P) := y_j\left(\frac{r_d}{3n}\right) = y_j\left(P, \frac{r_d}{3n}\right), \quad 1 \le j \le N,$$

we see that $P \mapsto (x_1(P),\dots,x_N(P))$ is a continuous mapping from $\Pi_{n,0}^d$ to $(\mathbb{S}^{d-1})^N$. We now verify Eq. (6.5.6) for $P \in \Pi_{n,0}^d$ with $\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} ||\nabla_0 P(x)||\mathrm{d}\sigma(x) = 1$. We write

$$\frac{1}{N}\sum_{j=1}^{N} P(x_j(P)) = \frac{1}{N}\sum_{j=1}^{N}\Big[P(x_j(P)) - P(x_j)\Big] + \frac{1}{N}\sum_{j=1}^{N} P(x_j) =: \Sigma_1 + \Sigma_2.$$

First, we estimate Σ_1 from below. Since $y_j(0) = x_j$ and $y_j(s) \in \mathbb{S}^{d-1}$,

$$\Sigma_1 = \frac{1}{N}\sum_{j=1}^{N}\left[P\left(y_j(\tfrac{r_d}{3n})\right) - P(y_j(0))\right] = \frac{1}{N}\sum_{j=1}^{N}\int_0^{\frac{r_d}{3n}} \frac{d}{\mathrm{d}s}\,[P(y_j(s))]\,\mathrm{d}s$$

$$= \frac{1}{N} \sum_{j=1}^{N} \int_0^{\frac{r_d}{3n}} \langle \nabla_0 P(y_j(s)), y_j'(s) \rangle \mathrm{d}s = \int_0^{\frac{r_d}{3n}} \left[\frac{1}{N} \sum_{j=1}^{N} \langle \nabla_0 P(y_j(s)), y_j'(s) \rangle \right] \mathrm{d}s.$$

However, by Eq. (6.5.7) and the definition of h_ε,

$$\frac{1}{N} \sum_{j=1}^{N} \nabla_0 P(y_j(s)) \cdot y_j'(s) = \frac{1}{N} \sum_{j=1}^{N} \frac{\|\nabla_0 P(y_j(s))\|^2}{h_\varepsilon(\|\nabla_0 P(y_j(s))\|)}$$

$$\geq \frac{1}{N} \sum_{\substack{\|\nabla_0 P(y_j(s))\| \geq \varepsilon \\ 1 \leq j \leq N}} \|\nabla_0 P(y_j(s))\| \geq \frac{1}{N} \sum_{j=1}^{N} \|\nabla_0 P(y_j(s))\| - \varepsilon.$$

For each $0 \leq s \leq \frac{r_d}{3n}$, since $\|y_j'(s)\| \leq 1$ by Eq. (6.5.7), we have

$$\|y_j(s) - x_j\| \leq s \leq \frac{r_d}{3n}.$$

Let $\mathscr{R}' = \{R_1', \ldots, R_N'\}$, where $R_j' = R_j \cup \{y_j(s)\}$, for $0 \leq s \leq \frac{r_d}{3n}$. For fixed s, $\mathscr{R}'$ is an area-regular partition of $\mathbb{S}^{d-1}$ with

$$\|\mathscr{R}'\| \leq \|\mathscr{R}\| + \frac{r_d}{3n} \leq c_d K_d^{-\frac{1}{d-1}} n^{-1} + \frac{r_d}{3n} \leq \frac{r_d}{n},$$

where we choose K_d large enough that $c_d K^{-\frac{1}{d-1}} \leq r_d/2$. Thus, using Corollary 6.4.5,

$$\frac{1}{N} \sum_{j=1}^{N} \|\nabla_0 P(y_j(s)\| \geq \frac{1}{2\sqrt{d}} \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \|\nabla_0 P(y)\| \mathrm{d}\sigma(y) = \frac{1}{2\sqrt{d}},$$

which in turn implies, recalling $\varepsilon = \frac{1}{4\sqrt{d}}$, that

$$\Sigma_1 \geq \left(\frac{1}{2\sqrt{d}} - \varepsilon \right) \frac{r_d}{3n} \geq \frac{1}{12\sqrt{d}} \frac{r_d}{n}.$$

Next we estimate the sum Σ_2 from above. Since $\int_{\mathbb{S}^{d-1}} P(y) \mathrm{d}\sigma(y) = 0$, we have

$$\Sigma_2 = \frac{1}{N} \sum_{j=1}^{N} P(x_j) - \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} P(x) \mathrm{d}\sigma(x) = \frac{1}{\omega_d} \sum_{j=1}^{N} \int_{R_j} (P(x_j) - P(x)) \mathrm{d}\sigma(x)$$

$$\leq \frac{1}{\omega_d N} \sum_{j=1}^{N} \max_{x \in \mathsf{c}(x_j, \|\mathscr{R}\|)} \|\nabla_0 P(x)\| \|\mathscr{R}\| = \|\mathscr{R}\| \frac{1}{\omega_d N} \sum_{j=1}^{N} \|\nabla_0 P(z_j)\|,$$

where $z_j \in \mathsf{c}(x_j, \|\mathscr{R}\|)$ such that $\|\nabla_0 P(z_j)\| = \max_{x\in \mathsf{c}(x_j,\|\mathscr{R}\|)} \|\nabla_0 P(x)\|$. Let $\mathscr{R}'' = \{R_1'', \dots, R_N''\}$, where $R_i'' = R_i \cup \{z_i\}$. Then $\mathscr{R}''$ is an area-regular partition of $\mathbb{S}^{d-1}$ with

$$\|\mathscr{R}''\| \le 2\|\mathscr{R}\| \le 2c_d K_d^{-\frac{1}{d-1}} n^{-1} \le \frac{r_d}{n}.$$

Thus, using Corollary 6.4.5,

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^{N} P(x_j) &\le \|\mathscr{R}\| \frac{3}{2}\sqrt{d}\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} \|\nabla_0 P(z)\| \mathrm{d}\sigma(z)\\ &\le \frac{3}{2}\frac{\sqrt{d}\,c_d}{\omega_d} K_d^{-\frac{1}{d-1}} n^{-1} = c_d' K_d^{-\frac{1}{d-1}} n^{-1}.\end{aligned}$$

Putting the above together, we conclude that

$$\frac{1}{N}\sum_{j=1}^{N} P(x_j(P)) = \Sigma_1 + \Sigma_2 \ge \frac{1}{12\sqrt{d}}\frac{r_d}{n} - \frac{1}{n} c_d' K_d^{-\frac{1}{d-1}} > 0,$$

provided that K_d is large enough. This completes the proof. □

We are now in a position to give the proof of the main theorem.

Proof of Theorem 6.5.1. Let $\Omega := \{P \in \Pi_{n,0}^d : \int_{\mathbb{S}^{d-1}} \|\nabla_0 P(x)\| \mathrm{d}\sigma(x) < 1\}$. Then $0 \in \Omega$, and since $P \mapsto \int_{\mathbb{S}^{d-1}} \|\nabla_0 P(x)\|\, \mathrm{d}\sigma(x)$ is a norm on $\Pi_{n,0}^d$, Ω is an open set in $\Pi_{n,0}^d$ with boundary

$$\partial\Omega = \left\{ P \in \Pi_{n,0}^d : \int_{\mathbb{S}^{d-1}} \|\nabla_0 P(x)\|\, \mathrm{d}\sigma(x) = 1 \right\}. \tag{6.5.8}$$

Recall that $Z_n(x,y)$ denotes the reproducing kernel of $\mathscr{H}_n^d$ and that it is a zonal function. Let $G_n(\langle x,y\rangle) = \sum_{j=1}^{N} Z_j(x,y)$. Then G_n is a reproducing kernel for the space $\Pi_{n,0}^d$, that is,

$$P(x) = \big\langle P, G_n(\langle x, \cdot\rangle)\big\rangle_{L^2(\mathbb{S}^{d-1})}, \quad x \in \mathbb{S}^{d-1}, \quad P \in \Pi_{n,0}^d.$$

Let $P \mapsto (x_1(P), \dots, x_N(P))$ be the continuous mapping from $\Pi_{n,0}^d$ to $(\mathbb{S}^{d-1})^N$ in Lemma 6.5.4. Define the mapping $\mathscr{F} : \Pi_{n,0}^d \to \Pi_{n,0}^d$ by

$$\mathscr{F}P(y) := \sum_{j=1}^{N} G_n(\langle x_j(P), y\rangle), \quad y \in \mathbb{S}^{d-1}.$$

Since G_n is continuously differentiable on $[-1,1]$, for $1 \le j \le N$ and $P, Q \in \Pi_{n,0}^d$,

$$\int_{\mathbb{S}^{d-1}} |G_n(\langle x_j(P), y\rangle) - G_n(\langle x_j(Q), y\rangle)|^2 \, \mathrm{d}\sigma(y)$$
$$\le C_n \int_{\mathbb{S}^{d-1}} |\langle x_j(P), y\rangle - \langle x_j(Q), y\rangle|^2 \, \mathrm{d}\sigma(y) \le C_n \|x_j(P) - x_j(Q)\|^2,$$

which implies, in particular, that $\mathscr{F}$ is continuous on $\Pi^d_{n,0}$.

Finally, we can apply the Brouwer degree theorem with $V = \Pi^d_0$. For each $P \in \partial\Omega$, by Eqs. (6.5.6) and (6.5.8), we have

$$\langle \mathscr{F}P, P\rangle_{L^2(\mathbb{S}^{d-1})} = \sum_{j=1}^{N} \langle G_n(x_j(P), \cdot\rangle), P\rangle_{L^2(\mathbb{S}^{d-1})} = \sum_{j=1}^{N} P(x_j(P)) > 0.$$

Hence by Lemma 6.5.2, there exists $P_0 \in \Omega$ such that

$$\mathscr{F}P_0(y) = \sum_{j=1}^{N} G_n(\langle x_j(P_0), y\rangle) = 0, \quad \forall y \in \mathbb{S}^{d-1}.$$

Now, for each $f \in \Pi_n(\mathbb{S}^{d-1})$, let $P_f := f - m_f$ with $m_f = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)\,\mathrm{d}\sigma(y)$. Then $P_f \in \Pi^d_{n,0}$, and for all $f \in \Pi_n(\mathbb{S}^{d-1})$,

$$\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N} f(x_j(P_0)) &= m_f + \frac{1}{N}\sum_{j=1}^{N} P_f(x_j(P_0)) \\
&= m_f + \frac{1}{N}\sum_{j=1}^{N} \big\langle P_f, G(\langle x_j(P_0), \cdot\rangle)\big\rangle_{L^2(\mathbb{S}^{d-1})} \\
&= m_f + \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} P_f(y)\mathscr{F}P_0(y)\mathrm{d}\sigma \\
&= m_f = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)\,\mathrm{d}\sigma(y).
\end{aligned}$$

This completes the proof. □

6.6 Notes and Further Results

For the general theory of cubature formulas, we refer to [163] and also [126]. The lower bound in Eq. (6.1.2) is classical. The lower bound in Eq. (6.1.4) and its proof in Eq. (6.1.3) were first given in [53]. For an unweighted integral, these lower bounds are sharp for only a few special cases [11, 157]. By choosing a more refined function F in Lemma 6.1.3, a stronger lower bound for the number of nodes for an unweighted integral was given in [171], which shows that for a cubature formula of degree n for $\int_{\mathbb{S}^{d-1}} f(x)\mathrm{d}\sigma$,

$$N \geq 2\frac{\int_0^1 (1-t^2)^{(d-3)/2}\mathrm{d}t}{\int_{\gamma_n}^1 (1-t^2)^{(d-3)/2}\mathrm{d}t},$$

where γ_n is the largest zero of $C_{n-1}^{d/2}(t)$. In particular, for $d=3$, this shows that

$$N \geq \frac{2}{1-\gamma_n} = \frac{4}{j_1^2} n^2(1+\mathscr{O}(n^{-1})) \geq 0.27244 n^2(1+\mathscr{O}(n^{-1})), \quad n \to \infty,$$

where j_1 is the first positive zero of the Bessel function $J_1(t)$ on using the known asymptotic formula for γ_s ([1, p. 787]). This asymptotic order is stronger than the one in Eq. (6.1.5), which states, for $d=3$, that $N \geq 0.25n^2(1+\mathscr{O}(n^{-1}))$.

Despite their drawback of nodes concentrated around poles, the product cubature formulas in Sect. 6.2 are widely used in applications because of their simplicity and lack of a viable alternative. For heavy numerical computations, especially in high-dimensional spheres, numerical methods such as quasi-Monte Carlo are often used; see, for example, [28, 87]. Computing good cubature rules on the sphere is a complicated problem that is hard to tackle; see [87, 153].

Theorem 6.1.7 was proved by Sobolev in [154], and it has been used to construct cubature formulas on $\mathbb{S}^2$ numerically. For example, positive cubature formulas invariant under the octahedral group with relatively small number of nodes were constructed in [105, 106], and the references therein, up to degree 131; see also [84], where such formulas are constructed using a connection to cubature formulas on the triangle. If positivity is not required, a family of cubature formulas invariant under the octahedral symmetry on $\mathbb{S}^{d-1}$, with remarkably small numbers of nodes, was constructed explicitly in [85] via a combinatorial method.

The first proof of the existence of positive cubature formulas via the Marcinkiewicz–Zygmund inequality on the unweighted sphere $\mathbb{S}^{d-1}$ was given in [122]. Further refinements of making the weights almost equal were developed in [23, 129]. For similar results for cubature formulas on spherical caps, we refer to [46, 119].

The area-regular partition of the sphere in Theorem 6.4.2 is intuitively clear, though a detailed proof was not given until recently; see [107] for the history and an iterative algorithm that produces a partition. Such partitions have been used in [21, 149] and in [19].

The concept of spherical design was introduced in [53]. It is closely related to other problems in coding theory, combinatorics, and geometry (cf. [36]), and to embeddings of classical Banach spaces (cf. [151]). The existence of a spherical design for all d and n was established in [152]. That a spherical design of degree n with $\mathscr{O}(n^{d-1})$ nodes exists was conjectured by Korevaar and Meyers [98]. The conjecture was proved only recently by Bondarenko, Radchenko, and Viazovska [19]. This is Theorem 6.5.1, and its proof follows [19]. Because of their wide interest, spherical designs with smallest number of points have been searched for numerically; see, for example, the webpage of Sloane, mostly for $d = 3$ and 4. Based on computational evidence, it has been conjectured in [82] that there exists a spherical design of degree n with $n^2/2 + \mathcal{O}(n)$ nodes for $d = 3$. For further results on numerical computation of spherical designs, see [2] and its references.

Chapter 7
Harmonic Analysis Associated with Reflection Groups

In this chapter, we introduce a far-reaching extension of spherical harmonics, in which the surface-area measure, the only rotation-invariant measure, on the sphere is replaced by a family of weighted measures invariant under a finite reflection group, and the Laplace operator is replaced by a sum of squares of Dunkl operators, a family of commuting first-order differential–difference operators. Our goal is to lay the foundation for developing weighted approximation and harmonic analysis on the sphere, which turn out to be indispensable for the corresponding theory, even for unweighted approximation and harmonic analysis, on the unit ball and on the simplex, as will be seen in later chapters.

To avoid heavy algebraic preparations, we shall limit ourselves, in the first two sections, to the simplest nontrivial case, namely $\mathbb{Z}_2^d$, so that we can develop the theory fully with minimal algebraic requirements. In the first section, the Dunkl operators and h-harmonics are defined, and the basic properties of h-harmonics are developed. The intertwining operator, a linear operator that intertwines between the partial derivative operators and the Dunkl operators, is discussed in the second section; this operator is essential for studying projection operators. The reproducing kernels of the spaces of h-harmonics can be expressed in terms of the Gegenbauer polynomials via the intertwining operator, in analogy to the addition formula for ordinary spherical harmonics. The theory of h-harmonics for general finite reflection groups is summarized in the third section. A convolution structure for the weighted integral is discussed in the fourth section, where it is used to study the convergence of Fourier orthogonal expansions, including convergence of the Poisson integrals and a sufficient condition for the convergence of the Cesàro means. A maximal function for the weighted integral is discussed in the fifth section and shown to satisfy the usual bounded properties in weighted space. In the case of $\mathbb{Z}_2^d$, this maximal function is dominated by the Hardy–Littlewood maximal function, as shown in the sixth section.

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7.1 Dunkl Operators and h-Spherical Harmonics

A function $f : \mathbb{R}^d \to \mathbb{C}$ is invariant under the group $\mathbb{Z}_2^d$ if it is invariant under a change of sign in each of its variables. For $1 \le j \le d$, let σ_j denote the reflection of x with respect to the coordinate plane $x_j = 0$; that is,

$$x\sigma_j := (x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_d).$$

Then f is invariant under $\mathbb{Z}_2^d$ if $f(x) = f(x\sigma_j)$ for all $1 \le j \le d$. Define now a family of difference operators by

$$E_j f(x) := \frac{f(x) - f(x\sigma_j)}{x_j}, \qquad 1 \le j \le d.$$

Let again ∂_j be the jth partial derivative.

Definition 7.1.1. Let κ_j, $1 \le j \le d$, be nonnegative numbers. For $1 \le j \le d$, the Dunkl operators $\mathscr{D}_j$ with respect to the group $\mathbb{Z}_2^d$ are defined by

$$\mathscr{D}_j := \partial_j + \kappa_j E_j. \tag{7.1.1}$$

Recall that $\mathscr{P}_n^d$ denotes the space of homogeneous polynomials of degree n in d variables. The Dunkl operators are first-order operators in the sense that they map $\mathscr{P}_n^d$ into $\mathscr{P}_{n-1}^d$. A remarkable property of these operators is that they commute.

Theorem 7.1.2. *For $1 \le j \le d$, $\mathscr{D}_j$ maps $\mathscr{P}_n^d$ into $\mathscr{P}_{n-1}^d$. Furthermore, these operators commute; that is,*

$$\mathscr{D}_i \mathscr{D}_j = \mathscr{D}_j \mathscr{D}_i, \qquad 1 \le i, j \le d.$$

Proof. Let $f \in \mathscr{P}_n^d$. It is easy to see that $E_j f$ is in $\mathscr{P}_{n-1}^d$, so that $\mathscr{D}_j f$ is in $\mathscr{P}_{n-1}^d$. A straightforward computation shows that for $i \ne j$,

$$\mathscr{D}_i \mathscr{D}_j f(x) = \partial_i \partial_j f(x) + \frac{\kappa_i}{x_i}(\partial_j f(x) - \partial_j f(x\sigma_i)) + \frac{\kappa_j}{x_j}(\partial_i f(x) - \partial_i f(x\sigma_j))$$
$$+ \frac{\kappa_i \kappa_j}{x_i x_j}(f(x) - f(x\sigma_j) - f(x\sigma_i) + f(x\sigma_j\sigma_i)),$$

from which $\mathscr{D}_i \mathscr{D}_j = \mathscr{D}_j \mathscr{D}_i$ follows immediately. □

The Laplace operator Δ is a central element of the algebra generated by the differentials. There is an analogous element, Δ_h, in the center of the commutative algebra generated by the Dunkl operators, and it is defined by

$$\Delta_h := \mathscr{D}_1^2 + \cdots + \mathscr{D}_d^2. \tag{7.1.2}$$

This is a second-order differential–difference operator, and it reduces to the usual Laplacian when all κ_i are equal to 0. It has the following explicit formula:

Lemma 7.1.3. *Let $h_\kappa(x) := \prod_{j=1}^d |x_j|^{\kappa_j}$. Then $\Delta_h = D_h - E_h$, where*

$$D_h f := \Delta f + 2\sum_{j=1}^d \frac{\kappa_j}{x_j}\partial_j f = \frac{1}{h_\kappa}[\Delta(fh_\kappa) - f\Delta h_\kappa] \quad \textit{and} \quad E_h := \sum_{j=1}^d \frac{\kappa_j}{x_j}E_j. \tag{7.1.3}$$

Proof. A straightforward computation shows that $E_j^2 = 0$ and

$$\mathscr{D}_j^2 = \partial_j^2 f + \kappa_j\partial_j E_j + \kappa_j E_j\partial_j = \partial_j^2 + 2\frac{\kappa_j}{x_j}\partial_j - \frac{\kappa_j}{x_j}E_j.$$

Summing over j proves that $\Delta_h = D_h - E_h$. The second equality in D_h follows from a simple verification. □

The h-Laplace operator plays the role of the ordinary Laplace operator when the rotation group is replaced by the reflection group.

Definition 7.1.4. Let $Y \in \mathscr{P}_n^d$ be a homogeneous polynomial of degree n. If $\Delta_h Y = 0$, then Y is called an h-harmonic polynomial of degree n.

The h-harmonics of different degrees turn out to be orthogonal with respect to the weighted inner product

$$\langle f, g\rangle_\kappa := \frac{1}{\omega_d^\kappa}\int_{\mathbb{S}^{d-1}} f(x)g(x)h_\kappa^2(x)\mathrm{d}\sigma(x), \tag{7.1.4}$$

where h_κ is the weight function, invariant under the group $\mathbb{Z}_2^d$, defined by

$$h_\kappa(x) := \prod_{j=1}^d |x_j|^{\kappa_j}, \qquad x \in \mathbb{S}^{d-1}, \tag{7.1.5}$$

and ω_κ^d is the constant chosen such that $\langle 1, 1\rangle_\kappa = 1$, whose value is given by

$$\omega_d^\kappa := \int_{\mathbb{S}^{d-1}} h_\kappa^2(x)\mathrm{d}\sigma = \frac{2\Gamma(\kappa_1 + \frac{1}{2})\cdots\Gamma(\kappa_d + \frac{1}{2})}{\Gamma(|\kappa| + \frac{d}{2})}. \tag{7.1.6}$$

Theorem 7.1.5. *Let f and g be h-harmonic polynomials of degree n and m, respectively, with $n \neq m$. Then $\langle f, g\rangle_\kappa = 0$.*

Proof. We first claim that the following identity holds:

$$\int_{\mathbb{S}^{d-1}} \frac{\partial f}{\partial n} g h_\kappa^2 \mathrm{d}\sigma = \int_{\mathbb{B}^d} (gD_h f + \nabla f \cdot \nabla g)h_\kappa^2 \mathrm{d}x, \tag{7.1.7}$$

where $\partial f/\partial n$ denotes the normal derivative of f. This identity is the analogue of the ordinary Green's identity

$$\int_{\mathbb{S}^{d-1}} \frac{\partial f}{\partial n} g \mathrm{d}\sigma = \int_{\mathbb{B}^d} (g\Delta f + \nabla f \cdot \nabla g)\, \mathrm{d}x,$$

from which it follows. Indeed, by the product rule of $\partial/\partial n$, we can write

$$\int_{\mathbb{S}^{d-1}} \frac{\partial f}{\partial n} g h_\kappa^2 \mathrm{d}\sigma = \int_{\mathbb{S}^{d-1}} \frac{\partial (f h_\kappa)}{\partial n} g h_\kappa \mathrm{d}\sigma - \int_{\mathbb{S}^{d-1}} \frac{\partial h_k}{\partial n} f g h_\kappa \mathrm{d}\sigma,$$

and Eq. (7.1.7) follows from applying Green's identity to the right-hand side of the equation. Now, if f is homogeneous, then $\frac{\partial f}{\partial n} = (\deg f) f$ on $\mathbb{S}^{d-1}$. Thus, if f and g are both homogeneous and $\Delta_h f = 0$ and $\Delta_h g = 0$, then Eq. (7.1.7) shows that

$$\begin{aligned}(\deg f - \deg g) \int_{\mathbb{S}^{d-1}} f g h_\kappa^2 \mathrm{d}\sigma &= \int_{\mathbb{B}^d} (g D_h f - f D_h g) h_\kappa^2 \mathrm{d}x \\ &= \int_{\mathbb{B}^d} (g E_h f - f E_h g) h_\kappa^2 \mathrm{d}x = 0,\end{aligned}$$

where the last step follows from a simple verification that the difference operator E_h is symmetric with respect to the integral over $\mathbb{B}^d$. □

For $n = 0, 1, 2, \ldots$, let $\mathscr{H}_n^d(h_\kappa^2)$ denote the linear space of h-harmonic polynomials of degree n. As a consequence of the above theorem, the following theorem follows exactly as in the case of ordinary spherical harmonics.

Theorem 7.1.6. *For $n = 0, 1, 2, \ldots$, there is a decomposition of $\mathscr{P}_n^d$,*

$$\mathscr{P}_n^d = \bigoplus_{0 \le j \le n/2} \|x\|^{2j} \mathscr{H}_{n-2j}^d(h_\kappa^2). \tag{7.1.8}$$

Furthermore, for $n = 0, 1, 2, \ldots$,

$$\dim \mathscr{H}_n^d(h_\kappa^2) = \dim \mathscr{P}_n^d - \dim \mathscr{P}_{n-2}^d = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1}. \tag{7.1.9}$$

In spherical coordinates (1.5.1), a basis of $\mathscr{H}_n^d(h_\kappa^2)$ can be given in terms of the generalized Gegenbauer polynomials $C_n^{(\lambda,\mu)}(t)$ in Eq. (B.3.1), which have normalization constants $h_n^{(\lambda,\mu)}$ given in Eq. (B.3.3). We use the same multi-index notation as in Theorem 1.5.1 and the convention that $\prod_{j=1}^0 = 1$.

Proposition 7.1.7. *For $d \ge 2$ and $\alpha \in \mathbb{N}_0^d$ with $\alpha_d = 0$ or 1, define*

$$Y_\alpha(x) := [h_\alpha]^{-1} r^{|\alpha|} g_\alpha(\theta_1) \prod_{j=1}^{d-2} (\sin\theta_{d-j})^{|\alpha^{j+1}|} C_{\alpha_j}^{(\lambda_j,\kappa_j)}(\cos\theta_{d-j}), \tag{7.1.10}$$

where $g_\alpha(\theta) = C_{\alpha_{d-1}}^{(\kappa_d,\kappa_{d-1})}(\cos\theta)$ *for* $\alpha_d = 0$*,* $\sin\theta C_{\alpha_{d-1}-1}^{(\kappa_d+1,\kappa_{d-1})}(\cos\theta)$ *for* $\alpha_d = 1$*,* $|\alpha^j| = \alpha_j + \cdots + \alpha_{d-1}$*,* $|\kappa^j| = \kappa_j + \cdots + \kappa_d$*,* $\lambda_j = |\alpha^{j+1}| + |\kappa^{j+1}| + \frac{d-j-1}{2}$*, and*

$$[h_\alpha]^2 = \frac{a_\alpha}{(|\kappa| + \frac{d}{2})_{|\alpha|}} \prod_{i=1}^{d-1} h_{\alpha_i}^{(\lambda_i,\kappa_i)}(\kappa_i + \lambda_i)_{\alpha_i}, \quad a_\alpha = \begin{cases} 1 & \text{if } \alpha_d = 0, \\ \kappa_d + \frac{1}{2} & \text{if } \alpha_d = 1. \end{cases}$$

Then $\{Y_\alpha : |\alpha| = n, \alpha_d = 0, 1\}$ *is an orthonormal basis of* $\mathscr{H}_n^d(h_\kappa^2)$ *under* $\langle\cdot,\cdot\rangle_\kappa$*.*

Proof. For $d = 2$, the orthonormality of this basis of two elements follows immediately, after changing variables $x = \cos\theta$, from the orthonormality of $C_n^{(\lambda,\mu)}(t)$. For $d > 2$, using the product nature of h_κ^2, the orthonormality follows along exactly the same lines as that of Theorem 1.5.1. That Y_α is a homogeneous polynomial of degree n, hence an element of $\mathscr{H}_n^d(h_\kappa^2)$, follows easily from the fact that $C_n^{(\lambda,\mu)}(t)$ is even if n is even and odd if n is odd. □

There is also an analogue of the Laplace–Beltrami operator, denoted by $\Delta_{h,0}$, associated with the reflection-invariant weight function. For the rest of this section and the next section, we set for $\kappa = (\kappa_1, \ldots, \kappa_d)$ with $\kappa_i \geq 0$,

$$\lambda_k := |\kappa| + \frac{d-2}{2} = \kappa_1 + \cdots + \kappa_d + \frac{d-2}{2}. \tag{7.1.11}$$

Lemma 7.1.8. *In the spherical–polar coordinates* $x = r\xi$*,* $r > 0$*,* $\xi \in \mathbb{S}^{d-1}$*, the Laplace operator satisfies*

$$\Delta_h = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2\lambda_\kappa + 1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r^2}\Delta_{h,0}, \tag{7.1.12}$$

where, with Δ_0 *denoting the usual Laplace–Beltrami operator,*

$$\Delta_{h,0} f = \frac{1}{h_\kappa}\left[\Delta_0(f h_\kappa) - f\Delta_0 h_\kappa\right] - \sum_{j=1}^d \frac{\kappa_j}{\xi_j} E_j^{(\xi)} f, \tag{7.1.13}$$

where $E_j^{(\xi)}$ *means that* E_j *is acting on the* ξ *variable.*

Proof. We use the decomposition $\Delta_h = D_h + E_h$ and the spherical polar form of Δ and the partial derivatives δ_j. By Eq. (1.4.6),

$$\sum_{j=1}^d \frac{\kappa_j}{x_j}\partial_j = \frac{1}{r}\sum_{j=1}^d \frac{\kappa_j}{\xi_j}\left(\frac{1}{r}(\nabla_0)_j - \xi_j\frac{\partial}{\partial r}\right) = \frac{1}{r^2}\sum_{j=1}^d \frac{\kappa_j}{\xi_j}(\nabla_0)_j - \frac{1}{r}|\kappa|\frac{\partial}{\partial r},$$

and hence, by Eqs. (1.4.2) and (7.1.3), we obtain

$$D_h = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2|\kappa| + d - 1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r^2}\left(\Delta_0 + 2\sum_{j=1}^d \frac{\kappa_j}{\xi_j}(\nabla_0)_j\right).$$

Furthermore, it is easy to see that $E_j f(r\xi) = r^{-1}E_j^{(\xi)} f(r\xi)$, and the same relation holds for E_h. Consequently, the proof is completed by using $\Delta_h = D_h + E_h$. □

Theorem 7.1.9. *The h-spherical harmonics are eigenfunctions of* $\Delta_{h,0}$,

$$\Delta_{h,0} Y_n^h(\xi) = -n(n+\lambda_k) Y_n^h(\xi), \quad \forall Y_n^h \in \mathscr{H}_n^d(h_\kappa^2), \quad \xi \in \mathbb{S}^{d-1}. \tag{7.1.14}$$

Proof. Using Eq. (7.1.12), the proof is exactly like that of Theorem 1.4.5. □

7.2 Projection Operator and Intertwining Operator

There is a linear operator V_κ on the space of spherical polynomials, called the intertwining operator, that acts between ordinary harmonics and h-harmonics and encodes essentially information on the action of the reflection group.

Definition 7.2.1. A linear operator V_κ on the space of algebraic polynomials on $\mathbb{R}^d$ is called an intertwining operator if it satisfies

$$\mathscr{D}_i V_\kappa = V_\kappa \partial_i, \quad 1 \le i \le d, \quad V_\kappa 1 = 1, \quad V_\kappa \mathscr{P}_n \subset \mathscr{P}_n, \ \ n \in \mathbb{N}_0. \tag{7.2.1}$$

From Eq. (7.2.1), it follows immediately that $\Delta_h V_\kappa = V_\kappa \Delta$, and consequently, if P is an ordinary harmonic polynomial, then $V_\kappa P$ is an h-harmonic. In the case of $\mathbb{Z}_2^d$, V_κ is given explicitly as an integral operator.

Theorem 7.2.2. *Let* $\kappa_i \ge 0$. *The intertwining operator for* $\mathbb{Z}_2^d$ *is given by*

$$V_\kappa f(x) = c_\kappa \int_{[-1,1]^d} f(x_1 t_1, \ldots, x_d t_d) \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} \mathrm{d}t, \tag{7.2.2}$$

where $c_k = c_{\kappa_1} \cdots c_{\kappa_d}$ *with* $c_\mu = \Gamma(\mu + 1/2)/(\sqrt{\pi}\Gamma(\mu))$, *and if any* κ_i *is equal to* 0, *then the formula holds under the limit*

$$\lim_{\mu \to 0} c_\mu \int_{-1}^1 f(t)(1-t^2)^{\mu-1} \mathrm{d}t = \frac{f(1)+f(-1)}{2}. \tag{7.2.3}$$

Proof. The integrals are normalized so that $V_\kappa 1 = 1$. Recall that $\mathscr{D}_j = \partial_j + \kappa_j E_j$. Taking derivatives shows that

$$\partial_j V_\kappa f(x) = c_\kappa \int_{[-1,1]^d} \partial_j f(x_1 t_1, \ldots, x_d t_d) t_j \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} \mathrm{d}t.$$

Taking into account the parity of the integrand, an integration by parts shows that

$$\kappa_j E_j V_\kappa f(x) = \frac{\kappa_j}{x_j} c_\kappa \int_{[-1,1]^d} f(x_1 t_1, \ldots, x_d t_d) 2t_j \prod_{i \neq j} (1+t_i) \prod_{i=1}^{d} (1-t_i^2)^{\kappa_i - 1} \mathrm{d}t$$

$$= c_\kappa \int_{[-1,1]^d} \partial_j f(x_1 t_1, \ldots, x_d t_d)(1-t_j) \prod_{i=1}^{d} (1+t_i)(1-t_i^2)^{\kappa_i - 1} \mathrm{d}t.$$

Adding the last two equations gives $\mathscr{D}_j V_\kappa = V_\kappa \partial_j$ for $1 \le j \le d$. $\square$

Note that Eq. (7.2.2) implies, in particular, that V_κ is a positive operator, and we can use Eq. (7.2.2) to extend the definition of $V_\kappa f(x)$ to all $f \in L^1(R_x)$ with $R_x := [-|x_1|, |x_1|] \times \cdots \times [-|x_d|, |x_d|]$.

With the help of the intertwining operator, a large part of the theory for h-harmonics can be developed in parallel to the theory for ordinary spherical harmonics. Because the proof follows similar ideas, we shall be brief and emphasize mostly the differences.

We write, for $\alpha \in \mathbb{N}_0^d$, $\mathscr{D}^\alpha = \mathscr{D}_1^{\alpha_1} \ldots \mathscr{D}_d^{\alpha_d}$ and use the notation $V_\kappa^{(x)}$, when necessary, to indicate that V_κ is acting on the x variable.

Theorem 7.2.3. *For $p, q \in \mathscr{P}_n^d$, define a bilinear form*

$$\langle p, q \rangle_{\mathscr{D}} := p(\mathscr{D}) q,$$

where $p(\mathscr{D})$ is the operator defined by replacing x^α in $p(x)$ by $\mathscr{D}^\alpha$. Then

1. *$\langle p, q \rangle_{\mathscr{D}}$ is an inner product on $\mathscr{P}_n^d$;*
2. *the reproducing kernel of $\langle \cdot, \cdot \rangle_{\mathscr{D}}$ is $k_n^h(x,y) := V_\kappa^{(x)}(\langle x, y \rangle^n)/n!$; that is,*

$$\langle k_n^h(x, \cdot), p \rangle_{\mathscr{D}} = p(x), \qquad \forall p \in \mathscr{P}_n^d;$$

3. *for $p \in \mathscr{P}_n^d$ and $q \in \mathscr{H}_n^d$, with $\langle \cdot, \cdot \rangle_\kappa$ defined in Eq. (7.1.4),*

$$\langle p, q \rangle_{\mathscr{D}} = 2^n (\lambda_\kappa + 1)_n \langle p, q \rangle_\kappa.$$

Proof. This is the analogue of Theorem 1.1.8, and the proof follows along the same lines. We first observe that it is easy to verify, by induction, that

$$\mathscr{D}_j^{2m} x_j^{2m} = 2^m m! (2\kappa_j + 1)_{2m-1} \quad \text{and} \quad \mathscr{D}_j^{2m+1} x_j^{2m+1} = 2^m m! (2\kappa_j + 1)_{2m+1}.$$

Both are positive numbers, since $\kappa_j \ge 0$ and $\mathscr{D}_j x_k^m = 0$ if $j \neq k$. This shows that if $\alpha, \beta \in \mathbb{N}_0^d$ and $|\alpha| = |\beta|$, then $\mathscr{D}^\alpha x^\beta = A_\alpha \delta_{\alpha,\beta}$ with $A_\alpha > 0$. Consequently, if $p, q \in \mathscr{P}_n^d$ are given by $p(x) = \sum_{|\alpha|=n} a_\alpha x^\alpha$ and $q(x) = \sum_{|\alpha|=n} b_\alpha x^\alpha$, then

$$\langle p, q \rangle_{\mathscr{D}} = \sum_{|\alpha|=n} a_\alpha \mathscr{D}^\alpha \sum_{|\beta|=n} b_\beta x^\beta = \sum_{|\alpha|=n} A_\alpha a_\alpha b_\alpha,$$

by the commutativity of $\mathscr{D}_j$, which proves that $\langle \cdot, \cdot \rangle_{\mathscr{D}}$ is an inner product on $\mathscr{P}_n^d$.

By Theorem 1.1.8, $p(x) = \langle x, \partial^{(y)}\rangle^n p(y)/n!$, so that $V_\kappa p(x) = k_n^h\left(x, \partial^{(y)}\right) p(y)$, which implies, applying $V_k^{(y)}$ and using $V_\kappa 1 = 1$,

$$V_k p(x) = V_k^{(y)} k_n^h\left(x, \partial^{(y)}\right) p(y) = k_n^h\left(x, \mathscr{D}^{(y)}\right) V_\kappa p(y).$$

This establishes the reproducing property for $V_\kappa p$, hence for all $q \in \mathscr{P}_n^d$, since Eq. (7.2.2) implies that $V_\kappa(\mathscr{P}_n^d) = \mathscr{P}_n^d$.

For the proof of item 3, let $\mathrm{d}\psi(x) := h_\kappa^2(x) e^{-\frac{\|x\|^2}{2}} \mathrm{d}x$. Integration by parts shows that

$$\int_{\mathbb{R}^d} \partial_i f(x) g(x) \mathrm{d}\psi = -\int_{\mathbb{R}^d} f(x) \left(\partial_i - x_i + \frac{2\kappa_i}{x_i}\right) g(x) \mathrm{d}\psi,$$

since $\partial_i h_\kappa(x) = \kappa_i h_\kappa(x)/x_i$, and a change of variables shows

$$\kappa_i \int_{\mathbb{R}^d} E_i f(x) g(x) \mathrm{d}\psi = \kappa_i \int_{\mathbb{R}^d} f(x) \frac{g(x) + g(x\sigma_i)}{x_i} \mathrm{d}\psi.$$

Consequently, adding the two integrals shows that

$$\int_{\mathbb{R}^d} \mathscr{D}_i f(x) g(x) \mathrm{d}\psi = -\int_{\mathbb{R}^d} f(x) \left(\mathscr{D}_i g(x) - x_i g(x)\right) \mathrm{d}\psi.$$

At this point, the proof follows from that of Theorem 1.1.8 in a straightforward manner. At the last step, we end up with, for $p \in \mathscr{P}_n^d$ and $q \in \mathscr{H}_n^d(h_\kappa^2)$,

$$\langle p, q\rangle_{\mathscr{D}} = c \int_0^\infty r^{2n+2|\kappa|+d-1} e^{-r^2/2} \mathrm{d}r \int_{\mathbb{S}^{d-1}} q(x') p(x') h_\kappa^2(x') \mathrm{d}\sigma(x'),$$

where the constant c is determined by $\langle 1, 1\rangle_{\mathscr{D}} = 1$. Evaluating the value of c and the integral in r concludes the proof. □

Theorem 7.2.4. *For $\alpha \in \mathbb{N}_0^d$, $n = |\alpha|$, define*

$$p_\alpha(x) := \frac{(-1)^n}{2^n (\lambda_\kappa)_n} \|x\|^{2|\alpha|+2\lambda_\kappa} \mathscr{D}^\alpha \{\|x\|^{-2\lambda_k}\}. \tag{7.2.4}$$

Then

1. *$p_\alpha \in \mathscr{H}_n^d(h_\kappa^2)$, and p_α is the monic spherical harmonic of the form*

$$p_\alpha(x) = x^\alpha + \|x\|^2 q_\alpha(x), \qquad q_\alpha \in \mathscr{P}_{n-2}^d.$$

2. *p_α satisfies the recurrence relation*

$$p_{\alpha+e_i}(x) = x_i p_\alpha(x) - \frac{1}{2n + 2\lambda_\kappa} \|x\|^2 \partial_i p_\alpha(x).$$

3. *$\{p_\alpha : |\alpha| = n, \alpha_d = 0 \text{ or } 1\}$ is a basis of $\mathscr{H}_n^d(h_\kappa^2)$.*

Proof. This is an analogue of Theorem 1.1.9, and it is proved along the same lines. In fact, once we establish the relation that for $g \in \mathscr{P}_n^d$ and $\rho \in \mathbb{R}$,

$$\Delta_h(\|x\|^\rho g) = \rho(2n + 2\lambda_k + \rho)\|x\|^{\rho-2} g + \|x\|^\rho \Delta_h g, \tag{7.2.5}$$

the proof follows from that of Theorem 1.1.9 with obvious small modifications. To prove Eq. (7.2.5), we use $\Delta_h = D_h - E_h$. By Eq. (7.1.3), $D_h(\|x\|^\rho g(x))$ can be derived from Eq. (1.1.12) and $\partial_j(\|x\|^\rho g(x)) = \rho x_j g(x) + \|x\|^\rho \partial_j g(x)$, whereas $E_h(\|x\|^\rho g(x)) = \|x\|^\rho E_h g(x)$, by the invariance of $\|x\|$ under $\mathbb{Z}_2^d$. Putting these together proves Eq. (7.2.5). □

The statements of the above two theorems and their proofs demonstrate a parallel between the theory of h-harmonics and that of ordinary harmonics. The parallel goes further, since the above two theorems are key ingredients for dealing with the projection operator and the reproducing kernels.

Let us denote by

$$\operatorname{proj}_n^\kappa : L^2(\mathbb{S}^{d-1}, h_\kappa^2) \mapsto \mathscr{H}_n^d(h_\kappa^2)$$

the orthogonal projection operator from $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$ onto $\mathscr{H}_n^d(h_\kappa^2)$. Just like the projection operator for ordinary spherical harmonics as in Lemma 1.2.4, the operator $\operatorname{proj}_n^\kappa$ can be expressed as an integral

$$\operatorname{proj}_n^\kappa f(x) = \frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} f(y) Z_n^\kappa(x,y) h_\kappa^2(y) \mathrm{d}\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{7.2.6}$$

where $Z_n^\kappa(\cdot,\cdot)$ is the reproducing kernel of $\mathscr{H}_n^d(h_\kappa^2)$ defined by $Z_n^\kappa(\cdot,z) \in \mathscr{H}_n^d(h_\kappa^2)$ for each fixed $z \in \mathbb{S}^{d-1}$, and

$$\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} Z_n^\kappa(x,y) p(y) h_\kappa^2(y) \mathrm{d}\sigma(y) = p(x), \quad \forall p \in \mathscr{H}_n^d(h_\kappa^2), \quad x \in \mathbb{S}^{d-1}. \tag{7.2.7}$$

In analogy to Lemma 1.2.5 for ordinary spherical harmonics, this kernel can be expressed in terms of an orthonormal basis $\{Y_j^h : 1 \le j \le N\}$ of $\mathscr{H}_n^d(h_\kappa^2)$ as

$$Z_n^\kappa(x,y) = \sum_{j=1}^N Y_j^h(x) Y_j^h(y), \quad N = \dim \mathscr{H}_n^d(h_\kappa^2), \quad x,y \in \mathbb{S}^{d-1}, \tag{7.2.8}$$

and it is independent of the particular choice of the basis of $\mathscr{H}_n^d(h_\kappa^2)$.

Lemma 7.2.5. *Let $p \in \mathscr{P}_n^d$. Then*

$$\operatorname{proj}_n^\kappa p = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{4^j j! (1 - n - \lambda_\kappa)_j} \|x\|^{2j} \Delta^j p. \tag{7.2.9}$$

Proof. By Theorem 7.2.4, p_α defined in Eq. (7.2.4) is the orthogonal projection of the function $q_\alpha(x) = x^\alpha$; that is, $p_\alpha = \mathrm{proj}_n^\kappa q_\alpha$. Since $\mathscr{D}_i f(\|x\|) = \partial_i f(\|x\|)$ and $E_j(\|x\|^\rho g(x)) = \|x\|^\rho E_j g(x)$, it is easy to verify that

$$\mathscr{D}_i(\|x\|^\rho g(x)) = \rho x_i \|x\|^{\rho-2} g(x) + \|x\|^\rho \mathscr{D}_i g(x).$$

Using this identity, the proof of Eq. (7.2.9) can be carried out as in the proof of Lemma 1.2.1, with obvious modifications. □

Theorem 7.2.6. *For $\kappa_i \ge 0$ and $\|y\| \le \|x\| = 1$,*

$$Z_n^\kappa(x,y) = \|y\|^n \frac{n+\lambda_\kappa}{\lambda_\kappa} V_\kappa \left[C_n^{\lambda_\kappa}\left(\left\langle \cdot, \frac{y}{\|y\|} \right\rangle \right) \right](x). \tag{7.2.10}$$

Proof. For a fixed $y \in \mathbb{S}^{d-1}$, let $p(x) = k_n^h(x,y)$ and $p_n = \mathrm{proj}_n^\kappa p$. If $f \in \mathscr{H}_n^d(h_\kappa^2)$, then by Theorem 7.2.3, $f(y) = \langle p, f \rangle_{\mathscr{D}}$, and furthermore,

$$f(y) = \langle p, f \rangle_{\mathscr{D}} = 2^n (\lambda_\kappa + 1)_n \langle p, f \rangle_\kappa = 2^n (\lambda_\kappa + 1)_n \langle p_n, f \rangle_\kappa.$$

Since the reproducing kernel is uniquely defined by the reproducing property, this shows that $Z_n^\kappa(x,y) = 2^n(\lambda_\kappa + 1)_n \mathrm{proj}_n^\kappa p(x)$. Consequently, it follows from Eq. (7.2.9) that

$$Z_n^\kappa(x,y) = 2^n (\lambda_\kappa + 1)_n \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{4^j j! (1-n-\lambda_\kappa)_j} \|x\|^{2j} \|y\|^{2j} k_{n-2j}^h(x,y),$$

where we have used $\Delta_h^j p(x) = \Delta_h^j k_n^h(x,y) = \|y\|^{2j} k_{n-2j}^h(x,y)$, which follows from the intertwining property of V_κ and taking derivatives of $\langle x,y\rangle^n/n!$ with respect to x. Using $n!/(n-2j)! = 2^{2j}(-\frac{n}{2})_j(\frac{-n+1}{2})_j$, we then conclude that

$$Z_n^\kappa(x,y) = \frac{2^n(\lambda_\kappa+1)_n}{n!} V_\kappa \left[\sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-\frac{n}{2})_j (\frac{-n+1}{2})_j}{j!(1-n-\lambda_\kappa)_j} \|x\|^{2j} \|y\|^{2j} \langle \cdot, y \rangle^{n-2j} \right](x).$$

If $\|x\| = 1$, then the sum is a $\|y\|^n$ multiple of a hypergeometric function ${}_2F_1$ in the variable $\langle \cdot, y/\|y\| \rangle$, which can be expressed as the Gegenbauer polynomial $C_n^{\lambda_\kappa}$ by Eq. (B.2.5). This proves Eq. (7.2.10) on using $\lambda_\kappa(\lambda_\kappa+1)_n = (\lambda_\kappa + n)(\lambda_\kappa)_n$. □

If all κ_i are equal to 0, then V_κ becomes the identity operator, and Eq. (7.2.10) becomes the addition formula (1.2.7) for ordinary spherical harmonics. Thus, Eq. (7.2.10) is an analogue of the addition formula for ordinary spherical harmonics.

The identity (7.2.10) also indicates that zonal functions, which depend only on $\langle x,y \rangle$, should be replaced by functions of the form $V_\kappa[f(\langle \cdot, y \rangle)](x)$ in the theory of h-harmonics. Indeed, we have an analogue of the Funk–Hecke formula.

Theorem 7.2.7. *Let f be an integrable function such that $\int_{-1}^{1} |f(t)|(1-t^2)^{\lambda_k - 1/2}\mathrm{d}t$ is finite. Then for every $Y_n^h \in \mathscr{H}_n^d(h_\kappa^2)$,*

$$\frac{1}{\omega_d^\kappa}\int_{\mathbb{S}^{d-1}} V_\kappa[f(\langle x,\cdot\rangle)](y)Y_n^h(y)h_\kappa^2(y)\mathrm{d}\sigma(y) = \Lambda_n(f)Y_n^h(x), \quad x\in\mathbb{S}^{d-1}, \tag{7.2.11}$$

where $\lambda_n(f)$ is a constant defined by

$$\Lambda_n(f) = c_{\lambda_\kappa}\int_{-1}^{1} f(t)\frac{C_n^{\lambda_\kappa}(t)}{C_n^{\lambda_\kappa}(1)}(1-t^2)^{\lambda_k-\frac{1}{2}}\mathrm{d}t,$$

where $c_\lambda = \Gamma(\lambda+1)/\sqrt{\pi}\Gamma(\lambda+1/2)$ such that $\Lambda_0(1)=1$.

Proof. The proof can be carried out exactly like that of Theorem 1.2.9. □

Corollary 7.2.8. *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be a function such that both integrals below are defined. Then for $x\in\mathbb{R}^d$,*

$$\frac{1}{\omega_d^\kappa}\int_{\mathbb{S}^{d-1}} V_k[f(\langle x,\cdot\rangle)](y)h_\kappa^2(y)\mathrm{d}\sigma(y) = c_{\lambda_\kappa}\int_{-1}^{1} f(\|x\|t)(1-t^2)^{\lambda_\kappa-\frac{1}{2}}\mathrm{d}t. \tag{7.2.12}$$

Proof. This follows from applying the Funk–Hecke formula (7.2.11) with $n=0$ to the function $\xi\mapsto f(\|x\|\xi)$, $\xi\in\mathbb{S}^{d-1}$. □

The identity (7.2.12) is a special case of a more general identity:

Theorem 7.2.9. *Let $f:\mathbb{B}^d\mapsto\mathbb{R}$ be a function such that both integrals below are defined. Then*

$$\frac{1}{\omega_d^\kappa}\int_{\mathbb{S}^{d-1}} V_k f(y)h_\kappa^2(y)\mathrm{d}\sigma(y) = a_\kappa\int_{\mathbb{B}^d} f(x)(1-\|x\|^2)^{|\kappa|-1}\mathrm{d}x, \tag{7.2.13}$$

where $a_k = \Gamma(|k|+\frac{d}{2})/(\pi^{d/2}\Gamma(|k|))$.

Proof. By Theorem 1.3.4, every polynomial f can be written as a linear sum of $p_j(\langle x,\xi_j\rangle)$, where $p_j:[-1,1]\mapsto\mathbb{R}$ and $\xi_j\in\mathbb{S}^{d-1}$, so that Eq. (7.2.13) follows from Eq. (7.2.12) for all polynomials. For general f, we can then pass to the limit, since the right-hand side of Eq. (7.2.13) is clearly closed under the limit. □

The identity (7.2.12) implies that the action of the reflection group embedded in V_κ dissipates on taking the average over the sphere. This identity has important implications in the study of h-harmonic series, as will be seen in later sections.

In the case $\mathbb{Z}_2^d$, the explicit formula (7.2.2) for V_κ gives the following corollary.

Corollary 7.2.10. *For h_κ^2 in Eq. (7.1.5) associated with $\mathbb{Z}_2^d$ and $x, y \in \mathbb{S}^{d-1}$,*

$$Z_n^\kappa(x,y) = \frac{n+\lambda_\kappa}{\lambda_\kappa} c_\kappa \int_{[-1,1]^d} C_n^{\lambda_\kappa}(x_1y_1t_1 + \cdots + x_dy_dt_d) \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} \mathrm{d}t.$$

In particular, for $e_1 = (1,0,\ldots,0)$, the identity in the corollary implies that

$$\begin{aligned} Z_n^\kappa(x,e_1) &= \frac{n+\lambda_\kappa}{\lambda_\kappa} c_{\kappa_1} \int_{-1}^1 C_n^{\lambda_\kappa}(x_1t_1)(1+t_1)(1-t_1^2)^{\kappa_1-1} \mathrm{d}t_1 \\ &= \frac{n+\lambda_\kappa}{\lambda_k} C_n^{(\lambda_\kappa - \kappa_1, \kappa_1)}(x_1), \end{aligned} \tag{7.2.14}$$

where the second equality follows from Eq. (B.3.4).

7.3 *h*-Harmonics for a General Finite Reflection Group

In this section, we summarize the theory of *h*-harmonics for a general reflection group G. All results stated in the previous two sections hold in the general setting. Reader who are unfamiliar with reflection groups can skip this section.

Let v be a nonzero vector in $\mathbb{R}^d$. The reflection σ_v along v is defined by

$$x\sigma_v := x - 2\langle x, v\rangle v/\|v\|^2, \qquad x \in \mathbb{R}^d,$$

where $\langle x, y\rangle$ denotes the usual Euclidean inner product.

Definition 7.3.1. A root system is a finite set R of nonzero vectors in $\mathbb{R}^d$ such that $u, v \in R$ implies $u\sigma_v \in R$. A reflection group G with root system R is the subgroup of the orthogonal group $O(d)$ generated by the reflections $\{\sigma_u : u \in R\}$.

If R is not the union of two nonempty orthogonal subsets, then the corresponding reflection group G is said to be irreducible. Fix $u_0 \in \mathbb{R}^d$ such that $\langle u, u_0\rangle \neq 0$ for all $u \in R$. The set of positive roots R_+ with respect to u_0 is defined by $R_+ = \{u \in R : \langle u, u_0\rangle > 0\}$ and $R = R_+ \cup (-R_+)$.

Definition 7.3.2. A multiplicity function $v \mapsto \kappa_v$ of $R_+ \mapsto \mathbb{R}$ is a function defined on R_+ with the property that $\kappa_u = \kappa_v$ if σ_u is conjugate to σ_v, that is, if there is a w in the reflection group G generated by R_+ such that $\sigma_u w = \sigma_v$.

By definition, a multiplicity function is invariant under the group G.

Definition 7.3.3. Let $v \mapsto \kappa_v$ be a multiplicity function associated with a finite reflection group G. The Dunkl operator is defined by, for $1 \le j \le d$,

$$\mathscr{D}_j f(x) = \partial_j f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} \langle v, e_j\rangle,$$

where $e_1 = (1,0,\dots,0),\dots,e_d = (0,\dots,0,1)$.

These are first-order differential–difference operators, and they commute:

Theorem 7.3.4. *The Dunkl operators commute:* $\mathscr{D}_j\mathscr{D}_k = \mathscr{D}_k\mathscr{D}_j$, $1 \le j,k \le d$.

The analogue of the Laplace operator Δ_h is again defined by

$$\Delta_h := \mathscr{D}_1^2 + \cdots + \mathscr{D}_d^2.$$

An h-harmonic polynomial Y is a homogeneous polynomial such that $\Delta_h Y = 0$. Furthermore, h-harmonic polynomials are orthogonal with respect to the inner product $\langle f,g\rangle_\kappa$ defined in Eq. (7.1.4), with, however, h_κ defined by

$$h_\kappa(x) := \prod_{v\in R_+} |\langle x,v\rangle|^{\kappa_v}, \qquad x \in \mathbb{R}^d, \tag{7.3.1}$$

which is a homogeneous function of degree $\gamma_\kappa := \sum_{v\in R_+} \kappa_v$ and invariant under G. For such a weight function, λ_κ in Eq. (7.1.11) is replaced by

$$\lambda_\kappa := \sum_{v\in R_+} \kappa_v + \frac{d-2}{2}. \tag{7.3.2}$$

There again exists an intertwining operator V_κ on the space of algebraic polynomials on $\mathbb{R}^d$, which is linear and uniquely defined by Eq. (7.2.1), acting between the partial derivatives and Dunkl operators. The operator V_κ is known to be positive, in the sense that $V_\kappa f(x) \ge 0$ if $f(x) \ge 0$. Under these definitions, most results stated in the previous two sections for the special case of $\mathbb{Z}_2^d$, in particular Theorems 7.2.7 and 7.2.9, hold with G a finite reflection group. However, in the case that $G \ne \mathbb{Z}_2^d$, explicit formulas for h-harmonics in spherical coordinates such as Eq. (7.1.10) are unknown, and although $\{p_\alpha : |\alpha| = n, \alpha_d = 0 \,\text{or}\, 1\}$ with p_α defined in Eq. (7.2.4) is still a basis of $\mathscr{H}_n^d(h_\kappa^2)$, the $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$-norm of p_α is unknown. In fact, no orthonormal basis of $\mathscr{H}_n^d(h_\kappa^2)$ is known for $d > 2$.

For studying h-harmonic series on the sphere, what we will need most is the representation of the reproducing kernel $Z_n^\kappa(\cdot,\cdot)$ in Eq. (7.2.10), which holds for all reflection groups. However, for a general reflection group, explicit formulas for V_κ in analogy to Eq. (7.2.2) are no longer available. Since V_κ encodes essential information about the group action, the lack of an explicit expression means that many results can at the moment be established only in the case $\mathbb{Z}_2^d$.

We end this section with a few words and examples on reflection groups and associated weight functions. The group $\mathbb{Z}_2^d$ is reducible, a product of d copies of the irreducible group $\mathbb{Z}_2$. There is a complete classification of irreducible finite reflection groups. The list consists of root systems of infinite families A_{d-1} of the symmetric group on d elements, B_d of the hyperoctahedral group, D_d of a subgroup of the hyperoctahedral group with an even number of sign changes, dihedral systems $I_2(m)$ of the symmetric group of regular m-gons in $\mathbb{R}^2$, and several other individual systems H_3, H_4, F_4 and E_6, E_7, E_8. Let $e_1,\dots,e_d$ be the standard canonical basis of $\mathbb{R}^d$. In the case of A_{d-1}, the group G has $R_+ = \{e_i - e_j : i > j\}$ and

$$h_\kappa(x) = \prod_{1\le i<j\le d} |x_i - x_j|^\kappa, \quad \kappa \ge 0.$$

In the case of B_d, the group G is the symmetric group of $\{\pm e_1, \ldots, \pm e_d\}$, for which $R_+ = \{e_i - e_j, e_i + e_j : i < j\} \cup \{e_i : 1 \le i \le d\}$ and

$$h_\kappa(x) = \prod_{i=1}^{d} |x_i|^{\kappa_0} \prod_{1\le i<j\le d} |x_i^2 - x_j^2|^{\kappa_1}, \quad \kappa_0, \kappa_1 \ge 0.$$

In the rest of the book, we will need little specific information on the reflection groups. To be sure, some of the results will be stated for an arbitrary reflection group; such results, however, will involve only rudimentary properties of the group, and the reader can simply assume that the group is $\mathbb{Z}_2^d$.

7.4 Convolution and *h*-Harmonic Series

Recall that $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$, $x \in (-1,1)$. Theorem 7.2.9, which holds for a general finite reflection group G, allows us to extend the intertwining operator V_κ to a linear positive operator from $L^1(w_{\lambda_\kappa}, [-1,1])$ to $L^1(h_\kappa^2, \mathbb{S}^{d-1})$. As suggested by Eqs. (7.2.7) and (7.2.10), we can define a convolution with respect to the h_κ^2 on the sphere.

Definition 7.4.1. For $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$ and $g \in L^1(w_{\lambda_\kappa}, [-1,1])$,

$$(f *_\kappa g)(x) := \frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} f(y) V_\kappa[g(\langle \cdot, y\rangle)](x) h_\kappa^2(y) \mathrm{d}\sigma(y). \tag{7.4.1}$$

Recall that the norm of the space $L^p(w_\lambda; [-1,1])$ is denoted by $\|\cdot\|_{\lambda,p}$. We denote by $\|\cdot\|_{\kappa,p}$ the norm of the space $L^p(h_\kappa, \mathbb{S}^{d-1})$ for $1 \le p < \infty$ and $\|\cdot\|_{\kappa,\infty} = \|\cdot\|_\infty$ the norm of the space $C(\mathbb{S}^{d-1})$.

Theorem 7.4.2. *Let $p,q,r \ge 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$. For $f \in L^q(h_\kappa^2, \mathbb{S}^{d-1})$ and $g \in L^r(w_{\lambda_\kappa}; [-1,1])$,*

$$\|f *_\kappa g\|_{\kappa,p} \le \|f\|_{\kappa,q} \|g\|_{\lambda_\kappa,r}. \tag{7.4.2}$$

In particular, for $1 \le p \le \infty$,

$$\|f * g\|_{\kappa,p} \le \|f\|_{\kappa,p} \|g\|_{\lambda_\kappa,1} \quad \textit{and} \quad \|f * g\|_{\kappa,p} \le \|f\|_{\kappa,1} \|g\|_{\lambda_\kappa,p}. \tag{7.4.3}$$

Proof. Following the proof of Theorem 2.1.2, we need to show only that

$$\|G(x,\cdot)\|_{\kappa,r} \le \|g\|_{\lambda_\kappa,r}, \qquad \text{where} \quad G(x,y) = V_\kappa[g(\langle x, \cdot\rangle)](y),$$

which can be proved as follows: the positivity of V_κ implies $|V_\kappa g| \le V_\kappa[|g|]$, so that $\|G(x,\cdot)\|_{\kappa,\infty} \le \|g\|_{\lambda_\kappa,\infty}$, and we deduce by Eq. (7.2.12) that

$$\|G(x,\cdot)\|_{\kappa,1} \le \frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} V_\kappa\left[|g(\langle x,\cdot\rangle)|\right](y) h_\kappa^2(y) \mathrm{d}\sigma = c_{\lambda_\kappa} \int_{-1}^1 |g(t)| w_\lambda(t) \mathrm{d}t = \|g\|_{\lambda_\kappa,1}.$$

The log-convexity of the L^p-norm implies then $\|G(x,\cdot)\|_{\kappa,r} \le \|g\|_{\lambda_\kappa,r}$. □

In particular, by Eqs. (7.2.6) and (7.2.10), the projection $\operatorname{proj}_n^\kappa$ is a convolution operator:

$$\operatorname{proj}_n^\kappa f = f *_\kappa Z_n^\kappa, \qquad Z_n^\kappa(t) := \frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(t), \tag{7.4.4}$$

and the following analogue of Theorem 2.1.3 holds.

Theorem 7.4.3. *For $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$ and $g \in L^1(w_{\lambda_\kappa}; [-1,1])$,*

$$\operatorname{proj}_n^\kappa (f *_\kappa g) = \hat{g}_n^{\lambda_\kappa} \operatorname{proj}_n^\kappa f, \quad n = 0,1,2\ldots, \tag{7.4.5}$$

where $\hat{g}_n^{\lambda_\kappa}$ is the Fourier coefficient of g in the Gegenbauer polynomial

$$\hat{g}_n^{\lambda_\kappa} = c_{\lambda_\kappa} \int_{-1}^1 g(t) \frac{C_n^{\lambda_\kappa}(t)}{C_n^{\lambda_\kappa}(1)} (1-t^2)^{\lambda_\kappa - \frac{1}{2}} \mathrm{d}t.$$

The Fourier orthogonal series in h-spherical harmonics are defined exactly as in the case of ordinary spherical harmonics. In analogy to Eq. (2.2.1), we have for $f \in L^2(h_\kappa^2, \mathbb{S}^{d-1})$ that

$$f(x) = \sum_{n=0}^\infty \operatorname{proj}_n^\kappa f(x), \tag{7.4.6}$$

and an analogue of Eq. (2.2.2) holds. Since the convergence of the series (7.4.6) does not go beyond L^2 convergence, we again need to consider summability methods.

For $\delta > -1$, let $S_n^\delta(h_\kappa^2; f)$ denote the Cesàro (C,δ) means of the series (7.4.6),

$$S_n^\delta\left(h_\kappa^2; f\right) := \frac{1}{A_n^\delta} \sum_{j=0}^n A_{n-j}^\delta \operatorname{proj}_j^\kappa f = f *_\kappa K_n^\delta\left(h_\kappa^2\right), \tag{7.4.7}$$

where $S_n^0(h_\kappa^2; f)$ is the nth partial sum and, as shown in Eq. (2.4.5),

$$K_n^\delta\left(h_\kappa^2; t\right) := \frac{1}{A_n^\delta} \sum_{k=0}^n A_{n-k}^\delta \frac{k+\lambda_\kappa}{\lambda_\kappa} C_k^{\lambda_\kappa}(t) = k_n^\delta\left(w_{\lambda_\kappa}; 1, t\right), \tag{7.4.8}$$

in which $k_n^\delta(w_{\lambda_\kappa}; \cdot, \cdot)$ is the kernel of the (C,δ) means of the Fourier orthogonal series in the Gegenbauer polynomials.

Theorem 7.4.4. *The Cesàro means of the h-spherical harmonic series satisfy*

1. *If* $\delta \ge 2\lambda_k + 1$, *then* $S_n^\delta(h_\kappa^2)$ *is a nonnegative operator;*
2. *If* $\delta > \lambda_\kappa$, *then*

$$\sup_{n\ge 0} \|S_n^\delta(h_\kappa^2; g)\|_{\kappa,p} \le c\|g\|_{\kappa,p}, \quad 1 \le p \le \infty. \tag{7.4.9}$$

In particular, $S_n^\delta(h_\kappa^2; f)$ *converges to* f *in* $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ *for* $1 \le p \le \infty$.

Proof. The positivity follows exactly as in Theorem 2.4.3. For the convergence in the second item, we need to show only, as in Corollary 2.4.5, that $\|S_n^\delta(h_k^2)\|_{\kappa,p}$ is bounded. By Young's inequality (7.4.3),

$$\|S_n^\delta(h_\kappa^2, f)\|_{\kappa,p} \le \|f\|_{\kappa,p} \|k_n^\delta(w_{\lambda_\kappa})\|_{\lambda_\kappa,1} \le c\|f\|_{\kappa,p},$$

whenever $\delta > \lambda_k$, as shown in the proof of Theorem 2.4.4. □

A major difference from the theory of ordinary spherical harmonic series is that in general, the condition $\delta > \lambda_k$ is only a sufficient condition, not a necessary and sufficient condition. In fact, the essential ingredient of the proof is Eq. (7.2.12), taking an average of $V_\kappa g$ over the sphere, which, however, removes V_κ from the scene and as a result, inadvertently erases the information about the reflection group. Determining the critical index of (C, δ) requires arduous work in drawing out information encoded in the intertwining operator V_κ, and this will be the focus of the next chapter.

As a corollary of Theorem 7.4.4, we have an analogue of Corollary 2.2.6:

Corollary 7.4.5. *If* $f, g \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$ *and* $\operatorname{proj}_n f = \operatorname{proj}_n g$ *for all* $n = 0, 1, \ldots,$ *then* $f = g$.

Next, we define an analogue of the translation operator $T_\theta f$ in Eq. (2.1.6), which, however, cannot be defined simply as an integral over the spherical cap. Instead, we shall take the analogue of Eq. (2.1.9) as the definition.

Definition 7.4.6. For $0 \le \theta \le \pi$, the translation operator T_θ^κ is defined by

$$\operatorname{proj}_n^\kappa(T_\theta^\kappa f) = \frac{C_n^{\lambda_\kappa}(\cos\theta)}{C_n^{\lambda_\kappa}(1)} \operatorname{proj}_n^\kappa f, \quad n = 0, 1, \ldots. \tag{7.4.10}$$

Proposition 7.4.7. *The operator* T_θ^κ *is well defined for all* $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$, *and it satisfies the following properties:*

(i) *For* $f \in L^2(h_\kappa^2, \mathbb{S}^{d-1})$ *and* $g \in L^1(w_{\lambda_\kappa}, [-1, 1])$,

$$(f *_\kappa g)(x) = c_{\lambda_\kappa} \int_0^\pi T_\theta^\kappa f(x) g(\cos\theta)(\sin\theta)^{2\lambda_\kappa} d\theta. \tag{7.4.11}$$

(ii) T_θ^κ *preserves positivity, i.e.,* $T_\theta^\kappa f \ge 0$ *if* $f \ge 0$.

(iii) *For* $f \in L^p(h_\kappa^2, \mathbb{S}^{d-1})$, *if* $1 \le p < \infty$, *or* $f \in C(\mathbb{S}^{d-1})$ *if* $p = \infty$,

$$\|T_\theta^\kappa f\|_{\kappa,p} \le \|f\|_{\kappa,p} \qquad \textit{and} \qquad \lim_{\theta\to 0} \|T_\theta^\kappa f - f\|_{\kappa,p} = 0. \tag{7.4.12}$$

(iv) $T_\theta^\kappa f(-x) = T_{\pi-\theta}^\kappa(x)$.

Proof. By Corollary 7.4.5, the translation operator T_θ^κ is well defined. Furthermore, Eq. (7.4.5) implies immediately that

$$\operatorname{proj}_n^\kappa (f *_\kappa g)(x) = c_{\lambda_\kappa} \int_0^\pi \operatorname{proj}_n^\kappa (T_\theta^\kappa f)(x) g(\cos\theta)(\sin\theta)^{2\lambda_\kappa} \mathrm{d}\theta,$$

which proves, again by Corollary 7.4.5, the identity (7.4.11).

To prove (ii) and (iii), for fixed θ, we let $B_n(\theta) := c_{\lambda_\kappa} \int_{\theta-1/n}^{\theta+1/n} (\sin t)^{2\lambda_\kappa} \mathrm{d}t$ and define g_n by $g_n(\cos\phi) = 1/B_n(\theta)$ if $|\phi-\theta| \le 1/n$, and $g_n(\cos\phi) = 0$ otherwise. Then $g_n(\cos\theta) \ge 0$ for $\theta \in [0,\pi]$ and

$$(f *_\kappa g_n)(x) = \frac{1}{B_n(\theta)} \int_{\theta-1/n}^{\theta+1/n} T_\phi^\kappa f(x)(\sin\phi)^{2\lambda_\kappa} \mathrm{d}\phi.$$

If, say, f is a polynomial, then $T_\theta^\kappa f$ is also a polynomial, hence continuous in θ, and $(f *_\kappa g_n)(x)$ converges to $T_\theta^\kappa f(x)$ as $n \to \infty$. In particular, $T_\theta^\kappa f \ge 0$ if $f \ge 0$. Furthermore, the construction shows that $\|g_n\|_{\lambda_\kappa,1} = 1$. Hence by (i) and Young's inequality, $\|f *_\kappa g_n\|_{\kappa,p} \le \|f\|_{\kappa,p}$, so that by the Fatou lemma,

$$\|T_\theta^\kappa f\|_{\kappa,p} \le \liminf_{n\to\infty} \|f *_\kappa g_n\|_{\kappa,p} \le \|f\|_{\kappa,p}.$$

Take, for example, $f_n = S_n^\delta(h_\kappa^2; f)$, the Cesàro (C,δ) means, with $\delta \ge 2\lambda_\kappa + 1$, so that f_n are positive and f_n converge to f in $L^p(h_\kappa^2, \mathbb{S}^{d-1})$. Passing to the limit then establishes the inequality in Eq. (7.4.12). Since Eq. (7.4.10) also implies that $T_\theta^\kappa g$ converges to g as $\theta \to 0$ for all polynomials g, the limit in Eq. (7.4.10) follows from the triangle inequality.

Finally, let $h(t) := g(-t)$. Then $(f *_\kappa g)(-x) = (f *_\kappa h)(x)$ directly from the definition. Since $h(\cos\theta) = g(\cos(\pi-\theta))$, a change of variable in Eq. (7.4.11) shows that

$$(f *_\kappa g)(-x) = c_{\lambda_\kappa} \int_0^\pi T_{\pi-\theta}^\kappa f(x) g(\cos\theta)(\sin\theta)^{2\lambda_\kappa} \mathrm{d}\theta.$$

Setting $g = g_n$ and letting $n \to \infty$ as in the previous paragraph then proves (iv). □

Definition 7.4.8. For $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$, the Poisson integral of f is defined by

$$P_r^\kappa f(\xi) := \frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} f(y) P_r^\kappa(\xi, y) h_\kappa^2(y) \mathrm{d}\sigma(y), \quad \xi \in \mathbb{S}^{d-1}, \tag{7.4.13}$$

where $0 < r < 1$, and the kernel $P_r^\kappa(x,\cdot)$ is given by

$$P_r^\kappa(x,y) := V_\kappa\left[\frac{1-r^2}{(1-2r\langle\cdot,y\rangle+r^2)^{\lambda_\kappa+1}}\right](x). \tag{7.4.14}$$

Lemma 7.4.9. *For $0<r<1$, the Poisson kernel has the following properties:*

(1) *For $x,y\in\mathbb{S}^{d-1}$, $P_r^\kappa(x,y)=\sum_{n=0}^\infty Z_n^\kappa(x,y)r^n$.*
(2) *$P_r^\kappa f=\sum_{n=0}^\infty r^n \operatorname{proj}_n^\kappa f$.*
(3) *$P_r^\kappa(x,y)\ge 0$ and $\frac{1}{\omega_d^\kappa}\int_{\mathbb{S}^{d-1}}P_r^\kappa(x,y)h_\kappa^2(y)\mathrm{d}\sigma(y)=1$.*

The lemma is an analogue of Lemma 2.2.4 and is proved in the same way.

Theorem 7.4.10. *For $f\in L^p(h_\kappa^2,\mathbb{S}^{d-1})$, if $1\le p<\infty$, or $f\in C(\mathbb{S}^{d-1})$ if $p=\infty$, then $\lim_{r\to 1^-}\|P_r^\kappa f-f\|_{\kappa,p}=0$.*

Proof. Directly from the definition of convolution, we can write

$$P_r^\kappa f = f *_\kappa p_r^{\lambda_\kappa} \qquad \text{with} \quad p_r^{\lambda_\kappa}(t)=\frac{1-r^2}{(1-2rt+r^2)^{\lambda_\kappa+1}}. \tag{7.4.15}$$

The function $p_r^\lambda(t)$ is also the generating function of $\frac{n+\lambda}{\lambda}C_n^\lambda(t)$, which implies, in particular, that $c_\lambda\int_0^\pi p_r^\lambda(\cos\theta)(\sin\theta)^{2\lambda}\mathrm{d}\theta=1$. By Eq. (7.4.12), for every $\varepsilon>0$, there is a $\delta>0$ such that $\|T_\theta^\kappa f-f\|_{\kappa,p}<\varepsilon$ whenever $|\theta|<\delta$. Hence, by Eq. (7.4.11) and the Minkowski inequality, we obtain

$$\begin{aligned}\|P_r^\kappa f-f\|_{\kappa,p} &= \left\|c_\lambda\int_0^\pi\left(T_\theta^\kappa f(x)-f(x)\right)p_r^{\lambda_\kappa}(\cos\theta)(\sin\theta)^{2\lambda_\kappa}\mathrm{d}\theta\right\|_{\kappa,p}\\ &\le c_{\lambda_\kappa}\int_0^\pi\|T_\theta^\kappa f-f\|_{\kappa,p}\,p_r^{\lambda_\kappa}(\cos\theta)(\sin\theta)^{2\lambda_\kappa}\mathrm{d}\theta\\ &\le \varepsilon+2\|f\|_{\kappa,p}c_{\lambda_\kappa}\int_\delta^\pi p_r^{\lambda_\kappa}(\cos\theta)(\sin\theta)^{2\lambda_\kappa}\mathrm{d}\theta.\end{aligned}$$

Since for $\delta<\theta\le\pi$, $p_r^{\lambda_k}(\cos\theta)\le p_r^{\lambda_\kappa}(\cos\delta)\to 0$ as $r\to 1^-$, taking the limit $r\to 1^-$ in the above inequality completes the proof. □

7.5 Maximal Functions and the Multiplier Theorem

A maximal function can be defined via the translation operator.

Definition 7.5.1. For h_κ associated with the reflection group and $f\in L^1(h_\kappa^2,\mathbb{S}^{d-1})$, define the maximal function

$$\mathscr{M}_\kappa f(x) := \sup_{0<\theta\le\pi}\frac{\int_0^\theta T_\phi^\kappa|f|(x)(\sin\phi)^{2\lambda_\kappa}\mathrm{d}\phi}{\int_0^\theta(\sin\phi)^{2\lambda_\kappa}\mathrm{d}\phi}. \tag{7.5.1}$$

Recall that the spherical cap is denoted by $\mathtt{c}(x,\theta) = \{y \in \mathbb{S}^{d-1} : \langle x,y\rangle \ge \cos\theta\}$. We define $\mathtt{b}(x,\theta)$ to be the convex hull of $\mathtt{c}(x,\theta)$ in $\mathbb{R}^d$,

$$\mathtt{b}(x,\theta) := \{y \in \mathbb{B}^d : \langle x,y\rangle \ge \cos\theta\}, \quad x \in \mathbb{S}^{d-1}, \quad 0 \le \theta \le \pi. \tag{7.5.2}$$

Let χ_E denote the characteristic function of the set E.

Proposition 7.5.2. *An alternative definition of $\mathscr{M}_\kappa f$ is given by*

$$\begin{aligned}\mathscr{M}_\kappa f(x) &= \sup_{0<\theta\le\pi} \frac{\int_{\mathbb{S}^{d-1}} |f(y)| V_\kappa[\chi_{\mathtt{b}(x,\theta)}](y) h_\kappa^2(y)\mathrm{d}\sigma(y)}{\int_{\mathbb{S}^{d-1}} V_\kappa[\chi_{\mathtt{b}(x,\theta)}](y) h_\kappa^2(y)\mathrm{d}\sigma(y)} \\ &= \sup_{0<\theta\le\pi} \frac{(|f| *_\kappa \chi_{[\cos\theta,1]})(x)}{c_{\lambda_\kappa}\int_0^\theta (\sin\phi)^{2\lambda_\kappa}\mathrm{d}\phi}.\end{aligned} \tag{7.5.3}$$

Proof. It is easy to verify that $\chi_{[\cos\theta,1]}(\langle x,y\rangle) = \chi_{\mathtt{b}(x,\theta)}(y)$ for $x \in \mathbb{S}^{d-1}$ and $y \in \mathbb{B}^d$. Hence, the alternative definition follows, by Eq. (7.4.11), from

$$c_{\lambda_k}\int_0^\theta T_\phi^\kappa |f|(x)(\sin\phi)^{2\lambda_\kappa}\mathrm{d}\phi = \left(|f| *_\kappa \chi_{[\cos\theta,1]}\right)(x)$$

and applying Eq. (7.2.12) to the denominator of Eq. (7.5.3). ◻

To state the weak-type inequality, we define, for any measurable subset E of $\mathbb{S}^{d-1}$, the measure with respect to h_κ^2 as

$$\mathrm{meas}_\kappa E := \int_E h_\kappa^2(y)\mathrm{d}\sigma(y).$$

The main result in this section is the boundedness of $\mathscr{M}_\kappa f$:

Theorem 7.5.3. *If $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$, then $\mathscr{M}_\kappa f$ satisfies*

$$\mathrm{meas}_\kappa\{x : \mathscr{M}_\kappa f(x) \ge \alpha\} \le c\frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0. \tag{7.5.4}$$

Furthermore, if $f \in L^p(h_\kappa^2, \mathbb{S}^{d-1})$ for $1 < p \le \infty$, then $\|M_\kappa f\|_{\kappa,p} \le c\|f\|_{\kappa,p}$.

In order to prove this theorem, we use Theorem 3.1.8. Recall that P_r^κ denotes the Poisson integral. By Lemma 7.4.9, it is easy to verify that $T^t := P_r^\kappa$ with $r = e^{-t}$ satisfies all the requirements in the definition. We will need another semigroup, one that is the discrete analogue of the heat operator,

$$H_t^\kappa f := f *_\kappa q_t^\kappa, \qquad q_t^\kappa(s) := \sum_{n=0}^\infty e^{-n(n+2\lambda_\kappa)t}\frac{n+\lambda_\kappa}{\lambda_\kappa}C_n^{\lambda_\kappa}(s). \tag{7.5.5}$$

Lemma 7.5.4. *The family of operators* $\{H_\kappa^t\}$ *is a symmetric diffusion semigroup.*

Proof. It will be shown that the kernel q_t^κ is nonnegative, from which it follows immediately that H_κ^t are positive and that $\|q_t^\kappa\|_{\lambda_\kappa,1} = 1$ by the orthogonality of the Gegenbauer polynomials. Hence, by Young's inequality, $\|H_t^\kappa f\|_{\kappa,p} \le \|f\|_{\kappa,p}$. Other requirements in Definition 3.1.7 can be directly verified.

To show the positivity of the kernel q_t^κ, we consider the differential operator

$$\mathscr{D}_\lambda = (1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (2\lambda_\kappa+1)x\frac{d}{\mathrm{d}x}, \quad x \in [-1,1], \tag{7.5.6}$$

which has the Gegenbauer polynomial as an eigenfunction (Section B.2),

$$\mathscr{D}_\lambda C_n^\lambda(x) = -n(n+2\lambda)C_n^\lambda(x), \quad x \in [-1,1], \tag{7.5.7}$$

and we set $\lambda = \lambda_k$ in the rest of the proof. We first show that the differential operator $\mathscr{D}_\lambda$ satisfies the following property:

$$\text{if } f \in C^2[-1,1] \text{ and } f(x_0) = \min_{x\in[-1,1]} f(x), \text{ then } \mathscr{D}_\lambda f(x_0) \ge 0. \tag{7.5.8}$$

Indeed, if $x_0 = 1$ or -1, then $f(x_0) = \min_{x\in[-1,1]} f(x)$ implies that $x_0 f'(x_0) \le 0$, so that by Eq. (7.5.6), $\mathscr{D}_\lambda f(x_0) = -(2\lambda+1)x_0 f'(x_0) \ge 0$. If $x_0 \in (-1,1)$, then $f'(x_0) = 0$, and by Taylor's theorem, $f''(x_0) \ge 0$. Hence Eq. (7.5.6) again implies that $\mathscr{D}_\lambda f(x_0) \ge 0$.

Next, we show that for every $a > 0$, $aI - \mathscr{D}_\lambda$ is an invertible operator on the space Π of algebraic polynomials of one variable, where I denotes the identity operator, and the inverse $(aI - \mathscr{D}_\lambda)^{-1}$ is positive on Π. The invertibility of the operator $aI - \mathscr{D}_\lambda$ follows directly from the facts that

$$(aI - \mathscr{D}_\lambda)C_n^\lambda = (a + n(n+2\lambda))C_n^\lambda, \quad n \in \mathbb{N},$$

and that $a + n(n+2\lambda) > 0$ for all $n \in \mathbb{N}$. To show the positivity of $(aI - \mathscr{D}_\lambda)^{-1}$, let f be a nonnegative polynomial on $[-1,1]$ and $g(x) = (aI - \mathscr{D}_\lambda)^{-1} f$. Assume that $x_0 \in [-1,1]$ and $\min_{x\in[-1,1]} g(x) = g(x_0)$. If $g(x_0) < 0$, then by Eq. (7.5.8), $f(x_0) = ag(x_0) - \mathscr{D}_\lambda g(x_0) < 0$, which contradicts the fact that $f \ge 0$ on $[-1,1]$. This proves the positivity of the operator $(aI - \mathscr{D}_\lambda)^{-1}$ on Π.

Third, we show that the operator $e^{t\mathscr{D}}$ is positive on Π for all $t > 0$. To see this, observe that

$$e^{t\mathscr{D}_\lambda} C_n^\lambda(x) = e^{-tn(n+2\lambda)} C_n^\lambda(x) = \lim_{m\to\infty}\left(1 + \frac{tn(n+2\lambda)}{m}\right)^{-m} C_n^\lambda(x),$$

which implies that for every $f \in \Pi$,

$$e^{t\mathscr{D}_\lambda} f = \lim_{m\to\infty}\left(\frac{t}{m}\left(mt^{-1}I - \mathscr{D}_\lambda\right)^{-1}\right)^m f.$$

Since $(mt^{-1}I - \mathscr{D}_\lambda)^{-1}$ is positive for each $m \in \mathbb{N}$, it follows that the operator $e^{t\mathscr{D}_\lambda}$ is positive on Π.

Finally, we prove the positivity of the kernel q_t^κ. Define

$$g_N(x) := \frac{1}{A_N^\delta} \sum_{j=0}^{N} A_{N-j}^\delta \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa}(x), \quad x \in [-1,1], \quad \delta > 2\lambda_\kappa + 1.$$

By Lemma B.1.2, g_N is nonnegative on $[-1,1]$. Thus,

$$e^{t\mathscr{D}_\lambda} g_N(x) = \frac{1}{A_N^\delta} \sum_{j=0}^{N} A_{N-j}^\delta e^{-tj(j+2\lambda)} \frac{j+\lambda}{\lambda} C_j^\lambda(x) \ge 0, \quad x \in [-1,1]. \tag{7.5.9}$$

Since the sum on the right-hand side of this last equation is the Cesàro (C,δ) mean of q_t^κ, which converges to q_t^κ pointwise as $N \to \infty$, it follows that $q_t^\kappa(x) \ge 0$ for all $x \in [-1,1]$. □

Lemma 7.5.5. *The Poisson and the heat semigroups are connected by*

$$P_{e^{-t}}^\kappa f(x) = \int_0^\infty \phi_t(s) H_s^\kappa f(x) \mathrm{d}s, \tag{7.5.10}$$

where

$$\phi_t(s) := \frac{t}{2\sqrt{\pi}} s^{-3/2} e^{-(\frac{t}{2\sqrt{s}} - \lambda_\kappa \sqrt{s})^2}.$$

Furthermore, if $f(x) \ge 0$ for all $x \in \mathbb{S}^{d-1}$, then

$$P_*^\kappa f(x) := \sup_{0<r<1} P_r^\kappa f(x) \le c \sup_{s>0} \frac{1}{s} \int_0^s H_u^\kappa f(x) \mathrm{d}u. \tag{7.5.11}$$

Consequently, $P_*^\kappa f$ is bounded on $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ for $1 < p \le \infty$ and is of weak type $(1,1)$.

Proof. The fact that $\{H_t^\kappa\}$ is a semigroup of operators allows us to apply the Hopf–Dunford–Schwartz ergodic theorem [157, p.48], which shows that the maximal operator $\sup_{s>0}\left(\frac{1}{s}\int_0^s H_u^\kappa f(x) \mathrm{d}u\right)$ is bounded on $L^p(h_\kappa^2, \mathbb{S}^{d-1})$ for $1 < p \le \infty$ and is of weak type $(1,1)$. Therefore, it is sufficient to prove Eqs. (7.5.10) and (7.5.11).

First, we prove Eq. (7.5.10). Applying the well-known identity [157, p. 46]

$$e^{-v} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-v^2/4u} \mathrm{d}u, \qquad v > 0,$$

with $v = (n + \lambda_\kappa)t$ and making a change of variables $s = t^2/4u$, we obtain that

$$\begin{aligned} e^{-nt} &= e^{\lambda_\kappa t} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{n(n+2\lambda_\kappa)t^2}{4u}} e^{-\frac{\lambda_\kappa^2 t^2}{4u}} \mathrm{d}u \\ &= \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-n(n+2\lambda_\kappa)s} s^{-3/2} e^{-(\frac{t}{2\sqrt{s}} - \lambda_\kappa \sqrt{s})^2} \mathrm{d}s \end{aligned}$$

$$= \int_0^\infty e^{-n(n+2\lambda_\kappa)s} \phi_t(s)\,\mathrm{d}s.$$

Multiplying by $\mathrm{proj}_n^\kappa f$ and summing over n proves the integral relation (7.5.10).

For the proof of Eq. (7.5.11), we use Eq. (7.5.10) and integration by parts to obtain

$$\begin{aligned} P_{e^{-t}}^\kappa f(x) &= -\int_0^\infty \left(\int_0^s H_u^\kappa f(x)\mathrm{d}u \right) \phi_t'(s)\mathrm{d}s \\ &\le \sup_{s>0} \left(\frac{1}{s} \int_0^s H_u^\kappa f(x)\mathrm{d}u \right) \int_0^\infty s|\phi_t'(s)|\mathrm{d}s, \end{aligned}$$

where the derivative of $\phi_t'(s)$ is taken with respect to s. Furthermore, since $\mathscr{P}_r^\kappa f = f *_k p_r^\kappa$ and $|p_r^\kappa(t)| \le c$ for $0 < r \le e^{-1}$, by Eq. (7.4.15), it follows that

$$\sup_{0<r\le e^{-1}} |P_r^\kappa f(x)| \le c\|f\|_{1,\kappa} = c \lim_{s\to\infty} \frac{1}{s} \int_0^s H_u^\kappa(|f|)(x)\,\mathrm{d}u.$$

Therefore, to finish the proof of Eq. (7.5.11), it suffices to show that $\sup_{0<t\le 1} \int_0^\infty s|\phi_t'(s)|\mathrm{d}s$ is bounded by a constant. A quick computation shows that $\phi_t'(s) > 0$ if $s < \alpha_t$ and $\phi_t'(s) < 0$ if $s > \alpha_t$, where

$$\alpha_t := \frac{t^2}{3+\sqrt{9+4\lambda_\kappa^2 t^2}} \sim t^2, \qquad 0 \le t \le 1.$$

Since the integral of $\phi_t(s)$ over $[0,\infty)$ is 1 and $\phi_t(s) \ge 0$, integration by parts gives

$$\begin{aligned} \int_0^\infty s|\phi_t'(s)|\mathrm{d}s &= 2\alpha_t\phi_t(\alpha_t) - \int_0^{\alpha_t} \phi_t(s)\mathrm{d}s + \int_{\alpha_t}^\infty \phi_t(s)\mathrm{d}s \\ &\le 2\alpha_t\phi_t(\alpha_t) + 1 = \frac{t}{\sqrt{\pi\alpha_t}} e^{-\frac{(t-2\lambda_\kappa\alpha_t)^2}{4\alpha_t}} + 1 \le c, \end{aligned}$$

as desired. □

We are now in a position to prove Theorem 7.5.3.

Proof of Theorem 7.5.3. From the definition of p_r^κ in Eq. (7.4.15), if $1-r \sim \theta$, then

$$\begin{aligned} p_r^\kappa(\cos\theta) &= \frac{1-r^2}{\left((1-r)^2 + 4r\sin^2\frac{\theta}{2}\right)^{\lambda_\kappa+1}} \\ &\ge c \frac{1-r^2}{\left((1-r)^2 + r\theta^2\right)^{\lambda_\kappa+1}} \ge c\,(1-r)^{-(2\lambda_\kappa+1)}. \end{aligned}$$

For $j \ge 0$, define $r_j := 1 - 2^{-j}\theta$ and set $B_j := \left\{y \in \mathbb{B}^d : 2^{-j-1}\theta \le \mathtt{d}(x,y) \le 2^{-j}\theta\right\}$. The lower bound of p_r^κ proved above shows that

$$\chi_{B_j}(y) \le c\,(2^{-j}\theta)^{2\lambda_k+1} p_{r_j}^\kappa(\langle x,y\rangle),$$

which implies immediately that

$$\chi_{\mathtt{b}(x,\theta)}(y) \le \sum_{j=0}^\infty \chi_{B_j}(y) \le c\,\theta^{2\lambda_k+1} \sum_{j=0}^\infty 2^{-j(2\lambda_\kappa+1)} p_{r_j}^\kappa(\langle x,y\rangle).$$

Since V_κ is a positive linear operator, applying V_κ to the above inequality gives

$$\begin{aligned}
&\int_{S^{d-1}} |f(y)| V_\kappa\left[\chi_{\mathtt{b}(x,\theta)}\right](y) h_\kappa^2(y) \mathrm{d}\sigma(y)\\
&\quad \le c\,\theta^{2\lambda_\kappa+1} \sum_{j=0}^\infty 2^{-j(2\lambda_\kappa+1)} \int_{S^{d-1}} |f(y)| V_\kappa\left[p_{r_j}(\langle x,y\rangle)\right](y) h_\kappa^2(y) \mathrm{d}\sigma(y)\\
&\quad = c\,\theta^{2\lambda_\kappa+1} \sum_{j=0}^\infty 2^{-j(2\lambda_\kappa+1)} P_{r_j}^\kappa(|f|;x)\\
&\quad \le c\,\theta^{2\lambda_\kappa+1} \sup_{0<r<1} P_r^\kappa(|f|;x).
\end{aligned}$$

Dividing by $\theta^{2\lambda_\kappa+1}$ and using the fact that by Eq. (7.2.12),

$$\frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} V_\kappa[\chi_{\mathtt{b}(x,\theta)}](y) h_\kappa^2(y) \mathrm{d}\sigma(y) = c_{\lambda_\kappa} \int_0^\theta (\sin\phi)^{2\lambda_\kappa} \mathrm{d}\phi \sim \theta^{2\lambda_\kappa+1},$$

we have proved that $\mathscr{M}_\kappa f(x) \le c P_*^\kappa |f|(x)$. The desired result now follows from Lemma 7.5.5. □

Theorem 7.5.6. *Assume that $g \in L^1(w_{\lambda_\kappa}, [-1,1])$ and $|g(\cos\theta)| \le k(\theta)$ for all θ, where $k(\theta)$ is a continuous, nonnegative, and decreasing function on $[0,\pi]$. Then for $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$,*

$$|(f *_\kappa g)(x)| \le c\mathscr{M}_\kappa(|f|)(x), \quad x \in \mathbb{S}^{d-1},$$

where $c = \int_0^\pi k(\theta)(\sin\theta)^{2\lambda_\kappa} \mathrm{d}\theta$.

Proof. Since T_θ^κ is a positive operator, it follows from Eq. (7.4.11) that

$$|f *_\kappa g(x)| \le \lambda_\kappa \int_0^\pi k(\theta) T_\theta(|f|)(x)(\sin\theta)^{2\lambda_\kappa} \mathrm{d}\theta,$$

from which the proof follows from Eq. (7.5.1) and integration by parts exactly as in the proof of Theorem 2.3.6. □

Corollary 7.5.7. *If $\delta > \lambda_\kappa$ and $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$, then for every $x \in \mathbb{S}^{d-1}$,*

$$\sup_{n\ge 0} |S_n^\delta(h_\kappa^2; f, x)| \le c\,[\mathscr{M}_\kappa f(x) + \mathscr{M}_\kappa f(-x)]\,. \tag{7.5.12}$$

If, in addition, $\delta \ge 2\lambda_k + 1$, then the term $\mathscr{M}_\kappa f(-x)$ in Eq. (7.5.12) can be dropped.

Proof. Using Eq. (7.4.7), the proof can be carried out exactly as in Theorem 2.4.7. □

Theorem 7.5.8. *For $\delta > \lambda_\kappa$, $1 < p < \infty$, and any sequence $\{n_j\}$ of positive integers,*

$$\left\| \left(\sum_{j=0}^\infty \left|S_{n_j}^\delta(h_\kappa^2; f_j)\right|^2 \right)^{1/2} \right\|_{\kappa,p} \le c \left\| \left(\sum_{j=0}^\infty |f_j|^2 \right)^{1/2} \right\|_{\kappa,p}. \tag{7.5.13}$$

Proof. Since the maximal function $M_\kappa f$ in Eq. (7.5.3) is not a Hardy–Littlewood maximal function when $\kappa \ne 0$, we cannot use the Fefferman–Stein inequality as in the proof of Theorem 3.2.3. We follow the approach of [157, pp.104–105], which uses a generalization of the Riesz convexity theorem for sequences of functions. Let $L^p(\ell^q)$ denote the space of all sequences $\{f_k\}$ of functions for which the norm

$$\|(f_k)\|_{L^p(\ell^q)} := \left(\int_{\mathbb{S}^{d-1}} \left(\sum_{j=0}^\infty |f_j(x)|^q \right)^{p/q} h_\kappa^2(x) \mathrm{d}\sigma(x) \right)^{1/p}$$

is finite. If T is bounded as an operator on $L^{p_0}(\ell^{q_0})$ and on $L^{p_1}(\ell^{q_1})$, then the Riesz convexity theorem states that T is also bounded on $L^{p_t}(\ell^{q_t})$, where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}, \quad 0 \le t \le 1.$$

We apply this theorem to the operator T that maps the sequence $\{f_j\}$ to the sequence $\{S_{n_j}^\delta(h_\kappa^2; f_j)\}$. From Eq. (7.5.12) and Theorem 7.5.3, it follows that T is bounded on $L^p(\ell^p)$. It is also bounded on $L^p(\ell^\infty)$, since

$$\left\| \sup_{j\ge 0} \left|S_{n_j}^\delta(h_\kappa^2; f_j)\right| \right\|_{\kappa,p} \le c \left\| \mathscr{M}_\kappa \Big(\sup_{j \ge 0} |f_j| \Big) \right\|_{\kappa,p} \le c \left\| \sup_{j\ge 0} |f_j| \right\|_{\kappa,p}.$$

Hence, the Riesz convexity theorem shows that T is bounded on $L^p(\ell^q)$ if $1 < p \le q \le \infty$. In particular, T is bounded on $L^p(\ell^2)$ if $1 < p \le 2$. The case $2 < p < \infty$ follows by a standard duality argument, since the dual space of $L^p(\ell^2)$ is $L^{p'}(\ell^2)$, where $1/p + 1/p' = 1$, under the pairing

$$\langle (f_j),(g_j)\rangle := \int_{\mathbb{S}^{d-1}} \sum_j f_j(x) g_j(x) h_\kappa^2(x) \mathrm{d}\sigma(x),$$

and T is self-adjoint under this pairing, since $S_n^\delta(h_\kappa^2)$ is self-adjoint in $L^2(h_\kappa^2;\mathbb{S}^{d-1})$. □

The proof of Theorem 7.5.8 actually yields the following Fefferman–Stein inequality for the maximal function $\mathscr{M}_\kappa f$ associated with a reflection group.

Corollary 7.5.9. *Let $1 < p \le 2$ and let f_j be a sequence of functions. Then*

$$\left\| \Big(\sum_j (\mathscr{M}_\kappa f_j)^2 \Big)^{1/2} \right\|_{\kappa,p} \le c \left\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \right\|_{\kappa,p}. \tag{7.5.14}$$

We do not know whether the inequality Eq. (7.5.14) holds for $2 < p < \infty$ for a general finite reflection group. It does, however, hold for $1 < p < \infty$ in the case of $\mathbb{Z}_2^d$, as will be shown in the next section.

We conclude this section with a Marcinkiewicz-type multiplier theorem for $L^p(h_\kappa^2,\mathbb{S}^{d-1})$ in analogy with Theorem 3.3.1.

Theorem 7.5.10. *Let $\{\mu_j\}_{j=0}^\infty$ be a bounded sequence of real numbers such that*

$$\sup_j 2^{j(k-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^k \mu_l| \le c < \infty,$$

where k is the smallest integer $\ge \lambda_\kappa + 1$. Then $\{\mu_j\}$ defines an $L^p(h_\kappa^2,\mathbb{S}^{d-1})$, $1 < p < \infty$, multiplier; that is,

$$\left\| \sum_{j=0}^\infty \mu_j \operatorname{proj}_j^\kappa f \right\|_{\kappa,p} \le c\|f\|_{\kappa,p}, \qquad 1 < p < \infty,$$

where c is independent of μ_j and f.

The proof of this theorem follows exactly the proof of Theorem 3.3.1 with obvious modifications. The inequality (7.5.13) plays an essential role, which requires the condition $k \ge \lfloor \lambda_k + 1 \rfloor$.

7.6 Maximal Function for $\mathbb{Z}_2^d$-Invariant Weight

In the case of $h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$, as in Eq. (7.1.5), invariant under the group $G = \mathbb{Z}_2^d$, an explicit formula of the intertwining operator V_κ, as shown in Eq. (7.2.2), is known. This allows us to show that the maximal function $\mathscr{M}_\kappa f$ in Eq. (7.5.3) is bounded by the weighted Hardy–Littlewood maximal function.

Definition 7.6.1. For $f \in L^1(h_\kappa^2;\mathbb{S}^{d-1})$, the weighted Hardy–Littlewood maximal function is defined by

$$M_\kappa f(x) := \sup_{0<\theta\le\pi} \frac{\int_{\mathtt{c}(x,\theta)} |f(y)| h_\kappa^2(y) \mathrm{d}\sigma(y)}{\int_{\mathtt{c}(x,\theta)} h_\kappa^2(y) \mathrm{d}\sigma(y)}. \tag{7.6.1}$$

Since h_κ is a doubling weight, $M_\kappa f$ enjoys the usual properties of maximal functions. We will show that the maximal function $\mathscr{M}_\kappa f$ is bounded by a sum of $M_\kappa f$, so that the properties of $\mathscr{M}_k f$ can be deduced from those of the Hardy–Littlewood function $M_\kappa f$. We shall need several lemmas. Let $\mathtt{b}(x,\theta)$ be the set defined in Eq. (7.5.2).

Lemma 7.6.2. *For $x \in \mathbb{S}^{d-1}$, let $\bar{x} := (|x_1|,\ldots,|x_d|)$. Then the support set of the function $V_\kappa\left[\chi_{\mathtt{b}(x,\theta)}\right](y)$ is $\{y \in \mathbb{S}^{d-1} : \mathtt{d}(\bar{x},\bar{y}) \le \theta\}$; in other words,*

$$V_\kappa\left[\chi_{\mathtt{b}(x,\theta)}\right](y) = 0 \qquad \text{if } \langle \bar{x},\bar{y}\rangle < \cos\theta.$$

Proof. The explicit formula (7.2.2) of V_κ shows that $V_\kappa\left[\chi_{\mathtt{b}(x,\theta)}\right](y) = 0$ if

$$\chi_{\mathtt{b}(x,\theta)}(t_1y_1,t_2y_2,\ldots,t_dy_d) = 0$$

for every $t \in [-1,1]^d$ or if $x_1y_1t_1 + \cdots + x_dy_dt_d < \cos\theta$, which clearly holds if $\langle \bar{x},\bar{y}\rangle < \cos\theta$. □

Our second lemma contains the essential estimate for an upper bound of $\mathscr{M}_\kappa f$.

Lemma 7.6.3. *For $0 \le \theta \le \pi$, $x = (x_1,\ldots,x_d) \in \mathbb{S}^{d-1}$, and $y \in \mathbb{S}^{d-1}$,*

$$\left|V_\kappa\left[\chi_{\mathtt{b}(x,\theta)}\right](y)\right| \le c \prod_{j=1}^d \frac{\theta^{2\kappa_j}}{(|x_j|+\theta)^{2\kappa_j}} \chi_{\mathtt{c}(\bar{x},\theta)}(\bar{y}). \tag{7.6.2}$$

Proof. The presence of $\chi_{\mathtt{c}(\bar{x},\theta)}(\bar{y})$ on the right-hand side of the stated estimate comes from Lemma 7.6.2. Hence, we need only to derive the upper bound of $V_\kappa\left[\chi_{\mathtt{b}(x,\theta)}\right](y)$ for $\mathtt{d}(\bar{x},\bar{y}) \le \theta$, which we assume in the rest of the proof. If $\pi/2 \le \theta \le \pi$, then $\theta/(|x_j|+\theta) \ge c$, and the inequality (7.6.2) is trivial. So we can assume $0 \le \theta \le \pi/2$ below. By the definition of V_κ,

$$V_\kappa\left[\chi_{\mathtt{b}(x,\theta)}\right](y) = c_\kappa \int_{\sum_{i=1}^d t_ix_iy_i \ge \cos\theta} \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i-1}\mathrm{d}t,$$

where t also satisfies $t \in [-1,1]^d$. We first enlarge the domain of integration to $\{t \in [-1,1]^d : \sum_{i=1}^d |t_ix_iy_i| \ge \cos\theta\}$ and replace $(1+t_i)$ by 2, so that we can use the $\mathbb{Z}_2^d$ symmetry of the integrand to replace the integral by one on $[0,1]^d$,

$$V_\kappa\left[\chi_{\mathtt{b}(x,\theta)}\right](y) \le c\int_{\sum_{i=1}^d |t_i x_i y_i|\ge\cos\theta} \prod_{i=1}^d (1-t_i^2)^{\kappa_i-1}\mathrm{d}t$$
$$\le c\int_{t\in[0,1]^d, \sum_{i=1}^d t_i|x_iy_i|\ge\cos\theta} \prod_{i=1}^d (1-t_i^2)^{\kappa_i-1}\mathrm{d}t.$$

To continue, we enlarge the domain of the integral to $\{t\in[0,1]^d : t_j|x_jy_j| + \sum_{i\ne j}|x_iy_i| \ge \cos\theta\}$ for each fixed j to obtain

$$V_\kappa\left[\chi_{\mathtt{b}(x,\theta)}\right](y) \le c\prod_{j=1}^d \int_{0\le t_j\le 1, t_j|x_jy_j|+\sum_{i\ne j}|x_iy_i|\ge\cos\theta} (1-t_j)^{\kappa_j-1}\mathrm{d}t_j.$$

For each j, we denote the last integral by I_j. First of all, there is the trivial estimate $I_j \le \int_0^1 (1-t_j)^{\kappa_j-1}\mathrm{d}t_j = \kappa_j^{-1}$. Second, for $\langle\bar{x},\bar{y}\rangle \ge \cos\theta$, we have the estimate

$$I_j \le \int_{\frac{\cos\theta-\sum_{i\ne j}|x_iy_i|}{|x_jy_j|}}^1 (1-t_j)^{\kappa_j-1}\mathrm{d}t_j = \kappa_j^{-1}\frac{(\langle\bar{x},\bar{y}\rangle-\cos\theta)^{\kappa_j}}{|x_jy_j|^{\kappa_j}}.$$

Together, we have established the estimate

$$I_j \le \kappa_j^{-1}\min\left\{1, \frac{(\langle\bar{x},\bar{y}\rangle-\cos\theta)^{\kappa_j}}{|x_jy_j|^{\kappa_j}}\right\}.$$

Recall that $d(\bar{x},\bar{y})\le\theta$. Assume first that $|x_j|\ge 2\theta$. Then $|x_j|\ge(|x_j|+\theta)/2$. The inequality $\big||x_j|-|y_j|\big| \le \|\bar{x}-\bar{y}\| \le \mathtt{d}(\bar{x},\bar{y}) \le \theta$ implies that $|y_j|\ge|x_j|-\theta\ge|x_j|/2$, so that $|y_j|\ge(|x_j|+\theta)/4$. Furthermore, write $t := \mathtt{d}(\bar{x},\bar{y})\le\theta$ and recall that $\theta\le\pi/2$. We have then

$$\langle\bar{x},\bar{y}\rangle-\cos\theta = \cos t-\cos\theta = 2\sin\frac{\theta-t}{2}\sin\frac{t+\theta}{2} \le (\theta-t)\theta \le \theta^2.$$

Putting these ingredients together, we arrive at an upper bound for I_j,

$$I_j \le c\frac{\theta^{2\kappa_j}}{(|x_j|+\theta)^{2\kappa_j}},$$

under the assumption that $|x_j|\ge 2\theta$. This estimate also holds for $|x_j|\le 2\theta$, since in that case, $\theta/(|x_j|+\theta)\ge 1/3$. Thus, the last inequality holds for all x and for all j, from which the stated inequality follows immediately. □

We are now ready to prove our first main result. For $x\in\mathbb{R}^d$ and $\varepsilon\in\mathbb{Z}_2^d$, we write $x\varepsilon := (x_1\varepsilon_1,\ldots,x_d\varepsilon_d)$.

Theorem 7.6.4. *Let $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$. Then for every $x \in \mathbb{S}^{d-1}$,*

$$\mathscr{M}_\kappa f(x) \le c \sum_{\varepsilon \in \mathbb{Z}_2^d} M_\kappa f(x\varepsilon). \tag{7.6.3}$$

Proof. Since $\{y \in \mathbb{S}^{d-1} : \mathtt{d}(\bar{x}, \bar{y}) \le \theta\} = \bigcup_{\varepsilon \in \mathbb{Z}_2^d} \{y \in \mathbb{S}^{d-1} : \mathtt{d}(x\varepsilon, y) \le \theta\}$, it follows from Lemma 7.6.2 that

$$\begin{aligned} J_\theta f(x) &:= \int_{\mathbb{S}^{d-1}} |f(y)| V_\kappa \left[\chi_{\mathtt{b}(x,\theta)}\right](y) h_\kappa^2(y) \mathrm{d}\sigma(y) \\ &= \int_{\langle \bar{x}, \bar{y}\rangle \ge \cos\theta} |f(y)| V_\kappa \left[\chi_{\mathtt{b}(x,\theta)}\right](y) h_\kappa^2(y) \mathrm{d}\sigma(y) \\ &\le \sum_{\varepsilon \in \mathbb{Z}_2^d} \int_{\langle x\varepsilon, y\rangle \ge \cos\theta} |f(y)| V_\kappa \left[\chi_{\mathtt{b}(x,\theta)}\right](y) h_\kappa^2(y) \mathrm{d}\sigma(y). \end{aligned}$$

Consequently, using Lemma 7.6.3 and Eq. (5.1.9), we conclude that

$$\begin{aligned} J_\theta f(x) &\le c \sum_{\varepsilon \in \mathbb{Z}_2^d} \prod_{j=1}^d \frac{\theta^{2\kappa_j}}{(|x_j| + \theta)^{2\kappa_j}} \int_{\langle x\varepsilon, y\rangle \ge \cos\theta} |f(y)| h_\kappa^2(y) \mathrm{d}\sigma(y) \\ &\le c\, \theta^{2|\kappa|+d-1} \sum_{\varepsilon \in \mathbb{Z}_2^d} M_\kappa f(x\varepsilon). \end{aligned}$$

Since the denominator in $\mathscr{M}_\kappa f$ is of order $\theta^{2|\kappa|+d-1} = \theta^{2\lambda_\kappa+1}$, dividing the above inequality by $\theta^{2|\kappa|+d-1}$ and taking the supremum over θ leads to Eq. (7.6.3). □

There are several applications of Theorem 7.6.4. First, we need some notation. For $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, we write $x < y$ if $x_j < y_j$ for all $1 \le j \le d$. We denote by $\mathbb{1}$ the vector $\mathbb{1} := (1, 1, \ldots, 1) \in \mathbb{R}^d$. Moreover, we extend the definitions of h_τ, meas_τ, $\|\cdot\|_{\tau,p}$, $L^p(h_\tau^2; \mathbb{S}^{d-1})$, and the Hardy–Littlewood maximal function M_τ to the full range of $\tau = (\tau_1, \ldots, \tau_d) > -\frac{\mathbb{1}}{2}$ from $\tau \ge 0$.

As an application of Theorem 7.6.4, we can prove the boundedness of $\mathscr{M}_\kappa f$ on the spaces $L^p(h_\tau^2; \mathbb{S}^{d-1})$ for a wider range of τ.

Theorem 7.6.5. *If $-\frac{\mathbb{1}}{2} < \tau \le \kappa$ and $f \in L^1(h_\tau^2; \mathbb{S}^{d-1})$, then $\mathscr{M}_\kappa f$ satisfies*

$$\mathrm{meas}_\tau\{x : \mathscr{M}_\kappa f(x) \ge \alpha\} \le c \frac{\|f\|_{\tau,1}}{\alpha}, \quad \forall \alpha > 0. \tag{7.6.4}$$

Furthermore, if $1 < p \le \infty$, $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$, and $f \in L^p(h_\tau^2; \mathbb{S}^{d-1})$, then

$$\|\mathscr{M}_k f\|_{\tau,p} \le c\|f\|_{\tau,p}. \tag{7.6.5}$$

Proof. We start with the proof of Eq. (7.6.4). If $\tau = (\tau_1, \ldots, \tau_d) \le \kappa$, then

$$\int_{\mathsf{c}(x,\theta)} |f(y)| h_\kappa^2(y)\,\mathrm{d}\sigma(y) \le c\left(\prod_{j=1}^d (|x_j|+\theta)^{2(\kappa_j-\tau_j)}\right)\int_{\mathsf{c}(x,\theta)} |f(y)| h_\tau^2(y)\,\mathrm{d}\sigma(y),$$

which, together with Eq. (5.1.9), implies

$$M_\kappa f(x) \le c M_\tau f(x), \quad x \in \mathbb{S}^{d-1}, \quad \tau \le \kappa.$$

Hence, using the inequality (7.6.3), we obtain that for $-\frac{\mathbb{1}}{2} < \tau \le \kappa$,

$$\begin{aligned}
\mathrm{meas}_\tau\{x : \mathscr{M}_k f(x) \ge \alpha\} &\le \sum_{\varepsilon\in\mathbb{Z}_2^d} \mathrm{meas}_\tau\left\{x : M_\kappa f(x\varepsilon) \ge c\,\alpha/2^d\right\} \\
&\le \sum_{\varepsilon\in\mathbb{Z}_2^d} \mathrm{meas}_\tau\left\{x : M_\tau f(x\varepsilon) \ge c'\,\alpha\right\} \\
&= \sum_{\varepsilon\in\mathbb{Z}_2^d} \int_{\{y:\ M_\tau f(y\varepsilon)\ge c'\,\alpha\}} h_\tau^2(y)\mathrm{d}\sigma(y).
\end{aligned}$$

Using the $\mathbb{Z}_2^d$-invariance of h_τ and the fact that M_τ is of weak type $(1,1)$ with respect to the doubling measure $h_\tau^2(y)\,\mathrm{d}\sigma(y)$, we then conclude that

$$\mathrm{meas}_\tau\{x : \mathscr{M}_k f(x) \ge \alpha\} = 2^d \int_{\{x: M_\tau f(x)\ge c'\,\alpha\}} h_\tau^2(y)\mathrm{d}\sigma(y) \le c\frac{\|f\|_{\tau,1}}{\alpha},$$

which proves Eq. (7.6.4). For the proof of Eq. (7.6.5), we choose a number $q \in (1,p)$ such that $\tau < q\kappa + \frac{q-1}{2}\mathbb{1}$ and claim that it is sufficient to prove

$$M_\kappa f(x) \le c\left(M_\tau(|f|^q)(x)\right)^{\frac{1}{q}}. \tag{7.6.6}$$

Indeed, using Eq. (7.6.6), the inequality (7.6.5) will follow from Eq. (7.6.3), the $\mathbb{Z}_2^d$-invariance of h_τ, and the boundedness of M_τ on the space $L^{\frac{p}{q}}(h_\tau^2;\mathbb{S}^{d-1})$.

To prove Eq. (7.6.6), we use Hölder's inequality with $q' = \frac{q}{q-1}$ and Eq. (5.1.9) to obtain

$$\begin{aligned}
&\int_{\mathsf{c}(x,\theta)} |f(y)| h_\kappa^2(y)\,\mathrm{d}\sigma(y) \\
&\quad\le \left(\int_{\mathsf{c}(x,\theta)} |f(y)|^q h_\tau^2(y)\,\mathrm{d}\sigma(y)\right)^{\frac{1}{q}} \left(\int_{\mathsf{c}(x,\theta)} h^2_{q'\kappa-\frac{q'}{q}\tau}(y)\,\mathrm{d}\sigma(y)\right)^{\frac{1}{q'}} \\
&\quad\sim \left(\int_{\mathsf{c}(x,\theta)} |f(y)|^q h_\tau^2(y)\,\mathrm{d}\sigma(y)\right)^{\frac{1}{q}} \left(\prod_{j=1}^d (|x_j|+\theta)^{2\kappa_j-\frac{2\tau_j}{q}}\right)\theta^{\frac{d}{q'}} \\
&\quad\sim \mathrm{meas}_\kappa(\mathsf{c}(x,\theta))\left(\frac{1}{\mathrm{meas}_\tau(\mathsf{c}(x,\theta))}\int_{\mathsf{c}(x,\theta)} |f(y)|^q h_\tau^2(y)\,\mathrm{d}\sigma(y)\right)^{\frac{1}{q}},
\end{aligned}$$

where we have also used the fact that the assumption $\tau < q\kappa + \frac{q-1}{2}\mathbb{1}$ is equivalent to $q'\kappa - \frac{q'}{q}\tau > -\frac{\mathbb{1}}{2}$. This proves Eq. (7.6.6) and completes the proof. □

For our next application of Theorem 7.6.4 we will need the following result.

Lemma 7.6.6. *Let $1 < p < \infty$ and let W be a nonnegative integrable function on $\mathbb{S}^{d-1}$. Then*

$$\int_{\mathbb{S}^{d-1}} |M_\kappa f(x)|^p W(x) h_\kappa^2(x)\,\mathrm{d}\sigma \le c_p \int_{\mathbb{S}^{d-1}} |f(x)|^p M_\kappa W(x) h_\kappa^2(x)\,\mathrm{d}\sigma. \tag{7.6.7}$$

Proof. Such a result was first proved in [74] for maximal functions on $\mathbb{R}^d$. The proof can be adapted to yield Lemma 7.6.6. Indeed, the fact that h_κ^2 is a doubling weight shows that the Hardy–Littlewood maximal function defined by Eq. (7.6.1) satisfies

$$M_\kappa f(x) \sim \sup_{x \in E \in \mathscr{C}} \frac{\int_E |f(y)| h_\kappa^2(y) \mathrm{d}\sigma(y)}{\int_E h_\kappa^2(y) \mathrm{d}\sigma(y)},$$

where $\mathscr{C}$ is the collection of all spherical caps in $\mathbb{S}^{d-1}$, which implies that

$$\int_{\mathsf{c}(x,\theta)} |f(y)| h_\kappa^2(y) \mathrm{d}\sigma(y) \le c\,(\mathrm{meas}_\kappa\,\mathsf{c}(x,\theta)) \inf_{z \in \mathsf{c}(x,\theta)} M_\kappa f(z)$$

for every spherical cap $\mathsf{c}(x,\theta)$. As a consequence, we can prove the key inequality

$$\mathrm{meas}_\kappa(E) \le \frac{c}{\alpha} \int_{\mathbb{S}^{d-1}} |f(y)| M_\kappa W(y) h_\kappa^2(y) \mathrm{d}\sigma(y)$$

for every compact set E in $\{x \in \mathbb{S}^{d-1} : M_k f(x) > \alpha\}$, as in the proof for the maximal function on $\mathbb{R}^d$ in [158, pp. 54–55]. In fact, Eq. (7.6.7) holds with $h_\kappa^2(y)\mathrm{d}\sigma$ replaced by any doubling measure $\mathrm{d}\mu$ on the sphere. □

We are now ready to establish the Fefferman–Stein inequality for the maximal function $\mathscr{M}_\kappa$ in the setting of $\mathbb{Z}_2^d$ for $1 < p < \infty$, whereas it was proved in Corollary 7.5.9 for a general reflection group but only for $1 < p \le 2$.

Theorem 7.6.7. *Let $1 < p < \infty$, $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$, and let $\{f_j\}_{j=1}^\infty$ be a sequence of functions. Then*

$$\left\| \left(\sum_{j=1}^\infty (\mathscr{M}_\kappa f_j)^2 \right)^{1/2} \right\|_{\tau,p} \le c \left\| \left(\sum_{j=1}^\infty |f_j|^2 \right)^{1/2} \right\|_{\tau,p}. \tag{7.6.8}$$

Proof. Using Theorem 7.6.4 and Minkowski's inequality, we obtain

$$\left\| \left(\sum_j (\mathscr{M}_\kappa f_j)^2 \right)^{1/2} \right\|_{\tau,p} \le c \left\| \left(\sum_j \left(\sum_{\varepsilon \in \mathbb{Z}_2^d} M_\kappa f_j(x\varepsilon) \right)^2 \right)^{1/2} \right\|_{\tau,p}$$
$$\le c \sum_{\varepsilon \in \mathbb{Z}_2^d} \left\| \left(\sum_j (M_\kappa f_j(x\varepsilon))^2 \right)^{1/2} \right\|_{\tau,p}$$
$$\le c \left\| \left(\sum_j (M_\kappa f_j)^2 \right)^{1/2} \right\|_{\tau,p}.$$

Thus, it is sufficient to prove

$$J := \left\| \left(\sum_j (M_\kappa f_j)^2 \right)^{1/2} \right\|_{\tau,p} \le c \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{\tau,p}. \tag{7.6.9}$$

We first consider the case $1 < p \le 2$. Let q be chosen such that $1 < q < p$ and $\tau < q\kappa + \frac{(q-1)\mathbb{1}}{2}$. We use the inequality (7.6.6) to obtain

$$J \le c \left\| \left(\sum_j (M_\tau(|f_j|^q))^{\frac{2}{q}} \right)^{\frac{q}{2}} \right\|_{\tau,\frac{p}{q}}^{\frac{1}{q}} \le c \left\| \left(\sum_j |f_j|^{q\cdot\frac{2}{q}} \right)^{\frac{q}{2}} \right\|_{\tau,\frac{p}{q}}^{\frac{1}{q}} = \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_{\tau,p},$$

where we have used the classical Fefferman–Stein inequality for the maximal function M_τ and the space $L^{\frac{p}{q}}(\ell^{\frac{2}{q}})$ in the second step. This proves Eq. (7.6.9) for $1 < p \le 2$.

Next, we consider the case $2 < p < \infty$. Noticing that

$$-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{(p-1)\mathbb{1}}{2} \Longleftrightarrow -\frac{\mathbb{1}}{2} < \frac{2}{p}\tau + \left(\frac{1}{p} - \frac{1}{2}\right)\mathbb{1} < 2\kappa + \frac{\mathbb{1}}{2},$$

we may choose a vector $\mu \in \mathbb{R}^d$ such that

$$-\frac{\mathbb{1}}{2} < \frac{2}{p}\tau + \left(\frac{1}{p} - \frac{1}{2}\right)\mathbb{1} < \mu < 2\kappa + \frac{\mathbb{1}}{2} \tag{7.6.10}$$

and a number $1 < q < 2$ such that $\mu < q\kappa + \frac{q-1}{2}\mathbb{1}$. Let g be a nonnegative function on $\mathbb{S}^{d-1}$ satisfying $\|g\|_{\tau,\frac{p}{p-2}} = 1$ and

$$\left\| \left(\sum_j |M_\kappa f_j|^2 \right)^{\frac{1}{2}} \right\|_{\tau,p}^2 = \int_{\mathbb{S}^{d-1}} \left(\sum_j |M_\kappa f_j(x)|^2 \right) g(x) h_\tau^2(x)\, d\sigma(x).$$

Then by the assumption $\mu < q\kappa + \frac{q-1}{2}\mathbb{1}$, Eqs. (7.6.6), (7.6.7) with $p = 2/q > 1$, and Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{S}^{d-1}}\Big(\sum_j |M_\kappa f_j(x)|^2\Big) g(x)h_\tau^2(x)\,\mathrm{d}\sigma &\le c\sum_j \int_{\mathbb{S}^{d-1}} \big(M_\mu\big(|f_j|^q\big)(x)\big)^{\frac{2}{q}} g(x)h_\tau^2(x)\,\mathrm{d}\sigma \\ &\le c\int_{\mathbb{S}^{d-1}}\Big(\sum_j |f_j(x)|^2\Big) M_\mu(gh_{\tau-\mu}^2)(x)h_\mu^2(x)\,\mathrm{d}\sigma \\ &\le c\Big\|\Big(\sum_j |f_j|^2\Big)^{\frac12}\Big\|_{\tau,p}^2 \Big\|M_\mu(gh_{\tau-\mu}^2)h_{\mu-\tau}^2\Big\|_{\tau,\frac{p}{p-2}}.\end{aligned}$$

Using the boundedness of $M_\kappa f$ and Eq. (7.6.10), we have

$$\begin{aligned}\Big\|M_\mu(gh_{\tau-\mu}^2)h_{\mu-\tau}^2\Big\|_{\tau,\frac{p}{p-2}} &= \Big\|M_\mu(gh_{\tau-\mu}^2)\Big\|_{\frac{p}{p-2}\mu-\frac{2}{p-2}\tau,\frac{p}{p-2}} \\ &\le c\Big\|gh_{\tau-\mu}^2\Big\|_{\frac{p}{p-2}\mu-\frac{2}{p-2}\tau,\frac{p}{p-2}} \\ &= c\|g\|_{\tau,\frac{p}{p-2}} = c.\end{aligned}$$

Putting these two inequalities together, we have proved the inequality (7.6.9) for the case $2 < p < \infty$. □

Remark 7.6.8. By Eq. (7.5.12), we can get a weighted inequality for the Cesàro means by replacing $\mathscr{M}_\kappa f_j$ in Eq. (7.6.8) by $S_{n_j}^\delta(h_\kappa^2; f_j)$, which gives a $\|\cdot\|_{\tau,p}$-weighted version of Theorem 7.5.8 that holds under the condition $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$.

7.7 Notes and Further Results

The theory of h-harmonics was pioneered by Dunkl. The Dunkl operators were introduced in [64], and the intertwining operators and the integral kernel appeared in [65]. For more results and the proof of the results in Sects. 7.1 and 7.2 for general reflection groups, we refer to [67].

The first study of h-harmonic expansions appeared in [177], which contains Eq. (7.2.2) and the formula for Z_n^κ in Corollary 7.2.10, proved by summing over a specific orthonormal basis using special function identities, and [176], which contains Eqs. (7.2.10) and (7.2.13). The proof of Eq. (7.2.13) in Theorem 7.2.9 is different from that of [176], which was based on the orthogonal expansion of $V_\kappa f$ on the unit ball. The Funk–Hecke formula (7.2.11) was established in [183].

The convolution and the translation operators were defined in [190], and used to study weighted best approximation on the sphere; see Chap. 10. The maximal function $\mathscr{M}_\kappa f$ was defined in [188]; the results in Sects. 7.5 and 7.6 were established in [47]. The positivity of the heat kernel for the Jacobi series, used in Lemma 7.5.4, was established in [17], our proof seems to be new.

The lack of an explicit formula for the intertwining operator V_κ has been an obstacle to deriving deeper results that rely on the essence of reflection groups. For the symmetric group S_3 with $h_\kappa(x) = |(x_1-x_2)(x_2-x_3)(x_3-x_1)|^\kappa$ for $x \in \mathbb{S}^2$ and the dihedral group $I(4)$ with $h_\kappa(x) = |x_1x_2|^{\kappa_0}|x_1^2-x_2^2|^{\kappa_1}$, some explicit integral formulas for V_κ are given in [66, 182], but neither is in a form strong enough for carrying out further analysis. The positivity of the intertwining operator was proved in [144]. At the moment, little is known about the intertwining operator for reflection groups other than $\mathbb{Z}_2^d$. For further information, we refer to [67].

Chapter 8
Boundedness of Projection Operators and Cesàro Means

The Cesàro (C,δ) means are important tools, and their boundedness in appropriate function spaces often serves as a gauge of our understanding of the underlying structure. In this chapter, we establish the boundedness of the Cesáro means for h-harmonic expansions with respect to the product weights $h_\kappa^2(x) = \prod_{i=1}^d |x_i|^{2\kappa_i}$ on the sphere. The main results are stated and discussed in the first section. The central piece of the proof is a pointwise estimate of the integral kernel of the means, which involves a multiple beta integral of the Jacobi polynomials. These integrals will be estimated in the second section, and the pointwise estimate of the kernels is given in the third section, from which the upper bound of the norm of (C,δ) means is deduced in the fourth section. Finally, a lower estimate of the norm is given in the fifth section.

8.1 Boundedness of Cesàro Means Above the Critical Index

Recall that the Cesàro (C,δ) means for h-harmonic expansions are denoted by $S_n^\delta(h_\kappa^2;f)$. Let $\|S_n^\delta(h_\kappa^2)\|_{\kappa,p}$ denote the operator norm of $S_n^\delta(h_\kappa^2)$ in $L^p(h_\kappa^2;\mathbb{S}^{d-1})$. By Theorem 7.4.4, the (C,δ) means are bounded in $L^p(h_\kappa^2,\mathbb{S}^{d-1})$, that is, $\|S_n^\delta(h_\kappa^2)\|_{\kappa,p}$ is bounded by a constant independent of n, if $\delta > \lambda_\kappa$, where λ_k is given in Eq. (7.3.2).

The condition $\delta > \lambda_\kappa$, however, is not sharp. In this section, we consider the case that

$$h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}, \quad \kappa_i \ge 0, \quad x \in \mathbb{S}^{d-1}, \tag{8.1.1}$$

which is invariant under $\mathbb{Z}_2^d$. In this case, the condition $\delta > \lambda_\kappa$ means that

$$\delta > \lambda_\kappa = |\kappa| + \frac{d-2}{2}.$$

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The sharp condition for the boundedness is given in the following theorem, which gives the asymptotic order of $S_n^\delta(h_\kappa^2)$ for all $\delta \ge -1$, where $S_n^\delta(h_\kappa^2; f)$ with $\delta = -1$ is understood to be $\operatorname{proj}_n^\kappa f$.

Theorem 8.1.1. *Let h_κ be as in Eq. (8.1.1). Let $\delta > -1$ and define*

$$\sigma_\kappa := \frac{d-2}{2} + |\kappa| - \min_{1\le i\le d} \kappa_i.$$

Then for $p = 1$ and $p = \infty$,

$$\|\operatorname{proj}_n(h_\kappa^2)\|_{\kappa,p} \sim n^{\sigma_\kappa} \quad \text{and} \quad \|S_n^\delta(h_\kappa^2)\|_{\kappa,p} \sim \begin{cases} 1, & \delta > \sigma_\kappa, \\ \log n, & \delta = \sigma_\kappa, \\ n^{-\delta+\sigma_\kappa}, & -1 < \delta < \sigma_\kappa. \end{cases}$$

By the Riesz interpolation theorem, the asymptotic order also serves as an upper bound of the operators in the $\|\cdot\|_{\kappa,p}$ norm for $1 < p < \infty$. Since $S_n^\delta(h_\kappa^2; f)$ converges to f as $n \to \infty$ whenever f is a fixed polynomial, the theorem has the following immediate corollary.

Corollary 8.1.2. *If $\delta > \sigma_\kappa$, then for $f \in L^p(h_\kappa^2, \mathbb{S}^{d-1})$ and $1 \le p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ when $p = \infty$,*

$$\sup_n \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,p} \le c\|f\|_{\kappa,p} \quad \text{and} \quad \lim_{n\to\infty} \|f - S_n^\delta(h_\kappa^2; f)\|_{\kappa,p} = 0.$$

Furthermore, for $p = 1$ and ∞, the convergence fails in general if $\delta = \sigma_\kappa$.

Because of these results, we call σ_κ the critical index of the (C, δ) means. When all κ_i are equal to 0, the h-harmonic expansions agree with the ordinary spherical harmonic expansions, and these results agree with Theorem 2.4.4 and Corollary 2.4.5. Furthermore, $\sigma_\kappa = \lambda_\kappa$ only when one of the κ_i equals zero; that is, whenever $\kappa_i > 0$ for all $1 \le i \le d$, we have $\lambda_\kappa > \sigma_\kappa$, and $\delta > \lambda_\kappa$ is a stronger condition for the boundedness of $S_n^\delta(h_\kappa^2; f)$.

When $\delta = 0$, the mean $S_n^0(h_\kappa; f)$ is the partial sum operator,

$$S_n(h_\kappa^2; f) = \sum_{j=0}^n \operatorname{proj}_j^\kappa f,$$

which is the best approximation to f from $\Pi_n(\mathbb{S}^{d-1})$ in $L^2(h_\kappa^2; \mathbb{S}^{d-1})$. By Theorem 8.1.1, $\|S_n(h_\kappa^2)\|_{\kappa,p} \sim \|\operatorname{proj}_n(h_\kappa^2)\|_{\kappa,p} \sim n^{\sigma_\kappa}$ for $p = 1$ or ∞.

Let us state another result that illustrates the role of the weight function. The impact of the weight $h_\kappa(x) = \prod_{i=1}^d |x_i|$ should be centered on the great circles that are defined by the intersection of $\mathbb{S}^{d-1}$ and the coordinate planes. In fact, these great circles act like boundaries on $\mathbb{S}^{d-1}$. Let us define

$$\mathbb{S}^{d-1}_{\text{int}} := \mathbb{S}^{d-1} \setminus \bigcup_{i=1}^{d} \{x \in \mathbb{S}^{d-1} : x_i = 0\},$$

which is the interior region bounded by these boundaries on $\mathbb{S}^{d-1}$.

Theorem 8.1.3. *Let h_κ be as in Eq. (8.1.1). Let f be continuous on $\mathbb{S}^{d-1}$. If $\delta > \frac{d-2}{2}$, then $S_n^\delta(h_\kappa^2; f)$ converges to f for every $x \in \mathbb{S}^{d-1}_{\text{int}}$, and the convergence is uniform over each compact subset of the interior $\mathbb{S}^{d-1}_{\text{int}}$.*

In other words, for the pointwise convergence away from the zero set of h_κ, the convergence holds if $\delta > (d-2)/2$, independent of the value κ, which is the same as the critical index for the Cesàro means of the ordinary harmonic expansions.

The proofs of these results are long and are given in the rest of the chapter. The means $S_n^\delta(h_\kappa^2; f)$ are integral operators with kernel $K_n^\delta(h_\kappa^2; \cdot, \cdot)$,

$$S_n^\delta(h_k^2; f, x) = \frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} f(y) K_n^\delta(h_\kappa^2; x, y) h_\kappa^2(y) \mathrm{d}\sigma(y), \tag{8.1.2}$$

where by Eqs. (7.4.1), (7.4.7), and (7.4.8),

$$K_n^\delta(h_\kappa^2; x, y) = V_\kappa[k_n^\delta(w_{\lambda_\kappa}; 1, \langle \cdot, y \rangle)](x).$$

In the case of $\mathbb{Z}_2^d$, the explicit formula for the intertwining operator in Eq. (7.2.2) then leads to

$$\begin{aligned} K_n^\delta(h_\kappa^2; x, y) = c_\kappa \int_{[-1,1]^d} & k_n^\delta(w_{\lambda_k}; 1, x_1 y_1 t_1 + \cdots + x_d y_d t_d) \\ & \times \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} \mathrm{d}t. \end{aligned} \tag{8.1.3}$$

For the proof of the case of $\delta > \lambda_\kappa$ in Theorem 7.4.4, we took the average of V_κ and erased the action of $\mathbb{Z}_2^d$ in the process, which means that we did not make use of the structure of $\mathbb{Z}_2^d$ that is embedded in the kernel. For the proof of the case $\delta > \sigma_\kappa$, we will dig into the structure of the kernel $K_n^\delta(h_\kappa; \cdot, \cdot)$ and establish first, in the next two sections, a pointwise estimate, which is of interest in itself.

8.2 A Multiple Beta Integral of the Jacobi Polynomials

Because of Eq. (8.1.3), the estimate of the kernel $K_n^\delta(h_\kappa^2; x, y)$ requires an estimate of the multiple beta integral, and the key component of the kernel turns out to be a Jacobi polynomial. In this section, we deal with a pointwise estimate of a multiple beta integral of a Jacobi polynomial.

Recall that throughout the book, we use the notation $\|x\|$ to denote the Euclidean norm of x in $\mathbb{R}^d$, and we write $|x| = |x_1| + \cdots + |x_d|$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

The properties of the Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ are summarized in Appendix B.1. In particular, we will make use of the following estimate (see Eq. (B.1.7)), which we restate as a lemma.

Lemma 8.2.1. *For an arbitrary real number α and $t \in [0,1]$,*

$$|P_n^{(\alpha,\beta)}(t)| \le cn^{-1/2}(1-t+n^{-2})^{-(\alpha+1/2)/2}. \tag{8.2.1}$$

The estimate on $[-1,0]$ follows from the fact that $P_n^{(\alpha,\beta)}(t) = (-1)^n P_n^{(\beta,\alpha)}(-t)$.

This estimate motivates the following definition.

Definition 8.2.2. Let $n, v \in \mathbb{N}_0$ and let $\rho, r, \mu \in \mathbb{R}$ such that $r > 0$ and $|\rho| + r \le 1$. A function $f : [-r,r] \to \mathbb{R}$ is said to be in the class $\mathscr{S}_n^v(\rho, r, \mu)$ if there exist functions F_j on $[-r,r]$, $j = 0, 1, \ldots, v$, such that $F_j^{(j)}(x) = f(x)$ for $x \in [-r,r]$ and

$$|F_j(x)| \le cn^{-2j}\left(1 + n\sqrt{1-|\rho+x|}\right)^{-\mu-\frac{1}{2}+j}, \quad x \in [-r,r]. \tag{8.2.2}$$

Using Eq. (8.2.1) and the fact that $\frac{\mathrm{d}}{\mathrm{d}t} P_n^{(\alpha,\beta)}(t) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(t)$, we observe that $n^{-\alpha}P_n^{(\alpha,\beta)} \in \mathscr{S}_n^v(0,1,\alpha)$ for all $v \in \mathbb{N}_0$.

Lemma 8.2.3. *Assume $\delta > 0$ and $0 < |a| \le r$. Let $f \in \mathscr{S}_n^v(\rho, r, \mu)$ with $v \ge |\mu| + 2\delta + \frac{3}{2}$, and let $\xi \in C^\infty[-1,1]$ be such that* $\operatorname{supp} \xi \subset [-\frac{1}{2}, 1]$. *Then for $|x| \le r - |a|$,*

$$\left| \int_{-1}^{1} f(at+x)(1-t)^{\delta-1}\xi(t)\,\mathrm{d}t \right| \le cn^{-2\delta}|a|^{-\delta}(1+n\sqrt{1-A})^{-\mu-\frac{1}{2}+\delta}, \tag{8.2.3}$$

where $A := |\rho + a + x|$.

Proof. To simplify the notation, we define

$$B := \frac{1+n\sqrt{1-A}}{2n^2|a|}.$$

First we claim that for $t \in [1-B, 1]$,

$$1 + n\sqrt{1-|at+x+\rho|} \sim 1 + n\sqrt{1-A} = 2n^2|a|B. \tag{8.2.4}$$

Indeed, if $t \in [1-B, 1]$ and $n\sqrt{1-A} \le 1$, then

$$\begin{aligned} n^2(1-|\rho+at+x|) &= n^2(1-A) + n^2(A - |\rho+at+x|) \\ &\le n^2(1-A) + n^2|a|B \le 1 + \frac{1+n\sqrt{1-A}}{2} \le 2, \end{aligned}$$

so that both sides of Eq. (8.2.4) are bounded above and below by a constant, whereas if $t \in [1-B, 1]$ and $n\sqrt{1-A} \ge 1$, then

$$\left| n\sqrt{1-|\rho+at+x|} - n\sqrt{1-A} \right| \le \frac{n|a||1-t|}{\sqrt{1-|at+x+\rho|}+\sqrt{1-A}} \le \frac{n|a|B}{\sqrt{1-A}} \le n^2|a|B = \frac{1+n\sqrt{1-A}}{2},$$

from which Eq. (8.2.4) follows by the triangle inequality. From Eqs. (8.2.4) and (8.2.2) with $j=0$, we obtain

$$\left| \int_{\max\{1-B,-1\}}^{1} f(at+x)(1-t)^{\delta-1}\xi(t)\,\mathrm{d}t \right| \le c(1+n\sqrt{1-A})^{-\mu-\frac{1}{2}} \int_{1-B}^{1} (1-t)^{\delta-1}\,\mathrm{d}t \le c\,n^{-2\delta}|a|^{-\delta}(1+n\sqrt{1-A})^{-\mu-\frac{1}{2}+\delta}.$$

If $B \ge \frac{3}{2}$, then the desired inequality (8.2.3) follows from the above inequality. Hence, we assume $B \le \frac{3}{2}$ from now on.

We now consider the integral over $[-1, 1-B]$. Set

$$\ell = \left\lfloor |\mu| + 2\delta + \frac{1}{2} \right\rfloor + 1.$$

Then $1 \le \ell \le \nu$ by our assumption. Since $\xi \in C^\infty[-1,1]$ with supp $\xi \subset [-\frac{1}{2}, 1]$, we use Eqs. (8.2.2), (8.2.4) and integration by parts ℓ times to obtain

$$\begin{aligned}
&\left| \int_{-1}^{1-B} f(at+x)(1-t)^{\delta-1}\xi(t)\,\mathrm{d}t \right| \\
&\quad \le c\sum_{j=1}^{\ell} |a|^{-j} n^{-2j}(1+n\sqrt{1-A})^{-\mu-\frac{1}{2}+j} B^{\delta-j} + c|a|^{-\ell} \int_{-\frac{1}{2}}^{1-B} |F_\ell(at+x)|(1-t)^{\delta-\ell-1}\,\mathrm{d}t \\
&\quad \le c\,n^{-2\delta}|a|^{-\delta}(1+n\sqrt{1-A})^{-\mu-\frac{1}{2}+\delta} \\
&\qquad + c|a|^{-\ell} n^{-2\ell} \int_{-\frac{1}{2}}^{1-B} (1+n\sqrt{1-|\rho+x+at|})^{-\mu-\frac{1}{2}+\ell}(1-t)^{\delta-\ell-1}\,\mathrm{d}t.
\end{aligned}$$

The first term is the desired upper bound in Eq. (8.2.3). We need to estimate only the second term, which we denote by L. A change of variable $s = |a|(1-t)$ shows that

$$\begin{aligned}
L &:= n^{-2\ell}|a|^{-\delta} \int_{B|a|}^{\frac{3}{2}|a|} (1+n\sqrt{1-|a+x+\rho-s\cdot \mathrm{sgn}\,a|})^{-\mu-\frac{1}{2}+\ell} s^{\delta-\ell-1}\,\mathrm{d}s \\
&= n^{-2\ell}|a|^{-\delta}(L_1+L_2),
\end{aligned}$$

where L_1 and L_2 are integrals over the intervals $I_1 = \left[|a|B,\ \frac{3}{2}|a|\right] \cap \left[0,\ \frac{1-A}{2}\right]$ and $I_2 = \left[|a|B,\ \frac{3}{2}|a|\right] \cap \left[\frac{1-A}{2},\ \infty\right)$, respectively. If $s \in I_1$, then

$$|A - |a+x+\rho - s\cdot \operatorname{sgn} a|| \le |s| \le (1-A)/2,$$

so that $1 - |a+x+\rho - s\cdot \operatorname{sgn} a| \sim 1-A$ by the triangle inequality. Consequently,

$$\begin{aligned} L_1 &:= \int_{I_1} (1+n\sqrt{1-|a+x+\rho-s\cdot \operatorname{sgn} a|})^{-\mu-\frac{1}{2}+\ell} s^{\delta-\ell-1}\,\mathrm{d}s \\ &\le c(1+n\sqrt{1-A})^{-\mu-\frac{1}{2}+\ell}\int_{B|a|}^{\infty} s^{\delta-\ell-1}\,\mathrm{d}t \\ &\le c(1+n\sqrt{1-A})^{-\mu-\frac{1}{2}+\ell}(|a|B)^{\delta-\ell} \\ &\le cn^{2\ell-2\delta}(1+n\sqrt{1-A})^{-\mu-\frac{1}{2}+\delta}. \end{aligned}$$

If $s \in I_2$, then $s \ge (1-A)/2$ and $1 - |a+x+\rho-s\cdot \operatorname{sgn} a| \le 1-A+s \sim s$ by the triangle inequality. Consequently, since $\ell \ge \mu + \frac{1}{2}$, it follows that

$$\begin{aligned} L_2 &:= \int_{I_2} (1+n\sqrt{1-|a+x+\rho-s\cdot \operatorname{sgn} a|})^{-\mu-\frac{1}{2}+\ell} s^{\delta-\ell-1}\,\mathrm{d}s \\ &\le c\int_{I_2} (1+n\sqrt{s})^{-\mu-\frac{1}{2}+\ell} s^{\delta-\ell-1}\,\mathrm{d}s \\ &\le cn^{-\mu-\frac{1}{2}+\ell}\int_{|a|B}^{\infty} s^{-\frac{\mu}{2}+\delta-\frac{\ell}{2}-\frac{5}{4}}\,\mathrm{d}s, \end{aligned}$$

since $n^2|a|B \ge \frac{1}{2}$. Using the fact that $\ell > -\mu + 2\delta - \frac{1}{2}$, we obtain

$$\begin{aligned} L_2 \le cn^{-\mu-\frac{1}{2}+\ell}(|a|B)^{-\frac{\mu}{2}+\delta-\frac{\ell}{2}-\frac{1}{4}} &= cn^{2\ell-2\delta}(1+n\sqrt{1-A})^{-\frac{\mu}{2}+\delta-\frac{1}{4}-\frac{\ell}{2}} \\ &\le cn^{2\ell-2\delta}(1+n\sqrt{1-A})^{-\mu+\delta-\frac{1}{2}}, \end{aligned}$$

using the inequality $\ell \ge \mu + \frac{1}{2}$. Putting these estimates together completes the proof of Eq. (8.2.3). □

Lemma 8.2.4. *Let* $\kappa_j > 0$, $a_j \ne 0$, $\xi_j \in C^\infty[-1,1]$ *with* $\operatorname{supp} \xi_j \subset [-\frac{1}{2}, 1]$ *for* $j = 1, 2, \ldots m$, *and let* $\sum_{j=1}^{m} |a_j| \le 1$. *Define*

$$f_m(x) := \int_{[-1,1]^m} P_n^{(\alpha,\beta)}\left(\sum_{j=1}^{m} a_j t_j + x\right) \prod_{j=1}^{m} \xi_j(t_j)(1-t_j)^{\kappa_j - 1}\,\mathrm{d}t$$

for $|x| \le 1 - \sum_{j=1}^{m} |a_j|$. *If* $\alpha \ge \beta$, *then*

$$|f_m(x)| \le cn^{\alpha - 2\tau_m} \prod_{j=1}^{m} |a_j|^{-\kappa_j} \left(1 + n\sqrt{1 - |A_m + x|}\right)^{-\alpha - \frac{1}{2} + \tau_m},$$

where $A_m := \sum_{j=1}^{m} a_j$ *and* $\tau_m := \sum_{j=1}^{m} \kappa_j$.

Proof. Since $n^{-\alpha} P_n^{(\alpha,\beta)}(x) \in \mathscr{S}_n^{v_1}(0,1,\alpha)$ for $v_1 := \lfloor |\alpha| + 2\kappa_1 \rfloor + 4$, we can apply Lemma 8.2.3 to conclude that

$$\begin{aligned} n^{-\alpha}|f_1(x)| &= n^{-\alpha} \left| \int_{-1}^{1} P_n^{(\alpha,\beta)}(a_1 t_1 + x)(1 - t_1)^{\kappa_1 - 1} \xi_1(t_1)\,\mathrm{d}t_1 \right| \\ &\le c|a_1|^{-\kappa_1} n^{-2\kappa_1} \left(1 + n\sqrt{1 - |a_1 + x|}\right)^{-\alpha - \frac{1}{2} + \kappa_1}, \end{aligned}$$

where $|a_1| + |x| \le 1$. Hence, the conclusion of the lemma holds when $m = 1$.

Assume that the conclusion of the lemma has been proved for a positive integer m, we now consider the case $m+1$. Let $v_{m+1} = \lfloor |\alpha - \tau_m| + 2\kappa_{m+1} \rfloor + 4$. For $i = 0, 1, \ldots, v_{m+1}$, we define

$$F_i(x) = C_{n,i} \int_{[-1,1]^m} P_{n+i}^{(\alpha - i, \beta - i)} \left(\sum_{j=1}^{m} a_j t_j + x \right) \prod_{j=1}^{m} (1 - t_j)^{\kappa_j - 1} \xi_j(t_j)\,\mathrm{d}t,$$

where $C_{n,0} = 1$ and $C_{n,i} = 2^i / \prod_{l=1}^{i} (n + \alpha + \beta + 1 - l) = \mathscr{O}(n^{-i})$ for $i = 1, \ldots, v_{m+1}$. Using Eq. (B.1.5), it is easy to verify that $F_i^{(i)}(x) = f(x)$ for $i = 0, 1, \ldots, v_{m+1}$. Furthermore, the induction hypothesis shows that

$$|F_i(x)| \le cn^{\alpha} - 2_i \left(\prod_{j=1}^{m} |a_j|^{-\kappa_j} n^{-2\kappa_j} \right) (1 + n\sqrt{1 - |A_m + x|})^{-\alpha - \frac{1}{2} + \tau_m + i}$$

for $i = 0, 1, \ldots, v_{m+1}$, where $|x| + \sum_{j=1}^{m} |a_j| \le 1$. By the definition of $\mathscr{S}_n^{v}(\rho, r, \mu)$, this shows that

$$n^{-\alpha} \left(\prod_{j=1}^{m} |a_j|^{\kappa_j} n^{2\kappa_j} \right) f_m(x) \in \mathscr{S}_n^{v_{m+1}} \left(A_m, 1 - \sum_{j=1}^{m} |a_j|, \alpha - \tau_m \right).$$

Since $v_{m+1} \ge |\alpha - \tau_m| + 2\kappa_{m+1} + \frac{3}{2}$, we can then apply Lemma 8.2.3 to the integral

$$f_{m+1}(x) = \int_{-1}^{1} f_m(a_{m+1} t_{m+1} + x)(1 - t_{m+1})^{\kappa_{m+1} - 1} \xi_{m+1}(t_{m+1})\,\mathrm{d}t_{m+1}$$

to conclude that

$$n^{-\alpha}\left(\prod_{j=1}^{m}|a_j|^{\kappa_j}n^{2\kappa_j}\right)|f_{m+1}(x)|$$
$$\le cn^{-2\kappa_{m+1}}|a_{m+1}|^{-\kappa_{m+1}}(1+n\sqrt{1-|A_{m+1}+x|})^{-\alpha-\frac{1}{2}+\tau_{m+1}},$$

where $|x|+|a_{m+1}|\le 1-\sum_{j=1}^{m}|a_j|$. This completes the induction and the proof. □

We are now ready to prove the main estimate that we need for establishing the pointwise upper bound of the Cesàro kernels.

Theorem 8.2.5. *Assume $\kappa_j>0$, $a_j\neq 0$, and $\varphi_j\in C^\infty[-1,1]$ for $j=1,2,\dots,m$. Let $|a|:=\sum_{j=1}^{m}|a_j|\le 1$. If $\alpha\ge\beta$, $\alpha\ge|\kappa|-\frac{1}{2}:=\sum_{j=1}^{m}\kappa_j-\frac{1}{2}$, and $|x|+|a|\le 1$, then*

$$\left|\int_{[-1,1]^m}P_n^{(\alpha,\beta)}\left(\sum_{j=1}^{m}a_jt_j+x\right)\prod_{j=1}^{m}\varphi_j(t_j)(1-t_j^2)^{\kappa_j-1}\,\mathrm{d}t\right| \tag{8.2.5}$$
$$\le cn^{\alpha-2|\kappa|}\frac{\prod_{j=1}^{m}(|a_j|+n^{-1}\sqrt{1-|a|-|x|}+n^{-2})^{-\kappa_j}}{\left(1+n\sqrt{1-|a|-|x|}\right)^{\alpha+\frac{1}{2}-|\kappa|}},$$

where the constant c satisfies $c\le c'\max_{1\le j\le m}\max_{0\le k\le|\alpha|+2|\kappa|+\frac{3}{2}}\|\varphi_j^{(k)}\|_\infty$.

Proof. Let $\psi\in C^\infty[-1,1]$ satisfy $\psi(t)=1$ for $\frac{1}{2}\le t\le 1$, and $\psi(t)=0$ for $-1\le t\le-\frac{1}{2}$. We define

$$\begin{aligned}\xi_{1,j}(t)&=\varphi_j(t)\psi(t)(1+t)^{\kappa_j-1},\\ \xi_{-1,j}(t)&=\varphi_j(-t)(1-\psi(-t))(1+t)^{\kappa_j-1},\end{aligned}\qquad j=1,\dots,m.$$

Evidently, $\xi_{1,j},\xi_{-1,j}\in C^\infty[-1,1]$ and $\operatorname{supp}\xi_{1,j},\operatorname{supp}\xi_{-1,j}\subset[-\frac{1}{2},1]$. Since

$$\int_{-1}^{1}g(t_j)\varphi_j(t_j)(1-t_j^2)^{\kappa_j-1}\,\mathrm{d}t_j$$
$$=\int_{-1}^{1}g(t_j)\xi_{1,j}(t_j)(1-t_j)^{\kappa_j-1}\,\mathrm{d}t_j+\int_{-1}^{1}g(-t_j)\xi_{-1,j}(t_j)(1-t_j)^{\kappa_j-1}\,\mathrm{d}t_j,$$

we can write

$$\begin{aligned}J:&=\int_{[-1,1]^m}P_n^{(\alpha,\beta)}\left(\sum_{j=1}^{m}a_jt_j+x\right)\prod_{j=1}^{m}\varphi_j(t_j)(1-t_j^2)^{\kappa_j-1}\,\mathrm{d}t\\ &=\sum_{\varepsilon\in\{1,-1\}^m}\int_{[-1,1]^m}P_n^{(\alpha,\beta)}\left(\sum_{j=1}^{m}\varepsilon_ja_jt_j+x\right)\prod_{j=1}^{m}\xi_{\varepsilon_j,j}(t_j)(1-t_j)^{\kappa_j-1}\,\mathrm{d}t\\ &=:\sum_{\varepsilon\in\{1,-1\}^m}I_\varepsilon(x).\end{aligned}$$

Recall that $|a| = \sum_{j=1}^m |a_j|$. For $\varepsilon \in \{1,-1\}^m$, we write $a(\varepsilon) := \sum_{j=1}^m a_j\varepsilon_j$. Applying Lemma 8.2.4 to I_ε gives

$$
\begin{aligned}
|I_\varepsilon(x)| &\le cn^{\alpha-2|\kappa|}\prod_{j=1}^m |a_j|^{-\kappa_j}\left(1+n\sqrt{1-|x+a(\varepsilon)|}\right)^{-\alpha-\frac{1}{2}+|\kappa|}\\
&\le n^{\alpha-2|\kappa|}\prod_{j=1}^m |a_j|^{-\kappa_j}\left(1+n\sqrt{1-|x|-|a|}\right)^{-\alpha-\frac{1}{2}+|\kappa|}
\end{aligned}
$$

for each $\varepsilon \in \{1,-1\}^m$, where we have used the assumption $\alpha \ge |\kappa| - \frac{1}{2}$ and the inequality $|x+\sum_{j=1}^m \varepsilon_j a_j| \le |x| + \sum_{j=1}^m |a_j|$ in the last step. Consequently,

$$
|J| \le c\,2^m n^{\alpha-2|\kappa|}\prod_{j=1}^m |a_j|^{-\kappa_j}\left(1+n\sqrt{1-|x|-|a|}\right)^{-\alpha-\frac{1}{2}+|\kappa|}. \tag{8.2.6}
$$

Finally, we claim that the desired inequality (8.2.5) is a consequence of Eq. (8.2.6). In fact, without loss of generality, we may assume that

$$
|a_j| \ge n^{-1}\sqrt{1-|a|-|x|}+n^{-2}, \quad \text{for } j=1,\dots,p \tag{8.2.7}
$$

and

$$
|a_j| < n^{-1}\sqrt{1-|a|-|x|}+n^{-2}, \quad \text{for } j=p+1,\dots,m. \tag{8.2.8}
$$

We then apply Eq. (8.2.6) with m and x replaced by p and $\sum_{j=p+1}^m a_j t_j + x$, respectively, to obtain

$$
\begin{aligned}
M_p(x,t') :&= \left|\int_{[-1,1]^p} P_n^{(\alpha,\beta)}\left(\sum_{j=1}^m a_j t_j + x\right)\prod_{j=1}^p \varphi_j(t_j)(1-t_j^2)^{\kappa_j-1}\,\mathrm{d}t\right|\\
&\le c\,2^p n^{\alpha-2\sum_{j=1}^p \kappa_j}\prod_{j=1}^p |a_j|^{-\kappa_j}(1+nA(x))^{-\alpha-\frac{1}{2}+\sum_{j=1}^p \kappa_j},
\end{aligned}
$$

where $t' := (t_{p+1},\dots,t_m) \in [-1,1]^{m-p}$ and $A(x) := \sqrt{1-|a|-|x|}$, and we have used the inequality $|\sum_{j=p+1}^m a_j t_j + x| \le \sum_{j=p+1}^m |a_j| + |x|$ as well as the fact that $\alpha \ge \sum_{j=1}^p \kappa_j - \frac{1}{2}$. Using the assumption (8.2.7), we then obtain

$$
\begin{aligned}
M_p(x,t') &\le cn^{\alpha-2|\kappa|}\prod_{i=1}^p |a_i|^{-\kappa_i}\prod_{j=p+1}^m (n^{-1}A(x)+n^{-2})^{-\kappa_j}(1+nA(x))^{-\alpha-\frac{1}{2}+|\kappa|}\\
&\le cn^{\alpha-2|\kappa|}\prod_{j=1}^m \left(|a_j|+n^{-1}A(x)+n^{-2}\right)^{-\kappa_j}(1+nA(x))^{-\alpha-\frac{1}{2}+|\kappa|}.
\end{aligned}
$$

Consequently, it follows that

$$\left|\int_{[-1,1]^m} P_n^{(\alpha,\beta)}\left(\sum_{j=1}^m a_j t_j + x\right) \prod_{j=1}^m \varphi_j(t_j)(1-t_j^2)^{\kappa_j-1}\,\mathrm{d}t\right|$$
$$\le \int_{[-1,1]^{m-p}} M_p(x,t') \prod_{i=p+1}^m \varphi_i(t_i)(1-t_i^2)^{\kappa_i-1}\,\mathrm{d}t_{p+1}\cdots\mathrm{d}t_m$$
$$\le cn^{\alpha-2|\kappa|}\prod_{j=1}^m \left(|a_j|+n^{-1}A(x)+n^{-2}\right)^{-\kappa_j}(1+nA(x))^{-\alpha-\frac{1}{2}+|\kappa|},$$

proving the desired inequality (8.2.5). □

8.3 Pointwise Estimation of the Kernel Functions

For estimating the kernel, we first show that the main term of $k_n^\delta(w_{\lambda_\kappa},1,u)$ is a Jacobi polynomial. We state a more general result on Jacobi expansions, which can be found in [162, p. 261, (9.4.13)].

Lemma 8.3.1. *For every $\alpha,\beta>-1$ such that $\alpha+\beta+\delta+3>0$,*

$$k_n^\delta(w_{\alpha,\beta},1,u) = \sum_{j=0}^J b_j(\alpha,\beta,\delta,n)P_n^{(\alpha+\delta+j+1,\beta)}(u) + G_n^\delta(u),$$

where J is a fixed integer and

$$G_n^\delta(u) = \sum_{j=J+1}^\infty d_j(\alpha,\beta,\delta,n)k_n^{\delta+j}(w_{\alpha,\beta},1,u).$$

Moreover, the coefficients satisfy the inequalities

$$|b_j(\alpha,\beta,\delta,n)| \le cn^{\alpha+1-\delta-j} \quad \text{and} \quad |d_j(\alpha,\beta,\delta,n)| \le cj^{-\alpha-\beta-\delta-4}.$$

Since the kernel function $K_n^{\delta+j}(w^{(\alpha,\beta)},1,u)$ contained in the G_n^δ term has larger indices, it could be handled by the estimate (B.1.13).

Theorem 8.3.2. *Let $x=(x_1,\ldots,x_d)\in\mathbb{S}^{d-1}$ and $y=(y_1,\ldots,y_d)\in\mathbb{S}^{d-1}$. Then for $\delta>-1$,*

$$|K_n^\delta(h_\kappa^2;x,y)| \le c\left[\frac{\prod_{j=1}^d(|x_jy_j|+n^{-1}\|\bar{x}-\bar{y}\|+n^{-2})^{-\kappa_j}}{n^{\delta-(d-2)/2}(\|\bar{x}-\bar{y}\|+n^{-1})^{\delta+d/2}} + \frac{\prod_{j=1}^d(|x_jy_j|+\|\bar{x}-\bar{y}\|^2+n^{-2})^{-\kappa_j}}{n(\|\bar{x}-\bar{y}\|+n^{-1})^d}\right], \tag{8.3.1}$$

where $\bar{z} = (|z_1|, \ldots, |z_{d+1}|)$ *for* $z = (z_1, \ldots, z_{d+1}) \in \mathbb{S}^{d-1}$. *Furthermore, for the kernel* $Z_n^\kappa(\cdot,\cdot)$ *of the projection operator,*

$$|Z_n^\kappa(x,y)| \le c \frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\|\bar{x}-\bar{y}\| + n^{-2})^{-\kappa_j}}{n^{-(d-2)/2}(\|\bar{x}-\bar{y}\| + n^{-1})^{(d-2)/2}}. \tag{8.3.2}$$

Proof. We start from the integral expression (8.1.3) of $K_n^\delta(h_\kappa^2; x, y)$. The first step of the proof is to replace the kernel $k_n^\delta(w_{(\lambda_\kappa - \frac12, \lambda_\kappa - \frac12)})$ by the expansion in Lemma 8.3.1. Let $\alpha = \beta = |\kappa| + (d-3)/2$ and let $J = \lfloor \alpha + \beta + 2 \rfloor = \lfloor 2|\kappa| + d - 1 \rfloor$. The choice of J guarantees that we can apply Eq. (B.1.13) the the G_n^δ term. Combining the formula (8.1.3) and Lemma 8.3.1, we obtain

$$K_n^\delta(h_\kappa^2; x, y) = \sum_{j=0}^J b_j(\alpha, \beta, \delta, n) \Omega_j(x,y) + \Omega_*(x,y),$$

where

$$\Omega_j(x,y) = c_\kappa \int_{[-1,1]^d} P_n^{(\alpha+\delta+j+1,\beta)}(u(x,y,t)) \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} \mathrm{d}t$$

and

$$\Omega_*(x,y) = c_\kappa \int_{[-1,1]^d} G_n^\delta(u(x,y,t)) \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} \mathrm{d}t,$$

in which $u(x,y,t) = x_1 y_1 t_1 + \cdots + x_d y_d t_d$.

Since the indices of the Jacobi polynomial in Ω_0 are $\alpha + \delta + 1 = \delta + |\kappa| + \frac{d-1}{2}$ and $|\kappa| + \frac{d-3}{2}$, we can use Theorem 8.2.5 with $m = d$, $x = 0$, and $a_j = x_j y_j$ to estimate Ω_0 for all $\delta > -1$. Using the fact that $1 - \langle \bar{x}, \bar{y} \rangle = \|\bar{x} - \bar{y}\|/2$ for $x, y \in \mathbb{S}^{d-1}$, this shows that $b_0(\alpha, \beta, \delta, n)\Omega_0$ is bounded by the first term on the right-hand of Eq. (8.2.5). The same estimate evidently holds for Ω_j.

Next we estimate Ω_*, which by Eq. (B.1.13) is bounded by

$$|\Omega_*| \le c n^{-1} \int_{[-1,1]^d} \frac{1}{(1 - u(x,y,t) + n^{-2})^{|\kappa| + d/2}} \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} \mathrm{d}t,$$

where if $\kappa_i = 0$ for some i, then under the usual limiting process, the integral against t_i becomes point evaluation at $t_i = 1$. The definition of $u(x,y,t)$ shows that

$$1 - u(x,y,t) \ge 1 - \sum_{i=1}^d |x_i y_i t_i| = \frac12 |\bar{x} - \bar{y}|^2 + \sum_{i=1}^d |x_i y_i|(1 - |t_i|),$$

from which it follows that

$$|\Omega_*| \le cn^{-1} \int_{[0,1]^d} \frac{\prod_{i=1}^d (1-t_i)^{\kappa_i - 1}}{(|\bar{x}-\bar{y}|^2/2 + \sum_{i=1}^d |x_i y_i|(1-t_i) + n^{-2})^{|\kappa|+d/2}} \mathrm{d}t.$$

Changing variables $t_j \to s_j$ by

$$s_j = \frac{|x_j y_j|}{|\bar{x}-\bar{y}|^2/2 + n^{-2}}(1-t_j), \quad 1 \le j \le d,$$

in the above integrals, we obtain

$$|\Omega_*| \le c \prod_{i=1}^d \left(\frac{|\bar{x}-\bar{y}|^2 + n^{-2}}{|x_i y_i|} \right)^{\kappa_i} \frac{1}{n(|\bar{x}-\bar{y}|^2 + n^{-2})^{|\kappa|+d/2}}$$
$$\times \int_0^{\frac{|x_1 y_1|}{|\bar{x}-\bar{y}|^2/2+n^{-2}}} \cdots \int_0^{\frac{|x_d y_d|}{|\bar{x}-\bar{y}|^2/2+n^{-2}}} \frac{1}{(1+s_1+\cdots+s_d)^{|\kappa|+d/2}} \prod_{i=1}^d s_i^{\kappa_i - 1} \mathrm{d}s.$$

Using the elementary inequality

$$\int_0^{r_1} \cdots \int_0^{r_d} \frac{\prod_{i=1}^d s_i^{\kappa_i-1} \mathrm{d}s}{(1+s_1+\cdots+s_d)^{|\kappa|_1+d/2}} \le \prod_{i=1}^d \int_0^{r_i} \frac{s_i^{\kappa_i-1}\mathrm{d}s_i}{(1+s_i)^{\kappa_i+1/2}} \le c \prod_{i=1}^d \left(\frac{r_i}{1+r_i} \right)^{\kappa_i},$$

it then follows that

$$|\Omega_*| \le c \frac{\prod_{i=1}^d (|x_i y_i| + |\bar{x}-\bar{y}|^2 + n^{-2})^{-\kappa_i}}{n(|\bar{x}-\bar{y}| + n^{-1})^d},$$

which is the second term on the right-hand side of the estimate (8.3.1).

Finally, we note that the kernel $Z_n^\kappa(x,y)$ can be regarded as the case $\delta = -1$ of $K_n^\delta(h_\kappa^2; x, y)$, and by Eqs. (7.2.10) and (7.2.2), Theorem 8.2.5 applies, which gives the pointwise estimate of Eq. (8.3.2). □

8.4 Proof of the Main Results

We first prove Theorem 8.1.3, since its proof is easier.

Proof of Theorem 8.1.3. By the estimate (8.3.1), for $x \in \mathbb{S}_{\mathrm{int}}^{d-1}$,

$$\left| K_n^\delta(h_\kappa^2; x, y) \right| h_\kappa(y) \le \; c(x) \left[\frac{1}{n^{\delta - \frac{d-2}{2}} (\|\bar{x}-\bar{y}\| + n^{-1})^{\delta + \frac{d}{2}}} + \frac{1}{n(\|\bar{x}-\bar{y}\| + n^{-1})^d} \right],$$

where $c(x) = c / \prod_{i=1}^d |x_i|_i^\kappa > 0$. Since the right-hand side of the estimate is invariant under sign changes, it follows that for the first term,

$$\int_{\mathbb{S}^{d-1}} \frac{1}{(\|\bar{x}-\bar{y}\|+n^{-1})^{\delta+\frac{d}{2}}} \mathrm{d}\sigma(y) \le 2^d \int_{\mathbb{S}^{d-1}} \frac{1}{(\|\bar{x}-y\|+n^{-1})^{\delta+\frac{d}{2}}} \mathrm{d}\sigma(y)$$
$$\le c\int_0^{\pi} \frac{(\sin\theta)^{d-2}}{(\sqrt{1-\cos\theta}+n^{-1})^{\delta+\frac{d}{2}}} \mathrm{d}\theta \le cn^{\delta-\frac{d-2}{2}},$$

which implies, together with a similar estimate for the second term, that

$$\int_{\mathbb{S}^{d-1}} \left|K_n^{\delta}(h_\kappa^2;x,y)\right| h_\kappa^2(y)\mathrm{d}\sigma(y) \le c(x).$$

Consequently, the pointwise convergence of $S_n^{\delta}(h_\kappa^2;f)$ follows from the triangle inequality and the fact that $S_n^{\delta}(h_\kappa^2;f)$ converges to f whenever f is a polynomial of fixed degree. Since $c(x)$ is evidently bounded over a compact set in $\mathbb{S}^{d-1}_{\mathrm{int}}$, the convergence is uniform over such a subset. $\square$

Proof of Theorem 8.1.1. A standard duality argument shows that $\|S_n^{\delta}(h_\kappa^2)\|_{\kappa,1} = \|S_n^{\delta}(h_\kappa^2)\|_{\kappa,\infty}$ and

$$\|S_n^{\delta}(h_\kappa^2)\|_{\kappa,1} = \sup_{x\in\mathbb{S}^{d-1}} \frac{1}{\omega_d^{\kappa}} \int_{\mathbb{S}^{d-1}} |K_n^{\delta}(h_\kappa^2;x,y)|h_\kappa^2(y)\mathrm{d}\sigma(y). \tag{8.4.1}$$

The proof is naturally divided into two cases, dealing with the upper bound and lower bound, respectively.

Case 1. Upper bound. We prove the upper bound for the norm of $S_n^{\delta}(h_\kappa^2)$ when $\delta > -1$. The norm of the projection operator can be treated similarly.

The main task of the proof is to establish the following claim:

$$|K_n^{\delta}(h_\kappa^2;x,y)|h_\kappa^2(y) \le cn^{d-1}(1+n\|\bar{x}-\bar{y}\|)^{-\beta(\delta)}, \quad x,y\in\mathbb{S}^{d-1}, \tag{8.4.2}$$

where $\beta(\delta) := \min\{d, \delta-\sigma_\kappa+d-1\}$. Indeed, once claim (8.4.2) is proven, we have

$$\int_{\mathbb{S}^{d-1}} |K_n^{\delta}(h_\kappa^2;x,y)|h_\kappa^2(y)\mathrm{d}\sigma(y) \le cn^{d-1}\int_0^{\frac{\pi}{2}} (1+n\theta)^{-\beta(\delta)}(\sin\theta)^{d-2}\mathrm{d}\theta$$
$$\sim \begin{cases} 1, & \delta > \sigma_\kappa, \\ \log n, & \delta = \sigma_\kappa, \\ n^{-\delta+\sigma_\kappa}, & -1 < \delta < \sigma_\kappa, \end{cases}$$

which together with Eq. (8.4.1) will prove the desired upper bound.

For the proof of Eq. (8.4.2), we shall use Theorem 8.3.2. Without loss of generality, we may assume $|x_1| = \max_{1\le j\le d}|x_j|$. Set

$$I_j(x,y) := (|x_j y_j| + n^{-1}\|\bar{x}-\bar{y}\| + n^{-2})^{-\kappa_j}|y_j|^{2\kappa_j}, \quad 1 \le j \le d.$$

Since $|x_1| = \max_{1\le j\le d}|x_j| \ge \frac{1}{\sqrt{d}}$, we have

$$I_1(x,y) \le |x_1|^{-\kappa_1}|y_1|^{\kappa_1} \le d^{\frac{\kappa_1}{2}}.$$

For $j \ge 2$, if $|x_j| \ge 2\|\bar{x}-\bar{y}\|$, then $|y_j| \le |x_j| + \|\bar{x}-\bar{y}\| \le \frac{3}{2}|x_j|$, and hence

$$I_j(x,y) \le |x_j y_j|^{-\kappa_j}|y_j|^{2\kappa_j} \le \left(\frac{3}{2}\right)^{\kappa_j} \le \left(\frac{3}{2}\right)^{\kappa_j}(1+n\|\bar{x}-\bar{y}\|)^{\kappa_j},$$

whereas if $|x_j| < 2\|\bar{x}-\bar{y}\|$, then $|y_j| \le |x_j| + \|\bar{x}-\bar{y}\| \le 3\|\bar{x}-\bar{y}\|$, and hence

$$I_j(x,y) \le (n^{-1}\|\bar{x}-\bar{y}\| + n^{-2})^{-\kappa_j}(3\|\bar{x}-\bar{y}\|)^{2\kappa_j} \le 3^{2\kappa_j}(1+n\|\bar{x}-\bar{y}\|)^{\kappa_j}.$$

Consequently, it follows that

$$\prod_{j=1}^{d} I_j(x,y) \le c\prod_{j=2}^{d} I_j(x,y) \le c(1+n\|\bar{x}-\bar{y}\|)^{|\kappa|-\kappa_1},$$

in which κ_1 can be replaced by $\min_{1\le i\le d}\kappa_i$. Thus, we obtain

$$\begin{aligned} I(x,y) &:= n^{d-1}(1+n\|\bar{x}-\bar{y}\|)^{-\delta-\frac{d}{2}}\prod_{j=1}^{d} I_j(x,y) \\ &\le cn^{d-1}(1+n\|\bar{x}-\bar{y}\|)^{-(\delta+d-\sigma_\kappa)}. \end{aligned} \tag{8.4.3}$$

Similarly, we can show that for $1 \le j \le d$,

$$J_j(x,y) := (|x_j y_j| + \|\bar{x}-\bar{y}\|^2 + n^{-2})^{-\kappa_j}|y_j|^{2\kappa_j} \le c,$$

which implies that

$$J(x,y) := \frac{\prod_{j=1}^{d} J_j(x,y)}{n(n^{-1}+\|\bar{x}-\bar{y}\|)^d} \le cn^{d-1}(1+n\|\bar{x}-\bar{y}\|)^{-d}. \tag{8.4.4}$$

Since the estimate in Theorem 8.3.2 shows that

$$|K_n^\delta(h_\kappa^2;x,y)|h_\kappa^2(y) \le c(I(x,y)+J(x,y)),$$

the claim (8.4.2) follows by Eqs. (8.4.3) and (8.4.4). □

Case 2. Lower bound. We first consider the projection operator. Without loss of generality, we can assume that $\kappa_1 = \min_{1\le i\le d} \kappa_i$. Let $e_1 = (1,0,\dots,0)$. By 7.2.14,

$$Z_n^\kappa(x,e_1) = \frac{n+\lambda_k}{\lambda_k} C_n^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1),$$

where $C_n^{(\lambda,\mu)}$ is the generalized Gegenbauer polynomial, which is orthogonal with respect to $v_{\lambda,\mu}(x) := |t|^{2\mu}(1-t^2)^{\lambda-1/2}$. Consequently, since $\sigma_\kappa = \lambda_\kappa - \kappa_1$, we obtain, on using Eq. (1.5.4),

$$\begin{aligned}\|\operatorname{proj}_n^\kappa(h_\kappa^2)\|_{\kappa,1} &\ge \frac{1}{\omega_d^\kappa}\int_{\mathbb{S}^{d-1}} |Z_n^\kappa(x,e_1)| h_\kappa^2(x)\mathrm{d}\sigma(x)\\ &= \frac{n+\lambda_k}{\lambda_k} a_\kappa \int_{-1}^1 \left|C_n^{(\sigma_\kappa,\kappa_1)}(s)\right| v_{\sigma_\kappa,\kappa_1}(s)\mathrm{d}s,\end{aligned}$$

where a_κ is the normalization constant of $v_{\sigma_\kappa,\kappa_1}$. The generalized Gegenbauer polynomials are related to the Jacobi polynomials. By Eq. (B.3.1), we can write the last integral as twice the integral over $[0,1]$ and then change variables $2x_j^2 - 1 \mapsto t$ to conclude that

$$\|\operatorname{proj}_{2n}^\kappa(h_\kappa^2)\|_{\kappa,1} \ge c n^{\sigma_\kappa+\frac12}\int_{-1}^1 \left|P_n^{(\sigma_\kappa-\frac12,\kappa_j-\frac12)}(t)\right| w_{\sigma_\kappa-\frac12,\kappa_j-\frac12}(t)\mathrm{d}t,$$

where $w_{\alpha,\beta}$ denotes the usual Jacobi weight. The last integral has asymptotic order n^{σ_κ}, as seen from the classical estimate in Eq. (B.1.8), which is the desired lower bound. The case of $\operatorname{proj}_{2n+1}^\kappa(h_\kappa^2)$ can be handled similarly.

We now consider $\|S_n^\delta(h_\kappa^2)\|_{\kappa,1}$ for $\delta > -1$. Let $k_n^\delta(v_{\lambda,\mu}; s,t)$ denote the kernel of the (C,δ) means of the generalized Gegenbauer expansions with respect to $v_{\lambda,\mu}$. From Eqs. (7.2.14) and (B.3.3), it follows that

$$K_n^\delta(h_\kappa^2; e_1, x) = k_n^\delta(v_{\sigma_\kappa,\kappa_1}; 1, x_1). \tag{8.4.5}$$

Hence, on using Eq. (1.5.4), we obtain the relation

$$\begin{aligned}\|S_n^\delta(h_\kappa^2)\|_{\kappa,\infty} &\ge \frac{1}{\omega_d^\kappa}\int_{\mathbb{S}^{d-1}} \left|K_n^\delta(h_\kappa^2; e_1, y)\right| h_\kappa^2(y)\mathrm{d}\sigma(y)\\ &= \frac{1}{\omega_d^\kappa}\int_{-1}^1 \left|k_n^\delta(v_{\sigma_\kappa,\kappa_1}; 1, s)\right| v_{\sigma_\kappa,\kappa_1}(s)\mathrm{d}s.\end{aligned}$$

Thus, it is sufficient to estimate the last integral from below, which, however, is harder than the estimation of the integral of the Jacobi polynomial in Eq. (B.1.8), and it is not classical. We give this estimate in Proposition 8.5.1 in the following section. □

8.5 Lower Bound for Generalized Gegenbauer Expansion

For the weight function $v_{\lambda,\mu}(t)$ on $[-1,1]$ and $\delta > -1$, we define

$$T_n^{\delta}(v_{\lambda,\mu};t) := \int_{-1}^{1} |k_n^{\delta}(v_{\lambda,\mu};s,t)| w_{\lambda,\mu}(s)\mathrm{d}s. \tag{8.5.1}$$

The following proposition contains what we need for finishing the last step in the proof of the lower bound of Theorem 8.1.1.

Proposition 8.5.1. *Assume $\mu \ge 0$ and $\delta \le \lambda$. If $\lambda \ge \mu$, then*

$$T_n^{\delta}(v_{\lambda,\mu};1), T_n^{\delta}(v_{\mu,\lambda};0) \ge c n^{-\delta+\lambda} \begin{cases} \log n, & \text{if } \delta = \lambda, \\ 1, & \text{if } -1 < \delta < \lambda. \end{cases}$$

Proof. First it follows from Eq. (B.3.5) that

$$\frac{1}{2}\Big[C_n^{(\lambda,\mu)}(1)C_n^{(\lambda,\mu)}(x) + C_n^{(\lambda,\mu)}(1)C_n^{(\lambda,\mu)}(-x)\Big] = C_n^{(\mu,\lambda)}(0)C_n^{(\mu,\lambda)}(\sqrt{1-x^2}),$$

which implies that

$$\begin{aligned} T_n^{\delta}(v_{\lambda,\mu};1) &\ge \frac{1}{2}\int_{-1}^{1} \Big| k_n^{\delta}(v_{\lambda,\mu};1,y) + k_n^{\delta}(v_{\lambda,\mu};1,-y)\Big| v_{\lambda,\mu}(y)\mathrm{d}y \\ &= \int_{-1}^{1} \Big| k_n^{\delta}(v_{\mu,\lambda};0,\sqrt{1-y^2})\Big| v_{\lambda,\mu}(y)\mathrm{d}y \\ &= \int_{-1}^{1} \Big| k_n^{\delta}(v_{\mu,\lambda};0,y)\Big| v_{\mu,\lambda}(y)\mathrm{d}y = T_n^{\delta}(v_{\mu,\lambda};0). \end{aligned}$$

Thus, to finish the proof, we will need only to prove that $T_n^{\mu}(v_{\lambda,\mu};0) \ge c\log n$ when $\mu \ge \lambda$. For $\delta = \mu \ge \lambda$, using Eq. (B.3.5) and Lemma 8.3.1, we obtain

$$\begin{aligned} k_n^{\delta}(v_{\mu,\lambda};0,y) &= c_\lambda \int_{-1}^{1} k_n^{\delta}(w_{\lambda+\mu};1,s\sqrt{1-y^2})(1-s^2)^{\lambda-1}\mathrm{d}s \\ &= c n^{\lambda+\mu+1/2-\delta}\int_{-1}^{1} P_n^{(\lambda+\mu+\delta+1/2,\lambda+\mu-1/2)}(s\sqrt{1-y^2})(1-s^2)^{\lambda-1}\mathrm{d}s \\ &\quad + \sum_{j=1}^{\infty} c_j(\lambda+\mu-1/2,\lambda+\mu-1/2,\delta,n)k_n^{\delta+j}(v_{\mu,\lambda};0,y), \end{aligned}$$

since $w_{\lambda+\mu}$ agrees with the Jacobi weight $w_{\lambda+\mu-1/2,\lambda+\mu-1/2}$. For $j \ge 1$, $\delta + j = \mu + j > \mu$, the sufficiency part of Theorem 8.1.1 shows that $\int_{-1}^{1} |k_n^{\delta+j}(v_{\mu,\lambda},0,y)| v_{\mu,\lambda}(y)\mathrm{d}y$ is uniformly bounded and

$$\sum_{j=1}^{\infty} |c_j(\lambda+\mu-1/2, \lambda+\mu-1/2, \delta, n)| \le c \sum_{j=1}^{\infty} j^{-2\lambda-2\mu-\delta-3} < \infty.$$

Hence, since $\delta = \mu$, it follows from the definition of T_n^{μ} that

$$T_n^{\delta}(v_{\mu,\lambda};0) = c n^{\lambda+\frac{1}{2}} \int_0^1 \left| \int_{-1}^1 P_n^{(\lambda+2\mu+\frac{1}{2}, \lambda+\mu-\frac{1}{2})}(s\sqrt{1-y^2})(1-s^2)^{\lambda-1} \mathrm{d}s \right| \\ \times |y|^{2\mu}(1-y^2)^{\lambda-\frac{1}{2}} \mathrm{d}y + \mathscr{O}(1),$$

where the outer integral is taken over $[0,1]$ instead of $[-1,1]$, since the function is even in y. Changing variables $t = \sqrt{1-y^2}$ in the outer integral, we obtain

$$T_n^{\delta}(w_{\mu,\lambda};0) = c\, n^{\lambda+\frac{1}{2}} \int_0^1 \left| \int_{-1}^1 P_n^{(\lambda+2\mu+\frac{1}{2}, \lambda+\mu-\frac{1}{2})}(st)(1-s^2)^{\lambda-1} \mathrm{d}s \right| \\ \times t^{2\lambda}(1-t^2)^{\mu-\frac{1}{2}} \mathrm{d}t + \mathscr{O}(1).$$

Consequently, we need to derive a lower bound on the double integral of the Jacobi polynomial, which is the content of Proposition 8.5.2 below. □

Proposition 8.5.2. *Assume $\lambda, \mu \ge 0$ and $\lambda \ge \delta > -1$. Let $a = \lambda+\mu+\delta$ and $b = \lambda+\mu-1$. Then*

$$\int_0^1 \left| \int_{-1}^1 P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(ty)(1-t^2)^{\mu-1} \mathrm{d}t \right| |y|^{2\mu}(1-y^2)^{\lambda-1/2} \mathrm{d}y \qquad (8.5.2)$$

$$\ge c\, n^{-\mu-1/2} \begin{cases} \log n, & \text{if } \delta = \lambda, \\ 1, & \text{if } -1 < \delta < \lambda, \end{cases}$$

where when $\mu = 0$, the inner integral is defined under the usual limit.

The proof of this proposition is fairly involved and will occupy the rest of this section. Let us denote the left-hand side of Eq. (8.5.2) by I_n. First, we assume that $0 < \mu < 1$. Changing variables $t = u/y$, followed by $y = \cos\phi$ and $u = \cos\theta$, and restricting the range of the outside integral leads to

$$I_n \ge c \int_{\sqrt{2}/2}^1 \left| \int_{-y}^{y} P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(u) y(y^2-u^2)^{\mu-1} \mathrm{d}u \right| (1-y^2)^{\lambda-1/2} \mathrm{d}y \\ \ge c \int_{n^{-1}}^{\pi/4} \left| \int_{\phi}^{\pi-\phi} P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(\cos\theta)(\cos^2\phi - \cos^2\theta)^{\mu-1} \sin\theta \mathrm{d}\theta \right| (\sin\phi)^{2\lambda} \mathrm{d}\phi.$$

We need the asymptotics of the Jacobi polynomials as given in [162, p. 198],

$$P_n^{(\alpha,\beta)}(\cos\theta)=\pi^{-\frac12}n^{-\frac12}\left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac12}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac12}\left[\cos(N\theta+\tau)+\mathscr{O}(1)(n\sin\theta)^{-1}\right]$$

for $n^{-1}\le\theta\le\pi-n^{-1}$, where $N=n+\frac{\alpha+\beta+1}{2}$ and $\tau=-\frac{\pi}{2}(\alpha+\frac12)$. Applying this asymptotic formula with $\alpha=a+1/2$ and $\beta=b+1/2$, we obtain

$$I_n\ge c\,n^{-1/2}\int_{n^{-1}}^{\pi/4}|M_n(\phi)|(\sin\phi)^{2\lambda}\mathrm{d}\phi-\mathscr{O}(1)E_n, \tag{8.5.3}$$

where $M_n(\phi)$ is the integral over the main term of the asymptotics

$$M_n(\phi):=\int_{\phi}^{\pi-\phi}\frac{(\cos^2\phi-\cos^2\theta)^{\mu-1}}{(\sin\frac{\theta}{2})^a(\cos\frac{\theta}{2})^b}\cos(N\theta+\tau)\mathrm{d}\theta, \tag{8.5.4}$$

and E_n comes from the remainder term in the asymptotics

$$E_n:=n^{-\frac32}\int_{n^{-1}}^{\pi/4}\int_{\phi}^{\pi-\phi}\frac{(\cos^2\phi-\cos^2\theta)^{\mu-1}}{(\sin\frac{\theta}{2})^{a+1}(\cos\frac{\theta}{2})^{b+1}}\mathrm{d}\theta\,(\sin\phi)^{2\lambda}\mathrm{d}\phi. \tag{8.5.5}$$

Here $N=n+\frac{a+b}{2}+1$ and $\tau=-\frac{\pi}{2}(a+1)$.

In order to handle the main part of Eq. (8.5.3), we first derive an asymptotic formula for $M_n(\phi)$. We need the following lemma, which follows directly from [68, p. 49].

Lemma 8.5.3. *If* $0<\mu<1$, $g(t)$ *is continuously differentiable on the interval* $[\alpha,\beta]$, *and* $\xi\in\mathbb{R}-\{0\}$, *then*

$$\int_\alpha^\beta g(t)\mathrm{e}^{\mathrm{i}\xi t}(t-\alpha)^{\mu-1}(\beta-t)^{\mu-1}\mathrm{d}t$$
$$=\Gamma(\mu)|\xi|^{-\mu}\left[\mathrm{e}^{-\frac{\mathrm{i}\pi\mu\xi}{2|\xi|}}g(\beta)\mathrm{e}^{\mathrm{i}\xi\beta}+e^{\frac{i\pi\mu\xi}{2|\xi|}}g(\alpha)\mathrm{e}^{\mathrm{i}\xi\alpha}\right]+R_\xi,$$

as $|\xi|\to+\infty$, *where*

$$|R_\xi|\le|\xi|^{-1}\int_\alpha^\beta|g'(t)|(t-\alpha)^{\mu-1}(\beta-t)^{\mu-1}\mathrm{d}t.$$

Lemma 8.5.4. *Assume* $0<\mu<1$, $\lambda\ge0$, *and* $\lambda\ge\delta\ge-1$. *Let* $M_n(\phi)$ *be defined by Eq.* (8.5.4)*. Then for* $0<\phi\le\pi/4$,

$$M_n(\phi)=K_n(\phi)+G_n(\phi), \tag{8.5.6}$$

where

$$K_n(\phi) = \Gamma(\mu)N^{-\mu}\frac{2^a(\sin(2\phi))^{\mu-1}}{(\pi-2\phi)^{\mu-1}(\sin\phi)^a} \times\left[(-1)^n\left(\sin\frac{\phi}{2}\right)^{a-b}\cos\left(N\phi+\gamma+\frac{(a-b)\pi}{2}\right)+\left(\cos\frac{\phi}{2}\right)^{a-b}\cos(N\phi+\gamma)\right], \tag{8.5.7}$$

$\gamma=\tau+\frac{\pi\mu}{2}$*, and the remainder satisfies*

$$|G_n(\phi)|\le cn^{-1}\phi^{\mu-\lambda-\delta-2}. \tag{8.5.8}$$

Proof. Writing $\cos(N\theta+\tau)=(\mathrm{e}^{\mathrm{i}(N\theta+\tau)}+e^{-i(N\theta+\tau)})/2$, we split $M_n(\phi)$ into two parts, $M_n^+(\phi)$ and $M_n^-(\phi)$, respectively, and apply Lemma 8.5.3 to these integrals. For $M_n^+(\phi)$, we define a function f_ϕ as

$$f_\phi(\theta)=\frac{(\cos^2\phi-\cos^2\theta)^{\mu-1}}{(\sin\frac{\theta}{2})^a(\cos\frac{\theta}{2})^b(\theta-\phi)^{\mu-1}(\pi-\phi-\theta)^{\mu-1}}$$

for $\phi<\theta<\pi-\phi$ and define its value at the boundary by the limit. Then it is easily seen that

$$f_\phi(\theta)=\begin{cases}\left(\frac{\sin(\pi-\phi-\theta)\sin(\theta-\phi)}{(\pi-\phi-\theta)(\theta-\phi)}\right)^{\mu-1}\frac{1}{(\sin\frac{\theta}{2})^a(\cos\frac{\theta}{2})^b}, & \text{if } \theta\in(\phi,\pi-\phi),\\ \left(\frac{\sin(\pi-2\phi)}{\pi-2\phi}\right)^{\mu-1}\frac{1}{(\sin\frac{\theta}{2})^a(\cos\frac{\theta}{2})^b}, & \text{if } \theta=\phi \text{ or } \pi-\phi,\end{cases}$$

is continuously differentiable on $[\phi,\pi-\phi]$. Hence, invoking Lemma 8.5.3 with $\xi=N$, and by a straightforward computation, we obtain

$$\begin{aligned}M_n^+(\phi) &= \frac{\mathrm{e}^{\mathrm{i}\tau}}{2}\int_\phi^{\pi-\phi} f_\phi(\theta)\mathrm{e}^{\mathrm{i}N\theta}(\theta-\phi)^{\mu-1}(\pi-\phi-\theta)^{\mu-1}\mathrm{d}\theta\\ &= \Gamma(\mu)N^{-\mu}\frac{(\sin(2\phi))^{\mu-1}}{(\pi-2\phi)^{\mu-1}}\frac{2^{a-1}}{(\sin\phi)^a}\\ &\quad\times\left[\left(\sin\frac{\phi}{2}\right)^{a-b}\mathrm{e}^{\mathrm{i}[N(\pi-\phi)-\frac{\pi\mu}{2}+\tau]}+\left(\cos\frac{\phi}{2}\right)^{a-b}\mathrm{e}^{\mathrm{i}[N\phi+\frac{\pi\mu}{2}+\tau]}\right]+R_n^+(\phi),\end{aligned}$$

in which

$$|R_n^+(\phi)|\le N^{-1}\int_\phi^{\pi-\phi}|f_\phi'(\theta)|(\theta-\phi)^{\mu-1}(\pi-\phi-\theta)^{\mu-1}\mathrm{d}\theta.$$

Since $0 < \phi \le \frac{\pi}{4}$, using the fact that $\sin x/x$ is analytic and that $\sin(\pi-\theta-\phi) = \sin(\theta+\phi)$, from the definition of f_ϕ we see easily that for $\phi < \theta < \pi-\phi$,

$$|f'_\phi(\theta)| \le c(\theta^{\mu-a-2} + (\pi-\theta)^{\mu-b-2}).$$

This implies that for $0 < \phi \le \frac{\pi}{4}$,

$$\begin{aligned}|R_n^+(\phi)| &\le cN^{-1}\left[\int_\phi^{\pi/2} \theta^{\mu-a-2}(\theta-\phi)^{\mu-1}\mathrm{d}\theta + \int_{\pi/2}^{\pi-\phi}(\pi-\theta)^{\mu-b-2}(\pi-\theta-\phi)^{\mu-1}\mathrm{d}\theta\right]\\ &\le cn^{-1}\int_\phi^{\pi/2}\theta^{\mu-a-2}(\theta-\phi)^{\mu-1}\mathrm{d}\theta,\end{aligned}$$

since $a \ge b$ and the first term dominates. A simple computation shows then that

$$\begin{aligned}|R_n^+(\phi)| &\le cn^{-1}\phi^{\mu-a-2}\int_\phi^{2\phi}(\theta-\phi)^{\mu-1}\mathrm{d}\theta + cn^{-1}\int_{2\phi}^{\pi/2}\theta^{2\mu-a-3}\mathrm{d}\theta\\ &\le cn^{-1}\phi^{2\mu-2-a} = cn^{-1}\phi^{\mu-\lambda-\delta-2}, \qquad (8.5.9)\end{aligned}$$

since $a = \lambda+\mu+\delta > 2\mu-2$.

Similarly, using Lemma 8.5.3 with $\xi = -N$, we derive a similar relation for $M_n^-(\phi)$:

$$\begin{aligned}M_n^-(\phi) &= \frac{e^{-i\tau}}{2}\int_\phi^{\pi-\phi} f_\phi(\theta)e^{-iN\theta}(\theta-\phi)^{\mu-1}(\pi-\phi-\theta)^{\mu-1}\mathrm{d}\theta\\ &= \Gamma(\mu)N^{-\mu}\frac{(\sin(2\phi))^{\mu-1}}{(\pi-2\phi)^{\mu-1}}\frac{2^{a-1}}{(\sin\phi)^a}\\ &\quad\times\left[\left(\sin\frac{\phi}{2}\right)^{a-b}e^{-i[N(\pi-\phi)-\frac{\pi\mu}{2}+\tau]} + \left(\cos\frac{\phi}{2}\right)^{a-b}e^{-i[N\phi+\frac{\pi\mu}{2}+\tau]}\right] + R_n^-(\phi),\end{aligned}$$

where the error term $R_n^-(\phi)$ satisfies the same upper bound as in Eq. (8.5.9). Since $M_n(\phi) = M_n^+(\phi) + M_n^-(\phi)$ and $N\pi + 2\tau = n\pi + \frac{b-a}{2}\pi$, the desired expression for $M_n(\phi)$ follows with $G_n(\phi) = R_n^+(\phi) + R_n^-(\phi)$, which satisfies the stated bound. □

Lemma 8.5.5. *Assume that* $0 < \mu < 1$, $\lambda \ge 0$ *and* $\lambda \ge \delta > -1$. *Then*

$$\int_{n^{-1}}^{\frac{\pi}{4}} |M_n(\phi)|(\sin\phi)^{2\lambda}\,\mathrm{d}\phi \ge cn^{-\mu}\begin{cases}\log n, & \text{if } \lambda = \delta,\\ 1, & \text{if } -1 < \delta < \lambda.\end{cases}$$

Proof. Since $a - b = \delta + 1 > 0$, we can choose an absolute constant $\varepsilon \in (0, \frac{\pi}{4})$ satisfying $(\tan\frac{\varepsilon}{2})^{a-b} \le \frac{1}{4}$. We then use Eq. (8.5.7), and obtain that for $\phi \in (0, \varepsilon)$,

$$
\begin{aligned}
|K_n(\phi)| &\geq cn^{-\mu}\phi^{-\lambda-\delta-1}\left(|\cos(N\phi+\gamma)| - \left(\tan\frac{\phi}{2}\right)^{a-b}\right) \\
&\geq cn^{-\mu}\phi^{-\lambda-\delta-1}\left(\cos^2(N\phi+\gamma) - \frac{1}{4}\right) \\
&= \frac{c}{4}n^{-\mu}\phi^{-\lambda-\delta-1} + \frac{c}{2}n^{-\mu}\phi^{-\lambda-\delta-1}\cos(2N\phi+2\gamma),
\end{aligned} \tag{8.5.10}
$$

where we have used the fact that $(\tan\frac{\phi}{2})^{a-b} \leq (\tan\frac{\varepsilon}{2})^{a-b} \leq \frac{1}{4}$ for $0 < \phi \leq \varepsilon$ in the second step, and the identity $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$ in the last step. It follows that

$$
\begin{aligned}
\int_{n^{-1}}^{\varepsilon} |K_n(\phi)|(\sin\phi)^{2\lambda}\,\mathrm{d}\phi &\geq cn^{-\mu}\int_{n^{-1}}^{\varepsilon}\phi^{\lambda-\delta-1}\,\mathrm{d}\phi \\
&\quad + cn^{-\mu}\int_{n^{-1}}^{\varepsilon}\phi^{\lambda-\delta-1}\cos(2N\phi+2\gamma)\,\mathrm{d}\phi \\
&\geq cn^{-\mu}\begin{cases}\log n, & \text{if } \lambda=\delta, \\ 1, & \text{if } -1<\delta<\lambda.\end{cases}
\end{aligned}
$$

where we have used an integration by parts in the last step.

To complete the proof, we just need to observe that by Eq. (8.5.6),

$$
\int_{n^{-1}}^{\frac{\pi}{4}} |M_n(\phi)|(\sin\phi)^{2\lambda}\,\mathrm{d}\phi \geq \int_{n^{-1}}^{\varepsilon}|K_n(\phi)|(\sin\phi)^{2\lambda}\,\mathrm{d}\phi - \int_{n^{-1}}^{\varepsilon}|G_n(\phi)|(\sin\phi)^{2\lambda}\,\mathrm{d}\phi,
$$

whereas by Eq. (8.5.8),

$$
\int_{n^{-1}}^{\varepsilon}|G_n(\phi)|(\sin\phi)^{2\lambda}\,\mathrm{d}\phi \leq cn^{-1}\log n + cn^{-\mu+\delta-\lambda},
$$

which is smaller in magnitude than the bound for the first term, since $0<\mu<1$. □

Lemma 8.5.6. *Assume* $0<\mu<1$, $\lambda\geq 0$, *and* $\lambda\geq\delta>-1$. *Let* E_n *be defined by Eq.* (8.5.5)*. Then*

$$
E_n \leq cn^{-\mu-\frac{1}{2}-(\lambda-\delta)} + cn^{-\frac{3}{2}}\log n.
$$

Proof. By Eq. (8.5.5) and the identity $\cos^2\theta - \cos^2\phi = \sin(\theta+\phi)\sin(\theta-\phi)$, we obtain

$$
\begin{aligned}
E_n &= n^{-\frac{3}{2}}\int_{n^{-1}}^{\frac{\pi}{4}}\int_{\phi}^{\pi-\phi}\frac{\sin^{\mu-1}(\theta+\phi)\sin^{\mu-1}(\theta-\phi)}{(\sin^{a+1}\frac{\theta}{2})(\cos^{b+1}\frac{\theta}{2})}\,\mathrm{d}\theta\sin^{2\lambda}\phi\,\mathrm{d}\phi \\
&\leq cn^{-\frac{3}{2}}\int_{n^{-1}}^{\frac{\pi}{4}}\int_{\phi}^{\frac{\pi}{2}}\theta^{\mu-a-2}(\theta-\phi)^{\mu-1}\,\mathrm{d}\theta\,\phi^{2\lambda}\,\mathrm{d}\phi.
\end{aligned}
$$

The inner integral can be estimated by splitting the integral into two parts, over $[\phi, 2\phi]$ and over $[2\phi, \pi/2]$, respectively. On considering the various cases and taking into the account that $a = \lambda + \mu + \delta$ and $\lambda \geq 0$, we conclude that

$$E_n \leq cn^{-\frac{3}{2}} \int_{n^{-1}}^{\frac{\pi}{4}} \left(\phi^{2\lambda} |\log \phi| + \phi^{\lambda+\mu-\delta-2} \right) \mathrm{d}\phi \leq cn^{-\mu-\frac{1}{2}-(\lambda-\delta)} + cn^{-\frac{3}{2}} \log n.$$

This completes the proof of Lemma 8.5.6. □

We now return to the proof of Proposition 8.5.2.

Proof of Proposition 8.5.2 (continued). We consider the following cases:

Case 1. $0 < \mu < 1$. This case follows directly from Eq. (8.5.3) and Lemmas 8.5.5 and 8.5.6.

Case 2. $\mu = 0$ or 1. In the case $\mu = 0$, I_n in limit form reduces to

$$\begin{aligned} I_n &= \int_0^1 \left| P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(y) + P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(-y) \right| (1-y^2)^{\lambda-1/2} \mathrm{d}y \\ &\geq \int_{n^{-1}}^{\pi/4} \left| P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(\cos\phi) + P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(\cos(\pi-\phi)) \right| (\sin\phi)^{2\lambda} \mathrm{d}\phi. \end{aligned}$$

The asymptotic formula of the Jacobi polynomial gives

$$\begin{aligned} &P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(\cos\phi) + P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(\cos(\pi-\phi)) = \frac{\pi^{-1/2} n^{-1/2}}{(\sin\frac{\phi}{2})^{a+1} (\cos\frac{\phi}{2})^{a+1}} \\ &\times \left[\left(\cos\frac{\phi}{2} \right)^{a-b} \cos(N\phi + \tau) + \left(\sin\frac{\phi}{2} \right)^{a-b} \cos(N(\pi-\phi)+\tau) \right] + \mathscr{O}\left((n\sin\phi)^{-1} \right), \end{aligned}$$

which is essentially the same as the asymptotic formula for $M_n(\phi)$ in Lemma 8.5.4 with $\mu = 0$ and a smaller remainder. Thus, a proof almost identical to that of Lemma 8.5.5 will yield Proposition 8.5.2 for $\mu = 0$. Proposition 8.5.2 for $\mu = 1$ can be proved in a similar way.

Case 3. $\mu > 1$. In this case, we denote by r the greatest integer less than μ. We then use Eq. (B.1.5) and integrate by parts r times to obtain

$$\begin{aligned} &\int_{-y}^{y} P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(u) (y^2-u^2)^{\mu-1} \mathrm{d}u \\ &= \frac{(-2)^r}{\prod_{i=1}^{r} (n+a+b+2-i)} \int_{-y}^{y} P_{n+r}^{(a+\frac{1}{2}-r, b+\frac{1}{2}-r)}(u) \frac{\mathrm{d}^r}{\mathrm{d}u^r} \left[(y^2-u^2)^{\mu-1} \right] \mathrm{d}u. \end{aligned}$$

Since $[(y^2-u^2)^{\mu-1}]^{(r)} = Aq(y,u)(y^2-u^2)^{\mu-r-1}$, where A is a nonzero constant and $q(y,u)$ is a polynomial in y and u that satisfies $q(y,y) = (-1)^r q(y,-y) = 1$, we conclude that

$$\left|\int_{-y}^{y} P_n^{(a+\frac{1}{2},b+\frac{1}{2})}(u)(y^2-u^2)^{\mu-1}\mathrm{d}u\right|$$
$$\geq cn^{-r}\left|\int_{-y}^{y} P_{n+r}^{(a'+\frac{1}{2},b'+\frac{1}{2})}(u)q(y,u)(y^2-u^2)^{\mu'-1}\mathrm{d}u\right|,$$

where $\mu' = \mu - r \in (0,1]$, $a' = \lambda + \mu' + \delta$ and $b' = \lambda + \mu' + \delta$. It follows that

$$I_n \geq cn^{-r}\int_{\sqrt{2}/2}^{1}\left|\int_{-y}^{y} P_{n+r}^{(a'+\frac{1}{2},b'+\frac{1}{2})}(u)q(y,u)y(y^2-u^2)^{\mu'-1}\mathrm{d}u\right|(1-y^2)^{\lambda-1/2}\mathrm{d}y$$
$$\geq cn^{-r}\int_{n^{-1}}^{\pi/4}\left|\int_{\phi}^{\pi-\phi} P_{n+r}^{(a'+\frac{1}{2},b'+\frac{1}{2})}(\cos\theta)q_\phi(\cos\theta)(\cos^2\phi-\cos^2\theta)^{\mu'-1}\sin\theta\mathrm{d}\theta\right|$$
$$\times(\sin\phi)^{2\lambda}\mathrm{d}\phi,$$

where $q_\phi(\cos\theta) = q(\cos\phi,\cos\theta)$. Since $\mu' \in (0,1]$, $q_\phi(\cos\phi) = (-1)^r q_\phi(-\cos\phi) = 1$ and $\sup_{\phi,\theta}|q'_\phi(\cos\theta)| \leq c < \infty$, the desired lower estimate in this case follows by a slight modification of the proofs in Cases 1 and 2.

Putting these cases together, we have completed the proof of Proposition 8.5.2.

□

8.6 Notes and Further Results

For $\delta > (d-2)/2$, the main estimate in Eq. (8.3.1) was established in [108] and used to establish Corollary 8.1.2 and Theorem 8.1.3. The general case of $\delta > -1$ and its proof were given in [49].

In contrast to Theorem 8.1.3, it was shown in [196] that if for $f \in L^1(h_\kappa^2)$, $S_n^\delta(h_\kappa^2;f)$ converges almost everywhere on $\mathbb{S}^{d-1}$, then it is necessary that $\delta \geq \sigma_\kappa = |\kappa| + \min_{1\leq i\leq d}\kappa_i + (d-2)/2$.

Chapter 9
Projection Operators and Cesàro Means in L^p Spaces

In analogy with the Bochner–Riesz means on $\mathbb{R}^d$, the Cesàro (C,δ) means of the spherical harmonic expansions on $\mathbb{S}^{d-1}$ can be bounded in L^p space for δ below the critical index $\frac{d-2}{2}$, and furthermore, they are bounded under the same condition as that of the Bochner–Riesz means. In this chapter, we establish such results for h-harmonic expansions with respect to the product $h_\kappa^2(x) = \prod_{i=1}^d |x_i|^{2\kappa_i}$, which cover results for ordinary spherical harmonic expansions. The proof of such results depends on the boundedness of projection operators, which will be established in the first section, assuming a critical estimate. The main results on the boundedness of the Cesàro means on L^p space are stated and proved in the second section. The critical estimate used in the first section is established in the third section, which is rather technical and can be bypassed with little impact on further reading.

9.1 Boundedness of Projection Operators

Throughout this section, we assume $h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$ and recall that

$$\sigma_\kappa := \frac{d-2}{2} + |\kappa| - \kappa_{\min} \quad \text{with} \quad \kappa_{\min} := \min_{1\le j\le d} \kappa_j.$$

For $1 \le p \le \infty$, we define a positive number $\delta_\kappa(p)$ by

$$\delta_\kappa(p) := \max\left\{(2\sigma_\kappa+1)\left|\frac{1}{p}-\frac{1}{2}\right| - \frac{1}{2}, 0\right\}. \tag{9.1.1}$$

The main result we need on the projection operator is the following.

F. Dai and Y. Xu, *Approximation Theory and Harmonic Analysis on Spheres and Balls*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-6660-4_9,

Theorem 9.1.1. *Let $d>2$ and $n\in\mathbb{N}$. Then*

(i) for $1\le p\le \frac{2(\sigma_k+1)}{\sigma_k+2}$,

$$\|\operatorname{proj}_n^\kappa f\|_{\kappa,2}\le cn^{\delta_\kappa(p)}\|f\|_{\kappa,p};$$

(ii) for $\frac{2(\sigma_k+1)}{\sigma_k+2}\le p\le 2$,

$$\|\operatorname{proj}_n^\kappa f\|_{\kappa,2}\le cn^{\sigma_k(\frac{1}{p}-\frac{1}{2})}\|f\|_{\kappa,p}.$$

Furthermore, the estimate (i) is sharp, and the estimate (ii) is sharp when $\kappa=0$.

Let χ_E denote the characteristic function of the set E. The proof of Theorem 9.1.1 depends on a sharp local estimate given below.

Theorem 9.1.2. *Let $\nu:=\frac{2\sigma_\kappa+2}{\sigma_\kappa+2}$ and $\nu':=\frac{\nu}{\nu-1}$. Let f be a function supported in a spherical cap $\mathsf{c}(\varpi,\theta)$ with $\theta\in(n^{-1},1/(8d)]$ and $\varpi\in\mathbb{S}^{d-1}$. Then*

$$\left\|(\operatorname{proj}_n^\kappa f)\chi_{c(\varpi,\theta)}\right\|_{\kappa,\nu'}\le cn^{\frac{\sigma_\kappa}{1+\sigma_\kappa}}\theta^{\frac{2\sigma_\kappa+1}{\sigma_\kappa+1}}\left[\int_{\mathsf{c}(\varpi,\theta)}h_\kappa^2(x)\mathrm{d}\sigma(x)\right]^{1-\frac{2}{\nu}}\|f\|_{\kappa,\nu}.$$

Since the norm of the left-hand side is taken over the spherical cap $\mathsf{c}(\varpi,\theta)$, the above estimate is a local one. The proof of Theorem 9.1.2 is long and will be given in later sections. We first use it to establish the following result.

Theorem 9.1.3. *Suppose that $1\le p\le\frac{2\sigma_\kappa+2}{\sigma_k+2}$ and f is supported in a spherical cap $\mathsf{c}(\varpi,\theta)$ with $\theta\in(n^{-1},\pi]$ and $\varpi\in\mathbb{S}^{d-1}$. Then*

$$\|\operatorname{proj}_n^\kappa f\|_{\kappa,2}\le cn^{\delta_\kappa(p)}\theta^{\delta_\kappa(p)+\frac{1}{2}}\left[\int_{\mathsf{c}(\varpi,\theta)}h_\kappa^2(x)\,\mathrm{d}\sigma(x)\right]^{\frac{1}{2}-\frac{1}{p}}\|f\|_{\kappa,p}.$$

Proof. Assume that f is supported in a spherical cap $\mathsf{c}(\varpi,\theta)$. Without loss of generality, we may assume $\theta<1/(8d)$, since otherwise, we can decompose f as a finite sum of functions supported on a family of spherical caps of radius less than $1/(8d)$.

We start with the case $p=1$. By the definition of the projection operator, it follows from the integral version of the Minkowski inequality and orthogonality that

$$\begin{aligned}\|\operatorname{proj}_n^\kappa f\|_{\kappa,2}&\le\sup_{y\in\mathsf{c}(\varpi,\theta)}\left(\int_{\mathbb{S}^{d-1}}|Z_n^\kappa(x,y)|^2h_\kappa^2(x)\,\mathrm{d}\sigma(x)\right)^{1/2}\|f\|_{\kappa,1}\\&=\left(\sup_{y\in\mathsf{c}(\varpi,\theta)}Z_n^\kappa(y,y)\right)^{1/2}\|f\|_{\kappa,1}.\end{aligned}$$

Using the pointwise estimate of the kernel in Eq. (8.3.2) and the fact that $n\theta \ge 1$, we then obtain

$$\begin{aligned}\|\operatorname{proj}_n^\kappa f\|_{\kappa,2} &\le cn^{\frac{d-2}{2}} \sup_{y\in c(\varpi,\theta)} \prod_{j=1}^d (|y_j|+n^{-1})^{-\kappa_j} \|f\|_{\kappa,1} \\ &\le cn^{\frac{d-2}{2}} (n\theta)^{\sigma_\kappa - \frac{d-2}{2}} \sup_{y\in c(\varpi,\theta)} \prod_{j=1}^d (|y_j|+\theta)^{-\kappa_j} \|f\|_{\kappa,1} \\ &\le cn^{\sigma_\kappa} \theta^{\sigma_\kappa+\frac12} \left(\int_{c(\varpi,\theta)} h_\kappa^2(y)\,\mathrm{d}\sigma(y)\right)^{-\frac12} \|f\|_{\kappa,1},\end{aligned}$$

where the last step follows from Eq. (5.1.9). This proves Theorem 9.1.3 for $p=1$.

Next, we use Hölder's inequality and Theorem 9.1.2 to obtain

$$\begin{aligned}\|\operatorname{proj}_n^\kappa f\|_{\kappa,2}^2 &= \int_{c(\varpi,\theta)} f(y) \operatorname{proj}_n^\kappa f(y) h_\kappa^2(y)\,\mathrm{d}\sigma(y) \\ &\le \|f\|_{\kappa,\nu} \left(\int_{c(\varpi,\theta)} |\operatorname{proj}_n^\kappa f(y)|^{\nu'} h_\kappa^2(y)\,\mathrm{d}\sigma(y)\right)^{\frac{1}{\nu'}} \\ &\le cn^{\frac{\sigma_\kappa}{\sigma_\kappa+1}} \theta^{\frac{2\sigma_\kappa+1}{\sigma_\kappa+1}} \left[\int_{c(\varpi,\theta)} h_\kappa^2(x)\,\mathrm{d}\sigma(x)\right]^{1-\frac{2}{\nu}} \|f\|_{\kappa,\nu}^2,\end{aligned}$$

which proves Theorem 9.1.3 for $p=\nu=\frac{2\sigma_\kappa+2}{\sigma_\kappa+2}$.

Finally, Theorem 9.1.3 for $1\le p\le\nu$ follows by applying the Riesz–Thorin convexity theorem to the linear operator $g \mapsto \operatorname{proj}_n(h_\kappa^2; g\chi_{c(\varpi,\theta)})$. □

For the proof of Theorem 9.1.1, we will also need the following duality result.

Lemma 9.1.4. *Assume* $1\le p\le 2\le q\le\infty$ *and* $\frac1p+\frac1q=1$. *Then the following are equivalent:*

(i) $\|\operatorname{proj}_n^\kappa f\|_{k,2} \le A\|f\|_{k,p}$,
(ii) $\|\operatorname{proj}_n^\kappa f\|_{k,q} \le A\|f\|_{k,2}$.

Proof. The proof is standard and follows from writing

$$\begin{aligned}\|\operatorname{proj}_n^\kappa f\|_{k,q} &= \sup_{\|g\|_{k,p}=1} \frac{1}{\omega_d^\kappa}\int_{\mathbb{S}^{d-1}} \operatorname{proj}_n^\kappa f(y) g(y) h_\kappa^2(y)\mathrm{d}\sigma(y) \\ &= \sup_{\|g\|_{k,p}=1} \frac{1}{\omega_d^\kappa}\int_{\mathbb{S}^{d-1}} f(y) \operatorname{proj}_n^\kappa g(y) h_\kappa^2(y)\mathrm{d}\sigma(y),\end{aligned}$$

where the second equation follows from the fact that $Z_n^\kappa(x,y)$ is symmetric in x and y. To prove that (i) implies (ii), we apply the Cauchy–Schwarz inequality on the right-hand side and then use (i). The proof that (ii) implies (i) follows similarly. □

Proof of Theorem 9.1.1. The inequality of (i) follows directly by invoking Theorem 9.1.2 with $\theta = \pi$. The inequality of (ii) follows from the Riesz–Thorin convexity theorem applied to the boundedness of $f \mapsto \operatorname{proj}_n^\kappa f$ in $(2,2)$ and in $(\nu,2)$.

To prove the sharpness of the estimates, we can assume without loss of generality that $\kappa_{\min} = \kappa_1$. In case (i), we define

$$f_n(x) := Z_n^\kappa(x,e), \qquad e = (1,0,0,\ldots,0).$$

Since $f_n \in \mathscr{H}_n^d(h_\kappa^2)$, we have

$$\|\operatorname{proj}_n^\kappa f_n\|_{\kappa,q} = \|f_n\|_{\kappa,q} = \left(\int_{\mathbb{S}^{d-1}} |Z_n^\kappa(x,e)|^q h_\kappa^2(x)\mathrm{d}\sigma(x)\right)^{1/q}.$$

Thus, it is sufficient to show that

$$\|f_n\|_{k,q} \sim n^{\sigma_k - \frac{2\sigma_\kappa+1}{q}} \|f\|_{k,2} \quad \text{for } q \ge \frac{2(\sigma_\kappa+1)}{\sigma_\kappa}. \tag{9.1.2}$$

Indeed, setting $p = q/(q-1)$ and using Lemma 9.1.4, Eq. (9.1.2) shows that

$$\|\operatorname{proj}_n^\kappa f_n\|_{\kappa,2} \sim c n^{\sigma_k - \frac{2\sigma_\kappa+1}{q}} \|f_n\|_{\kappa,p} = c n^{(2\sigma_k+1)\left(\frac{1}{p} - \frac{\sigma_\kappa+1}{2\sigma_k+1}\right)} \|f_n\|_{\kappa,p},$$

which proves the sharpness of (i).

Recall that $C_n^{(\lambda,\mu)}(t)$ denotes the generalized Gegenbauer polynomial. By Eq. (7.2.14),

$$Z_n^\kappa(x,e) = \frac{n+\lambda_\kappa}{\lambda_k} C_n^{(\sigma_\kappa,\kappa_1)}(x_1).$$

Hence by Eq. (B.3.1), in terms of Jacobi polynomials we have

$$Z_{2n}^\kappa(x,e) = \mathscr{O}(1) n^{\sigma_\kappa+\frac{1}{2}} P_n^{(\sigma_k-\frac{1}{2},\kappa_1-\frac{1}{2})}(2x_1^2-1). \tag{9.1.3}$$

Since this is a function that depends only on x_1, a standard change of variables leads to

$$\begin{aligned}
\|f_{2n}\|_{\kappa,q} &\sim n^{\sigma_\kappa+\frac{1}{2}} \left(\int_0^\pi |P_n^{(\sigma_k-\frac{1}{2},\kappa_1-\frac{1}{2})}(2\cos^2\theta-1)|^q |\cos\theta|^{2\kappa_1}(\sin\theta)^{2\sigma_\kappa}\mathrm{d}\theta\right)^{\frac{1}{q}} \\
&\sim n^{\sigma_\kappa+\frac{1}{2}} \left(\int_{-1}^1 |P_n^{(\sigma_\kappa-\frac{1}{2},\kappa_1-\frac{1}{2})}(t)|^q w^{(\sigma_k-\frac{1}{2},\kappa_1-\frac{1}{2})}(t)\mathrm{d}t\right)^{1/q} \\
&\sim n^{\sigma_\kappa} n^{\sigma_k - \frac{2\sigma_k+1}{q}},
\end{aligned}$$

where in the last step we have used Eq. (B.1.8) and the condition $q \ge 2(\sigma_k + 1)/\sigma_\kappa > (2\sigma_k + 1)/\sigma_\kappa$ to conclude that the integral on $[0,1]$ has the stated estimate, whereas the integral over $[-1,0]$, using $P_n^{(\alpha,\beta)}(t) = P_n^{(\beta,\alpha)}(-t)$, has an order dominated by the integral on $[0,1]$. For $q = 2$, using Eq. (9.1.3), we get

$$\|f_{2n}\|_{\kappa,2} = (Z_{2n}^{\kappa}(e,e))^{\frac{1}{2}} \sim n^{\sigma_\kappa}.$$

Together, these two relations establish Eq. (9.1.2) for even n. The proof for odd n is similar. This completes the proof of (i).

To show that the estimate of (ii) is sharp, we choose $f_n(x) = (x_{d-1} + ix_d)^n$, which is harmonic, so that it is an element of $\mathscr{H}_n^d$. Using Eq. (A.5.3), we see that for $d > 3$,

$$\begin{aligned}\|f_n\|_q^q &= \frac{\omega_{d-2}}{\omega_d} \int_{\mathbb{B}^2} |x_1^2 + x_2^2|^{\frac{nq}{2}} (1 - x_1^2 - x_2^2)^{\frac{d-4}{2}} \mathrm{d}x_1 \mathrm{d}x_2 \\ &= 2\pi \frac{\omega_{d-2}}{\omega_d} \int_{-1}^{1} r^{\frac{nq}{2}+1} (1 - r^2)^{\frac{d-4}{2}} \mathrm{d}r \sim n^{-\frac{d-2}{2}},\end{aligned}$$

whereas for $d = 3$, we use Eq. (A.5.4) instead, from which the sharpness of (ii) follows immediately. □

9.2 Boundedness of Cesàro Means in L^p Spaces

Our main results on the Cesàro summation of h-harmonic expansions are the following two theorems:

Theorem 9.2.1. *Suppose that* $f \in L^p(h_\kappa^2, \mathbb{S}^{d-1})$, $1 \le p \le \infty$, $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{2\sigma_\kappa + 2}$, *and*

$$\delta > \delta_\kappa(p) := \max\left\{(2\sigma_\kappa + 1)\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right\}. \qquad (9.2.1)$$

Then $S_n^\delta(h_\kappa^2; f)$ *converges to* f *in the* $L^p(h_\kappa^2, \mathbb{S}^{d-1})$ *norm and*

$$\sup_{n \in \mathbb{N}} \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,p} \le c\|f\|_{\kappa,p}.$$

Theorem 9.2.2. *Assume* $1 \le p \le \infty$ *and* $0 < \delta \le \delta_\kappa(p)$. *Then there exists a function* $f \in L^p(h_\kappa^2, \mathbb{S}^{d-1})$ *such that* $S_n^\delta(h_\kappa^2; f)$ *diverges in* $L^p(h_\kappa^2, \mathbb{S}^{d-1})$.

For $\kappa = 0, h_\kappa(x) \equiv 1$, and the spherical h-harmonic becomes the ordinary spherical harmonics, and the above two theorems become Theorems 2.5.1 and 2.5.3 in this case.

9.2.1 Proof of Theorem 9.2.1

The proof is rather involved, and we break it into several steps.

9.2.1.1 Decomposition

Let $\varphi_0 \in C^\infty[0,\infty)$ be a nonnegative function such that for all $x \ge 0$, $\chi_{[0,1]}(x) \le \varphi_0(x) \le \chi_{[0,2]}(x)$, and let $\varphi(t) := \varphi_0(t) - \varphi_0(2t)$. Then φ is a C^∞-function supported in $(\frac{1}{2}, 2)$, and it satisfies $\sum_{v=0}^{\infty} \varphi(2^v t) = 1$ for all $t > 0$. Setting

$$\hat{S}_{n,v}^{\delta}(j) := \varphi\Big(\frac{2^v(n-j)}{n}\Big)\frac{A_{n-j}^{\delta}}{A_n^{\delta}},$$

we define

$$S_{n,v}^{\delta} f := \sum_{j=0}^{n} \hat{S}_{n,v}^{\delta}(j)\operatorname{proj}_j^{\kappa} f, \qquad v = 0,1,\ldots,\lfloor \log_2 n \rfloor + 2.$$

Since $\sum_{v=0}^{\lfloor \log_2 n \rfloor + 2} \varphi\left(\frac{2^v(n-j)}{n}\right) = 1$ for $0 \le j \le n-1$, it follows that the Cesàro means are decomposed as

$$S_n^{\delta}(h_\kappa^2; f) = \sum_{v=0}^{\lfloor \log_2 n \rfloor + 2} S_{n,v}^{\delta} f + \frac{1}{A_n^{\delta}}\operatorname{proj}_n^{\kappa} f. \tag{9.2.2}$$

Using Theorem 9.1.1 and the fact that $\delta > \delta_\kappa(p)$, we have

$$\frac{1}{A_n^{\delta}}\|\operatorname{proj}_n^{\kappa} f\|_{\kappa,p} \le c n^{-\delta}\|\operatorname{proj}_n^{\kappa} f\|_{\kappa,2} \le c n^{\delta_\kappa(p)-\delta}\|f\|_{\kappa,p} \le c\|f\|_{\kappa,p}.$$

On the other hand, using summation by parts $\ell \ge 1$ times shows that

$$S_{n,v}^{\delta} f = \sum_{j=0}^{n} \Delta^{\ell}\left(\hat{S}_{n,v}^{\delta}(j)\right) A_j^{\ell-1} S_j^{\ell-1}(h_\kappa^2; f),$$

where Δ denotes the forward difference and $\Delta^{\ell+1} := \Delta\Delta^{\ell}$. Since $\hat{S}_{n,v}^{\delta}(j) = 0$ whenever $n-j > \frac{n}{2^{v-1}}$ or $n-j < \frac{n}{2^{v+1}}$, it is easy to verify by the Leibniz rule that

$$\left|\Delta^{\ell}(S_{n,v}^{\delta}(j))\right| \le c 2^{-v\delta}\left(\frac{2^v}{n}\right)^{\ell}, \quad \forall \ell \in \mathbb{N},\ 0 \le j \le n. \tag{9.2.3}$$

Hence, choosing $\ell > \lambda_\kappa$ and using the fact that $S_n^{\ell}(h_\kappa^2; f)$ is bounded in $L^p(h_\kappa^2, \mathbb{S}^{d-1})$ for all $1 \le p \le \infty$ if $\ell > \lambda_\kappa$, we conclude that for $v = 0$ and 1,

$$\|S_{n,v}^{\delta}f\|_{\kappa,p} \le cn^{-\ell}\sum_{j=0}^{n} j^{\ell-1}\|S_j^{\ell-1}(h_\kappa^2;f)\|_{\kappa,p} \le c\|f\|_{\kappa,p}.$$

Therefore, by Eq. (9.2.2), it is sufficient to prove that

$$\|S_{n,v}^{\delta}(f)\|_{\kappa,p} \le c2^{-v\varepsilon_0}\|f\|_{\kappa,p}, \quad v = 2,\dots,\lfloor \log_2 n \rfloor + 2, \tag{9.2.4}$$

where ε_0 is a sufficiently small positive constant depending on δ and p, but independent of n and v.

9.2.1.2 Estimate of the Kernel of $S_{n,v}^{\delta}$

Let

$$D_{n,v}^{\delta}(t) := \sum_{j=0}^{n} \hat{S}_{n,v}^{\delta}(j)\frac{\lambda_\kappa + j}{\lambda_\kappa}C_j^{\lambda_\kappa}(t).$$

The definition shows that $S_{n,v}^{\delta}f = f *_\kappa D_{n,v}^{\delta}$, so that the kernel of $S_{n,v}^{\delta}f$ is

$$K_{n,v}^{\delta}(x,y) := V_\kappa\left[D_{n,v}^{\delta}(\langle x,\cdot\rangle)\right](y).$$

Lemma 9.2.3. *Let $2 \le v \le \lfloor \log_2 n \rfloor + 2$. Then for any given positive integer ℓ,*

$$|K_{n,v}^{\delta}(x,y)|h_\kappa^2(y) \le cn^{d-1}2^{v(\ell-1-\delta)}\left(1 + n\mathsf{d}(\bar{x},\bar{y})\right)^{-\ell-d+\lambda_\kappa+2},$$

where $\bar{z} = (|z_1|,\dots,|z_d|)$ for $z = (z_1,\dots,z_d) \in \mathbb{R}^d$.

Proof. We first define a sequence of functions $\{a_{n,v,\ell}(\cdot)\}_{\ell=0}^{\infty}$ by

$$a_{n,v,0}(j) = 2(j+\lambda_\kappa)\hat{S}_{n,v}^{\delta}(j),$$
$$a_{n,v,\ell+1}(j) = \frac{a_{n,v,\ell}(j)}{2j+2\lambda_\kappa+\ell} - \frac{a_{n,v,\ell}(j+1)}{2j+2\lambda_\kappa+\ell+2}, \qquad \ell \ge 0.$$

Following the proof of Theorem 2.6.7, we can write, for any integer $\ell \ge 0$,

$$D_{n,v}^{\delta}(t) = c_\kappa \sum_{j=0}^{\infty} a_{n,v,\ell}(j)\frac{\Gamma(j+2\lambda_\kappa+\ell)}{\Gamma(j+\lambda_\kappa+\frac{1}{2})}P_j^{(\lambda_\kappa+\ell-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(t),$$

so that

$$K_{n,v}^{\delta}(x,y) = c_\kappa \sum_{j=0}^{\infty} a_{n,v,\ell}(j)\frac{\Gamma(j+2\lambda_\kappa+\ell)}{\Gamma(j+\lambda_\kappa+\frac{1}{2})}V_\kappa\left[P_j^{(\lambda_\kappa+\ell-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(\langle x,\cdot\rangle)\right](y).$$

Note that $a_{n,v,\ell}(j) = 0$ if $j+\ell \le (1-\frac{1}{2^{v-1}})n$ or $j \ge (1-\frac{1}{2^{v+1}})n$, so that the sum is over $j \sim n$. Furthermore, it follows from the definition, Eq. (9.2.3), and Leibniz's rule that

$$\left|a_{n,v,\ell}(j)\right| \le c2^{-v\delta} n^{-\ell+1} \left(\frac{2^v}{n}\right)^{\ell}, \qquad \ell = 0,1,\dots. \tag{9.2.5}$$

Consequently, using the pointwise estimate of Eq. (8.2.5), it follows that

$$\begin{aligned}|K^{\delta}_{n,v}(x,y)| &\le cn^{2\lambda_\kappa+2\ell-1-2|\kappa|} \sum_{\substack{j\sim n\\ n-j\sim \frac{n}{2^v}}} |a_{n,v,\ell}(j)| \frac{\prod_{i=1}^d (|x_i y_i| + n^{-1}\mathtt{d}(\bar{x},\bar{y}) + n^{-2})^{-\kappa_i}}{(1+n\mathtt{d}(\bar{x},\bar{y}))^{\lambda_\kappa+\ell-|\kappa|}} \\ &\le cn^{d-1}2^{v(\ell-1-\delta)} \frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\mathtt{d}(\bar{x},\bar{y}) + n^{-2})^{-\kappa_j}}{(1+n\mathtt{d}(\bar{x},\bar{y}))^{\lambda_\kappa+\ell-|\kappa|}} \\ &\le cn^{d-1}2^{v(\ell-1-\delta)} h_\kappa^{-2}(y)\,(1+n\mathtt{d}(\bar{x},\bar{y}))^{\lambda_\kappa-d+2-\ell},\end{aligned}$$

where in the last inequality we have used the fact that

$$\prod_{j=1}^d (|x_j y_j| + n^{-1}\mathtt{d}(\bar{x},\bar{y}) + n^{-2})^{-\kappa_j} \le ch_\kappa^{-2}(y)\mathtt{d}(\bar{x},\bar{y})^{|\kappa|},$$

which follows because if $|y_j| \ge 2\mathtt{d}(\bar{x},\bar{y})$, then $|\bar{x}_j - \bar{y}_j| \le \mathtt{d}(\bar{x},\bar{y}) \le |y_j|/2$, so that $|y_j|^2 \le 2|x_j y_j|$, whereas if $|y_j| < 2\mathtt{d}(\bar{x},\bar{y})$, then $|y_j|^2 \le 2(n^{-1}\mathtt{d}(\bar{x},\bar{y}))\cdot n\mathtt{d}(\bar{x},\bar{y})$. This completes the proof of Lemma 9.2.3. □

Corollary 9.2.4. *For every $\gamma > 0$, there exists an $\varepsilon_0 > 0$ independent of n and v such that*

$$\sup_{x\in\mathbb{S}^{d-1}} \int_{\{y\in\mathbb{S}^{d-1}:\ \mathtt{d}(\bar{x},\bar{y})>2^{(1+\gamma)v}/n\}} |K^{\delta}_{n,v}(x,y)|\, h_\kappa^2(y)\,\mathrm{d}\sigma(y) \le c2^{-v\varepsilon_0}.$$

Proof. Invoking Lemma 9.2.3 with $\ell > \lambda_\kappa + 1 + \frac{\lambda_\kappa-\delta}{\gamma}$, we see that the quantity to be estimated is bounded by

$$\begin{aligned}c \sup_{x\in\mathbb{S}^{d-1}} n^{d-1}2^{v(\ell-1-\delta)} &\int_{\{y:\ \mathtt{d}(\bar{x},\bar{y})>2^{(1+\gamma)v}/n\}} \frac{1}{(1+nd(\bar{x},\bar{y}))^{\ell+d-\lambda_k-2}}\mathrm{d}\sigma(y) \\ &\le c2^{v(\ell-1-\delta)} \int_{2^{(1+\gamma)v}/n}^{\pi} \frac{n(n\theta)^{d-2}}{(1+n\theta)^{\ell+d-\lambda_k-2}}\mathrm{d}\theta \\ &\le c2^{v(\ell-1-\delta-(1+\gamma)(\ell-\lambda_\kappa-1))} = c2^{-v\varepsilon_0},\end{aligned}$$

which proves the corollary. □

9.2.1.3 Proof of Eq. (9.2.4)

Now we are in a position to prove Eq. (9.2.4). Recall that

$$S_{n,\nu}^{\delta} f = \sum_{(1-2^{-\nu+1})n \le j \le (1-2^{-\nu-1})n} \hat{S}_{n,\nu}^{\delta}(j) \operatorname{proj}_j^{\kappa} f. \tag{9.2.6}$$

Assume $\delta > \delta_\kappa(p)$, and let $\gamma > 0$ be sufficiently small that $\delta > \delta_\kappa(p) + \gamma\left(\delta_\kappa(p) + \frac{1}{2}\right)$. Set $\nu_1 = \nu(1+\gamma)$. Let Λ be a maximal $\frac{2^{\nu_1}}{n}$-separated subset of $\mathbb{S}^{d-1}$; that is, $\min_{\varpi \ne \varpi' \in \Lambda} \mathtt{d}(\varpi, \varpi') \ge \frac{2^{\nu_1}}{n}$ and $\mathbb{S}^{d-1} \subset \cup_{\varpi \in \Lambda} \mathtt{c}(\varpi, \frac{2^{\nu_1}}{n})$. Define

$$f_\varpi(x) := f(x) \chi_{\mathtt{c}(\varpi, \frac{2^{\nu_1}}{n})}(x) [A(x)]^{-1}, \qquad A(x) := \sum_{\varpi \in \Lambda} \chi_{\mathtt{c}(\varpi, \frac{2^{\nu_1}}{n})}(x).$$

Then evidently $1 \le A(x) \le c$, $x \in \mathbb{S}^{d-1}$, $|f_\varpi| \le c|f|$, and $f(x) = \sum_{\varpi \in \Lambda} f_\varpi(x)$. Using the Minkowski inequality, we obtain

$$\|S_{n,\nu}^{\delta}(f)\|_{\kappa,p} \le \sum_{\varpi \in \Lambda} \|S_{n,\nu}^{\delta}(f_\varpi)\|_{\kappa,p}.$$

Thus, it is sufficient to show that for each $\varpi \in \Lambda$, we have

$$\|S_{n,\nu}^{\delta}(f_\varpi)\|_{\kappa,p} \le c 2^{-\nu\varepsilon_0} \|f_\varpi\|_{\kappa,p}. \tag{9.2.7}$$

To this end, we denote by $\mathtt{c}^*(\varpi, 2^{\nu_1+1}/n)$ the set

$$\mathtt{c}^*\left(\varpi, \frac{2^{\nu_1+1}}{n}\right) = \left\{x \in \mathbb{S}^{d-1} : \mathtt{d}(\bar{x}, \bar{\varpi}) \le 2^{\nu_1+1}/n\right\}$$

and further define $J(\nu,n) := \{j : (1-2^{-\nu+1})n \le j \le (1-2^{-\nu-1})n\}$. Using Eq. (9.2.6) and orthogonality, we obtain

$$\|S_{n,\nu}^{\delta}(f_\varpi)\|_{\kappa,2} = \left(\sum_{j \in J(\nu,n)} |\hat{S}_{n,\nu}^{\delta}(j)|^2 \|\operatorname{proj}_j^{\kappa} f_\varpi\|_{\kappa,2}^2\right)^{\frac{1}{2}}.$$

Hence, by Hölder's inequality, Theorem 9.1.3, and Eqs. (5.1.9), and (9.2.3) with $\ell = 0$, we have

$$\begin{aligned}
&\left(\int_{\mathtt{c}^*(\varpi, 2^{\nu_1+1}/n)} |S_{n,\nu}^{\delta}(f_\varpi)(x)|^p h_\kappa^2(x) \, d\sigma(x)\right)^{\frac{1}{p}} \\
&\quad \le c \left(\int_{\mathtt{c}(\varpi, 2^{\nu_1+1}/n)} h_\kappa^2(x) \, d\sigma(x)\right)^{\frac{1}{p} - \frac{1}{2}} \left(\sum_{j \in J(\nu,n)} |\hat{S}_{n,\nu}^{\delta}(j)|^2 \|\operatorname{proj}_j(h_\kappa^2; f_\varpi)\|_{\kappa,2}^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\le c2^{v_1(\delta_\kappa(p)+\frac12)}n^{-\frac12}\Big(\sum_{j\in J(v,n)}|\hat S^\delta_{n,v}(j)|^2\Big)^{\frac12}\|f_\varpi\|_{\kappa,p}$$
$$\le c2^{-v\left(\delta-\delta_\kappa(p)-\gamma(\delta_\kappa(p)+\frac12)\right)}\|f_\varpi\|_{\kappa,p}=c2^{-v\varepsilon_0}\|f_\varpi\|_{\kappa,p}.$$

Finally, using Hölder's inequality, we obtain, for $x\notin \mathtt{c}^*(\varpi,2^{v_1+1}/n)$,

$$|S^\delta_{n,v}(f_\varpi)(x)|^p=\left|\int_{\{y:\mathtt{d}(\varpi,y)\le 2^{v_1}/n\}}f_\varpi(y)K^\delta_{n,v}(x,y)h^2_\kappa(y)\,\mathrm{d}\sigma(y)\right|^p$$
$$\le\left(\int_{\{y:\ \mathtt{d}(\bar y,\bar x)\ge 2^{v_1}/n\}}|f_\varpi(y)|^p|K^\delta_{n,v}(x,y)|h^2_\kappa(y)\,\mathrm{d}\sigma(y)\right)$$
$$\times\left(\int_{\{y:\ \mathtt{d}(\bar y,\bar x)\ge 2^{v_1}/n\}}|K^\delta_{n,v}(x,y)|h^2_\kappa(y)\,\mathrm{d}\sigma(y)\right)^{p-1},$$

which, together with Corollary 9.2.4, implies

$$\left(\int_{\mathbb{S}^{d-1}\setminus\mathtt{c}^*(\varpi,2^{v_1+1}/n)}|S^\delta_{n,v}(f_\varpi)(x)|^p\,h^2_\kappa(x)\,\mathrm{d}\sigma(x)\right)^{\frac1p}$$
$$\le c(2^{-v\varepsilon_0})^{1-\frac1p}\sup_{y\in\mathbb{S}^{d-1}}\left(\int_{\{x:\mathtt{d}(\bar x,\bar y)\ge 2^{v_1}/n\}}|K^\delta_{n,v}(x,y)|h^2_\kappa(x)\,\mathrm{d}\sigma(x)\right)^{\frac1p}\|f_\varpi\|_{\kappa,p}$$
$$\le c2^{-v\varepsilon_0}\|f_\varpi\|_{\kappa,p}.$$

Putting the above together, we deduce the desired estimate (9.2.7), hence Eq. (9.2.4), and complete the proof of Theorem 9.2.1. □

9.2.2 *Proof of Theorem 9.2.2*

Our main objective is to show that

$$\sup_{n\in\mathbb{N}}\|S^\delta_n(h^2_\kappa;f)\|_{\kappa,p}\le c\|f\|_{k,p}\tag{9.2.8}$$

does not hold if $1\le p\le\frac{2\sigma_\kappa+1}{\sigma_\kappa+\delta+1}$ or $p\ge\frac{2\sigma_\kappa+1}{\sigma_\kappa-\delta}$. Let

$$p_1:=\frac{2\sigma_\kappa+1}{\sigma_\kappa-\delta}\quad\text{and}\quad q_1:=\frac{p_1}{p_1-1}=\frac{2\sigma_\kappa+1}{\sigma_k+1+\delta}.$$

It is sufficient to prove that Eq. (9.2.8) does not hold for p_1, since it then follows from the Riesz–Thorin convexity theorem that Eq. (9.2.8) fails for $p_1 \le p \le \infty$, and then that Eq. (9.2.8) fails for $1 \le p \le q_1$ follows by duality.

Let $e \in \mathbb{S}^{d-1}$ be fixed. Define a linear functional $T_n^\delta : L^p(h_\kappa^2;\mathbb{S}^{d-1}) \mapsto \mathbb{R}$ by

$$T_n^\delta f := S_n^\delta(h_\kappa^2; f, e) = a_\kappa \int_{\mathbb{S}^{d-1}} f(x) K_n^\delta(h_\kappa^2; x, e) h_\kappa^2(x) \mathrm{d}\sigma(x).$$

Since this is an integral operator, a standard argument shows that

$$\|T_n^\delta\|_{\kappa,p} = \|K_n^\delta(h_\kappa^2; x, e)\|_{\kappa,q}, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

where $\|T_n^\delta\|_{\kappa,p} = \sup_{\|f\|_{\kappa,p}=1} |T_n^\delta f|$. On the other hand, by the Nikolskii inequality (5.5.1),

$$\|S_n^\delta(h_\kappa^2; f)\|_{\kappa,\infty} \le c n^{(2\sigma_\kappa+1)/p} \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,p},$$

since for $w = h_\kappa^2$, $s_w = 2\sigma_\kappa + 1$ by Eq. (5.1.8), so that if Eq. (9.2.8) holds, then we will have

$$\begin{aligned} |T_n^\delta f| = |S_n^\delta(h_\kappa^2; f, e)| &\le \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,\infty} \\ &\le c n^{(2\sigma_\kappa+1)/p} \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,p} \le c n^{(2\sigma_\kappa+1)/p} \|f\|_{\kappa,p}. \end{aligned}$$

Consequently, the above two equations show that we will have

$$\|K_n^\delta(h_\kappa^2; \cdot, e)\|_{\kappa,q} \le c n^{(2\sigma_\kappa+1)/p}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{9.2.9}$$

To complete the proof of Theorem 9.2.2, we show that Eq. (9.2.9) does not hold for $p = p_1$.

Without loss of generality, assume $\kappa_1 = \min \kappa$. We use the explicit formula (8.4.5), which shows that

$$\|K_n^\delta(h_\kappa^2; \cdot, e)\|_{\kappa,q}^q = c \int_{-1}^1 \left| k_n^\delta(v_{\sigma_\kappa,\kappa_1}; 1, t) \right|^q w_{\sigma_\kappa,\kappa_1}(t) \mathrm{d}t.$$

Consequently, the proof of Theorem 9.2.2 follows from applying Proposition 9.2.5 below, an analogue of Proposition 8.5.1, with $\sigma = \sigma_\kappa$ and $\mu = \kappa_{\min} = \kappa_1$. □

Proposition 9.2.5. *Let $v_{\sigma,\mu}$ be the generalized Gegenbauer weight function and $\sigma \ge \mu \ge 0$. Define*

$$T_{n,q}^\delta(w_{\sigma,\mu}; s) := \int_{-1}^1 |k_n^\delta(v_{\sigma,\mu}; s, t)|^q w_{\sigma,\mu}(t) \mathrm{d}t.$$

Then for $q_1 = \frac{2\sigma+1}{\sigma+1+\delta}$ *and* $p_1 = \frac{2\sigma+1}{\sigma-\delta}$,

$$T^{\delta}_{n,q_1}(v_{\sigma,\mu},1) \geq T^{\delta}_{n,q_1}(v_{\mu,\sigma},0) \geq cn^{(2\sigma+1)q_1/p_1}\log n.$$

Proof. The case $q = 1$ and $\delta = \sigma$ has already been established in Proposition 8.5.1. By considering the even functions on $\mathbb{S}^1$, we can deduce the boundedness of (C,δ) means for the generalized Gegenbauer expansions in $L^p(w_{\lambda,\mu},[-1,1])$ from Theorem 9.2.1. Hence, following the proof of Proposition 8.5.1, we can deduce that

$$T^{\delta}_{n,q_1}(w_{\sigma,\mu};1) \geq T^{\delta}_{n,q_1}(w_{\mu,\sigma};0) = cn^{(\sigma+\mu-\delta+\frac{1}{2})q_1}$$
$$\times \int_0^1 \left| \int_{-1}^1 T_n^{(\sigma+\mu+\delta+\frac{1}{2},\sigma+\mu-\frac{1}{2})}(st)(1-s^2)^{\mu-1}\mathrm{d}s \right|^{q_1} t^{2\mu}(1-t^2)^{\sigma-\frac{1}{2}}\mathrm{d}t + \mathscr{O}(1).$$

Thus we see that that stated result is a consequence of the lower bound of a double integral of the Jacobi polynomial given in the next proposition. □

Proposition 9.2.6. *Assume* $\sigma,\mu \geq 0$ *and* $0 \leq \delta \leq \sigma+\mu$. *Let* $a = \sigma+\mu+\delta$, $b = \sigma+\mu-1$, *and* $q_1 = \frac{2\sigma+1}{\sigma+1+\delta}$. *Then*

$$\int_0^1 \left| \int_{-1}^1 P_n^{(a+\frac{1}{2},b+\frac{1}{2})}(st)(1-s^2)^{\mu-1}\mathrm{d}s \right|^{q_1} t^{2\mu}(1-t^2)^{\sigma-1/2}\mathrm{d}t$$
$$\geq cn^{-(\mu+1/2)q_1}\log n. \tag{9.2.10}$$

Proof. Denote the left-hand side of Eq. (9.2.10) by I_{n,q_1}. First assume that $0 < \mu < 1$. Following the proof of Eq. (8.5.3) in the proof of Proposition 8.5.2, we can conclude that

$$I_{n,q_1} \geq cn^{-q_1/2}\int_{n^{-1}}^{\pi/4} |M_n(\phi)|^{q_1}(\sin\phi)^{2\sigma}\mathrm{d}\phi - \mathscr{O}(1)E_{n,q_1},$$

where

$$E_{n,q_1} := n^{-\frac{3}{2}q_1}\int_{n^{-1}}^{\pi/4}\left[\int_{\phi}^{\pi-\phi}\frac{(\cos^2\phi-\cos^2\theta)^{\mu-1}}{(\sin\frac{\theta}{2})^{a+1}(\cos\frac{\theta}{2})^{b+1}}\mathrm{d}\theta\right]^{q_1}(\sin\phi)^{2\sigma}\mathrm{d}\phi$$

and $M_n(\phi) = K_n(\phi) + G_n(\phi)$ as in Eq. (8.5.6) with $\lambda = \sigma$. From the estimates (8.5.10) and (8.5.8), and the fact that $q_1(\sigma+1+\delta) = 2\sigma+1$, it follows that

$$\int_{n^{-1}}^{\varepsilon} |K_n(\phi)|^{q_1}(\sin\phi)^{2\sigma}\mathrm{d}\phi \geq cn^{-\mu q_1}\int_{n^{-1}}^{\varepsilon}\phi^{2\sigma-q_1(\sigma+1+\delta)}(1+\cos(2N\phi+2\gamma))^{q_1}\mathrm{d}\phi$$
$$= cn^{-\mu q_1}\int_{n^{-1}}^{\varepsilon}\phi^{-1}(1+\cos(2N\phi+2\gamma))^{q_1}\mathrm{d}\phi \geq cn^{-\mu q_1}\log n,$$

where in the last step we used $(1+A)^{q_1} \ge 1+q_1 A$ for $A \in [-1,1]$ and the fact that $\int_{n^{-1}}^{\varepsilon} \phi^{-1} \cos(2N\phi + 2\gamma))\mathrm{d}\phi \le c$, on integration by parts once. Furthermore,

$$\int_{n^{-1}}^{\varepsilon} |G_n(\phi)|^{q_1} (\sin\phi)^{2\sigma} \mathrm{d}\phi \le cn^{-q_1} \int_{n^{-1}}^{\varepsilon} \phi^{2\sigma + q_1(\mu-\sigma-\delta-2)} \mathrm{d}\phi$$
$$= cn^{-q_1} \int_{n^{-1}}^{\varepsilon} \phi^{q_1(\mu-1)-1} \mathrm{d}\phi \le cn^{-q_1\mu}.$$

Together, these estimates yield that for $0 < \mu < 1$,

$$\int_{n^{-1}}^{\pi/4} |M_n(\phi)|^{q_1} (\sin\phi)^{2\sigma} \mathrm{d}\phi \ge cn^{-q_1\mu} \log n.$$

Moreover, the remainder E_{n,q_1} term can be estimated as follows:

$$\begin{aligned} E_{n,q_1} &\le cn^{-\frac{3}{2}q_1} \int_{n^{-1}}^{\pi/4} \left[\int_{\phi}^{\pi/4} \theta^{\mu-a-2} (\theta-\phi)^{\mu-1} \mathrm{d}\theta \right]^{q_1} \phi^{2\sigma} \mathrm{d}\phi \\ &\le cn^{-\frac{3}{2}q_1} \int_{n^{-1}}^{\pi/4} \phi^{(\mu-\sigma-\delta-2)q_1 + 2\sigma} \mathrm{d}\phi \\ &\le cn^{-\frac{3}{2}q_1 - q_1(\mu-1)} = cn^{-(\mu+\frac{1}{2})q_1}, \end{aligned}$$

where in the second step, we divided the inner integral into two parts, over $[\phi, 2\phi]$ and over $[2\phi, \pi/2]$, respectively, to derive the stated estimate.

Putting these two terms together, we conclude the proof for the case $0 < \mu < 1$. The case $\mu = 1$ can be derived similarly on integrating the inner integral in Eq. (9.2.10) by parts. The case $\mu > 1$ reduces to the case $0 < \mu < 1$ on integration by parts $\lfloor \mu \rfloor$ times as in the proof of Proposition 8.5.2. □

9.3 Local Estimates of the Projection Operators

In this section, we establish the local estimate of the projection operator stated in Theorem 9.1.2.

Throughout this section, we shall fix the spherical cap $c(\varpi, \theta)$. Without loss of generality, we may assume $\varpi = (\varpi_1, \ldots, \varpi_d)$ satisfying $|\varpi_k| \ge 4\theta$ for $1 \le k \le v$ and $|\varpi_k| < 4\theta$ for $v < k \le d$. Accordingly, we define

$$\gamma = \gamma_{\varpi} := \begin{cases} 0, & \text{if } v = d, \\ \sum_{i=v+1}^{d} \kappa_i, & \text{if } v < d. \end{cases} \tag{9.3.1}$$

Since $\theta \in (0, 1/(8d)]$ and $\varpi \in \mathbb{S}^{d-1}$, it follows that

$$0 \le \gamma \le |\kappa| - \min_{1\le i\le d} \kappa_i = \sigma_\kappa - \frac{d-2}{2}.$$

The proof of Theorem 9.1.2 consists of two cases, one for $\gamma < \sigma_\kappa - \frac{d-2}{2}$ and the other for $\gamma = \sigma_\kappa - \frac{d-2}{2}$; they require different methods.

9.3.1 *Proof of Theorem 9.1.2, Case I: $\gamma < \sigma_\kappa - \frac{d-2}{2}$*

The proof is long and will be divided into several subsections.

9.3.1.1 Decomposition of the Projection Operator

Recall $\lambda_\kappa = \frac{d-2}{2} + |\kappa|$. Let $\xi_0 \in C^\infty[0,\infty)$ be such that $\chi_{[0,1/2]}(t) \le \xi_0(t) \le \chi_{[0,1]}(t)$, and define $\xi_1(t) := \xi_0(t/4) - \xi_0(t)$. Evidently, $\operatorname{supp}\xi_1 \subset (1/2, 4)$ and

$$\xi_0(t) + \sum_{j=1}^{\infty} \xi_1(4^{-j+1}t) = 1$$

whenever $t \in [0,\infty)$. Define, for $u \in [-1,1]$,

$$C_{n,0}(u) := \frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(u)\xi_0\left(n^2(1-u^2)\right)$$

$$C_{n,j}(u) := \frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(u)\xi_1\left(\frac{n^2(1-u^2)}{4^{j-1}}\right), \quad j = 1, 2, \ldots, N_n,$$

where $N_n := \lfloor \log_2 n \rfloor + 2$. By Eq. (7.4.4), $\operatorname{proj}_n(h_\kappa^2; f)$ can be decomposed as

$$\operatorname{proj}_n(h_\kappa^2; f) = \sum_{j=0}^{N_n} Y_{n,j} f, \quad \text{where} \quad Y_{n,j} f := f *_\kappa C_{n,j}. \tag{9.3.2}$$

By the definition of convolution, the kernel of $Y_{n,j}$ is $V_\kappa[C_{n,j}(\langle x, \cdot\rangle)](y)$.

9.3.1.2 Estimates of the Kernels $V_\kappa\left[C_{n,j}\langle x,\cdot\rangle\right](y)$ and L^∞ Estimates

Recall the definition of the class $\mathscr{S}_n^v(\rho, r, \mu)$ in Definition 8.2.2 and that $n^{-\alpha} P_n^{(\alpha,\beta)} \in \mathscr{S}_n^v(0, 1, \alpha)$ for all $v \in \mathbb{N}_0$.

Lemma 9.3.1. *Assume that* $\delta = (\delta_1,\dots,\delta_m) \in \mathbb{R}^m$ *satisfies* $\min_{1\le j\le m}\delta_j > 0$ *and* $\mu \in \mathbb{R}$. *Let* $F \in \mathscr{S}_n^v(\mu)$ *with* v *an integer satisfying* $v \ge 2m + \sum_{j=1}^m \delta_j + |\mu|$. *Let* ξ *be a* C^∞ *function, supported in* $[-8,8]$ *and constant in a neighborhood of* 0. *For* $\rho \in (n^{-1},4]$, *define*

$$G(u) := F(u)\xi\left(\frac{1-u^2}{\rho^2}\right), \qquad u \in [-1,1].$$

Then for $s \in [-1,1]$ *and* $a = (a_1,\dots,a_m) \in [-1,1]^m$ *satisfying* $\sum_{j=1}^m |a_j| + |s| \le 1$,

$$\left|\int_{[-1,1]^m} G\Big(\sum_{j=1}^m a_j t_j + s\Big)\prod_{j=1}^m (1-t_j^2)^{\delta_j-1}(1+t_j)\,\mathrm{d}t_j\right| \le cn^{-\frac12-|\delta|}\rho^{|\delta|-\mu-\frac12}\prod_{j=1}^m (|a_j|+n^{-1}\rho)^{-\delta_j}, \tag{9.3.3}$$

where $|\delta| = \sum_{j=1}^m \delta_j$.

Proof. Without loss of generality, we may assume that $|a_j| \ge n^{-1}\rho$ for $1 \le j \le m$, since otherwise, we can modify the proof by replacing s with $s + \sum_{\{j:|a_j|<n^{-1}\rho\}} a_j t_j$.

Let $\eta_0 \in C^\infty(\mathbb{R})$ be such that $\eta_0(t) = 1$ for $|t| \le \frac12$ and $\eta_0(t) = 0$ for $|t| \ge 1$, and let $\eta_1(t) = 1 - \eta_0(t)$. Set

$$B_j := \frac{\rho}{n|a_j|}, \quad j = 1,\dots,m.$$

Given $\varepsilon := (\varepsilon_1,\dots,\varepsilon_m) \in \{0,1\}^m$, we define $\psi_\varepsilon : [-1,1]^m \to \mathbb{R}$ by

$$\psi_\varepsilon(t) := \xi\left(\frac{1-(\sum_{j=1}^m a_j t_j + s)^2}{\rho^2}\right)\prod_{j=1}^m \eta_{\varepsilon_j}\Big(\frac{1-t_j^2}{B_j}\Big)(1+t_j)(1-t_j^2)^{\delta_j-1},$$

where $t = (t_1,\dots,t_m)$. We then split the integral in Eq. (9.3.3) into a finite sum:

$$\sum_{\varepsilon\in\{0,1\}^m}\int_{[-1,1]^m} F\Big(\sum_{j=1}^m a_j t_j + s\Big)\psi_\varepsilon(t)\,\mathrm{d}t =: \sum_{\varepsilon\in\{0,1\}^m} J_\varepsilon.$$

Thus, it is sufficient to prove that each term J_ε in the above sum satisfies the desired inequality. By symmetry and Fubini's theorem, we need only consider the case in which $\varepsilon_1 = \dots = \varepsilon_{m_1} = 0$ and $\varepsilon_{m_1+1} = \dots = \varepsilon_m = 1$ for some $0 \le m_1 \le m$.

Let m_1 and ε be fixed as in the last line. Fix $(t_1,\dots,t_{m_1}) \in [-1,1]^{m_1}$ momentarily, and write $s_1 = \sum_{j=1}^{m_1} a_j t_j + s$. Define

$$\phi(t) := \xi\left(\frac{1-(\sum_{j=1}^m a_j t_j + s)^2}{\rho^2}\right)\prod_{j=m_1+1}^m \eta_1\left(\frac{1-t_j^2}{B_j}\right)(1+t_j)(1-t_j^2)^{\delta_j-1}.$$

Since the support set of each $\eta_1\left(\frac{1-t_j^2}{B_j}\right)$ is a subset of $\{t_j : |t_j| \le 1-\frac{1}{4}B_j\}$, we can use integration by parts $|\mathbf{l}| = \sum_{j=m_1+1}^m \ell_j$ times to obtain

$$\begin{aligned}
&\left|\int_{[-1,1]^{m-m_1}} F\Big(\sum_{j=m_1+1}^m a_j t_j + s_1\Big)\phi(t)\,\mathrm{d}t\right| \\
&= \prod_{j=m_1+1}^m |a_j|^{-\ell_j}\left|\int_{[-1,1]^{m-m_1}} F_{|\mathbf{l}|}\Big(\sum_{j=m_1+1}^m a_j t_j + s_1\Big)\frac{\partial^{|\mathbf{l}|}\phi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1}\cdots\partial^{\ell_m} t_m}\,\mathrm{d}t\right| \\
&\le \prod_{j=m_1+1}^m |a_j|^{-\ell_j}\int_{[-1,1]^{m-m_1}}\left|F_{|\mathbf{l}|}\Big(\sum_{j=m_1+1}^m a_j t_j + s_1\Big)\right|\left|\frac{\partial^{|\mathbf{l}|}\phi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1}\cdots\partial^{\ell_m} t_m}\right|\mathrm{d}t,
\end{aligned}$$

where $F_{|\mathbf{l}|}^{(|\mathbf{l}|)} = F$ is as in Definition 8.2.2, and $\mathbf{l} = (\ell_{m_1+1},\ldots,\ell_m) \in \mathbb{N}^{m-m_1}$ satisfies $\ell_j > \delta_j$ and $|\mathbf{l}| \ge \mu + \frac{1}{2}$. Since ξ is supported in $(-8,8)$, the integrand of the last integral is zero unless

$$\begin{aligned}
8\rho^2 &\ge 1 - \Big|\sum_{k=m_1+1}^m a_k t_k + s_1\Big| \\
&\ge 1 - \sum_{k=m_1+1}^m |a_k| - |s_1| + (1-|t_j|)|a_j| \ge |a_j|(1-|t_j|),
\end{aligned} \tag{9.3.4}$$

for all $m_1+1 \le j \le m$; that is, $\frac{|a_j|}{\rho^2} \le 8(1-|t_j|)^{-1}$ for $j = m_1+1,\ldots,m$. Also, recall that ξ is constant near 0. Hence, taking the kth partial derivative with respect to t_j, the ξ part of ϕ is bounded by $c(1-t_j)^{-k}$, and likewise for the same derivative of the η_1 part of ϕ, since $B_j^{-1} \le (1-t_j^2)^{-1}$ in the support of η_1'. Consequently, by Leibniz's rule, we conclude that

$$\left|\frac{\partial^{|\mathbf{l}|}\phi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1}\cdots\partial^{\ell_m} t_m}\right| \le c\prod_{j=m_1+1}^m (1-|t_j|)^{\delta_j-\ell_j-1}$$

in the support of the integrand. Next, since $\rho \ge n^{-1}$ and $|\mathbf{l}| \ge \mu + \frac{1}{2}$, Eq. (9.3.4) together with Eq. (8.2.2) implies

$$\left|F_{|\mathbf{l}|}\Big(\sum_{k=m_1+1}^m a_k t_k + s_1\Big)\right| \le c n^{-\frac{1}{2}-|\mathbf{l}|}\rho^{-\mu-\frac{1}{2}+|\mathbf{l}|}.$$

It follows that

$$\int_{[-1,1]^{m-m_1}} \left|F_{|\mathbf{l}|}\Big(\sum_{j=m_1+1}^{m} a_j t_j + s_1\Big)\right| \left|\frac{\partial^{|\mathbf{l}|}\phi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1}\cdots\partial^{\ell_m} t_m}\right| \mathrm{d}t$$

$$\le cn^{-\frac{1}{2}-|\mathbf{l}|}\rho^{-\mu-\frac{1}{2}+|\mathbf{l}|}\prod_{j=m_1+1}^{m}\int_0^{1-\frac{B_j}{4}}(1-t_j)^{\delta_j-\ell_j-1}\,\mathrm{d}t_j$$

$$\le cn^{-\frac{1}{2}-|\mathbf{l}|}\rho^{-\mu-\frac{1}{2}+|\mathbf{l}|}\prod_{j=m_1+1}^{m} B_j^{\delta_j-\ell_j}$$

$$\le cn^{-\frac{1}{2}-\alpha}\rho^{\alpha-\mu-\frac{1}{2}}\prod_{j=m_1+1}^{m}|a_j|^{\ell_j-\delta_j},$$

where $\alpha=\sum_{j=m_1+1}^{m}\delta_j$. Thus, since

$$\psi_\varepsilon(t)=\phi(t)\prod_{j=1}^{m_1}\eta_0\Big(\frac{1-t_j^2}{B_j}\Big)(1+t_j)(1-t_j^2)^{\delta_j-1},$$

and $\eta_0\left(\frac{1-t_j^2}{B_j}\right)$ is supported in $\{t_j:\ \ 1-B_j\le|t_j|\le 1\}$, integrating with respect to $t_1,\ldots,t_{m_1}$ over $[-1,1]^{m_1}$ yields

$$J_\varepsilon \le \int_{[-1,1]^{m_1}}\left|\int_{[-1,1]^{m-m_1}} F\Big(\sum_{j=1}^{m} a_j t_j + s\Big)\phi(t)\,\mathrm{d}t_{m_1+1}\cdots\mathrm{d}t_m\right|$$

$$\times\prod_{j=1}^{m_1}\eta_0\left(\frac{1-t_j^2}{B_j}\right)(1+t_j)(1-t_j^2)^{\delta_j-1}\mathrm{d}t_j$$

$$\le cn^{-\frac{1}{2}-\alpha}\rho^{\alpha-\mu-\frac{1}{2}}\prod_{j=m_1+1}^{m}|a_j|^{-\delta_j}\prod_{j=1}^{m_1}\int_{1-B_j\le|t_j|\le 1}(1-|t_j|)^{\delta_j-1}\,\mathrm{d}t_j$$

$$\le cn^{-\frac{1}{2}-|\delta|}\rho^{|\delta|-\mu-\frac{1}{2}}\prod_{j=1}^{m}|a_j|^{-\delta_j},$$

where we have used $|a_j|^{\ell_j}\le 1$ in the second step. This completes the proof. □

Using the relation between the Gegenbauer and the Jacobi polynomials yields

$$C_{n,j}(u)=a_n P_n^{(\lambda_\kappa-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(u)\,\xi\left(\frac{1-u^2}{(2^{j-1}/n)^2}\right),$$

where $\xi=\xi_1$ or ξ_0, and $|a_n|\le cn^{\lambda_\kappa+\frac{1}{2}}$. Hence, using the fact that $c_{v,\kappa}P_n^{(\lambda_\kappa-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}\in\mathscr{S}_n^{v}(\lambda_\kappa-\frac{1}{2})$ for all $v\in\mathbb{N}$, Lemma 9.3.1 has the following corollary.

Corollary 9.3.2. *For $x,y \in \mathbb{S}^{d-1}$ and $j = 1,2,\dots,N_n$,*

$$\left|V_\kappa\left[C_{n,j}(\langle x,\cdot\rangle)\right](y)\right| \le cn^{d-2}2^{-j(d-2)/2}\prod_{i=1}^{d}\left(|x_i y_i| + 2^j n^{-2}\right)^{-\kappa_i}.$$

Recall that $c(\varpi,\theta)$ is a fixed spherical cap, θ is in $[n^{-1},\pi]$, and $\gamma = \gamma_\varpi$ is defined in Eq. (9.3.1). We are now in a position to prove the following L^∞ estimate:

Lemma 9.3.3. *If f is supported in $c(\varpi,\theta)$, then*

$$\sup_{x\in c(\varpi,\theta)}\left|Y_{n,j}(f)(x)\right| \le cn^{d-2+2\gamma}2^{-j(\frac{d-2}{2}+\gamma)}\theta^{2\gamma+d-1}\left[\int_{c(\varpi,\theta)} h_\kappa^2(x)\,\mathrm{d}\sigma(x)\right]^{-1}\|f\|_{\kappa,1}.$$

Proof. Note that if $x \in c(\varpi,\theta)$, then $|x_i - \varpi_i| \le \|x-\varpi\| \le d(x,\varpi) \le \theta$, so that $\frac{3}{4}|\varpi_i| \le |x_i| \le \frac{5}{4}|\varpi_i|$ for $1 \le i \le v$, and $|x_i| \le 5\theta$ for $v+1 \le i \le d$. It follows from Corollary 9.3.2 that for every $x,y \in c(\varpi,\theta)$,

$$\begin{aligned}
\left|V_\kappa\left[C_{n,j}(\langle x,\cdot\rangle)\right](y)\right| &\le cn^{d-2}2^{-j(d-2)/2}\prod_{i=1}^{v}|\varpi_i|^{-2\kappa_i}\prod_{i=v+1}^{d}n^{2\kappa_i}2^{-j\kappa_i} \\
&\le cn^{d-2}2^{-j(\frac{d-2}{2}+\gamma)}(n\theta)^{2\gamma}\prod_{i=1}^{d}(|\varpi_i|+\theta)^{-2\kappa_i} \\
&\le cn^{d-2}2^{-j(\frac{d-2}{2}+\gamma)}(n\theta)^{2\gamma}\theta^{d-1}\left[\int_{c(\varpi,\theta)} h_\kappa^2(z)\,\mathrm{d}\sigma(z)\right]^{-1},
\end{aligned}$$

where the last step follows from the relation (5.1.9). This implies that

$$\begin{aligned}
\sup_{x\in c(\varpi,\theta)}|Y_{n,j}(f)(x)| &\le \sup_{x\in c(\varpi,\theta)}\int_{c(\varpi,\theta)}|f(y)|\left|V_\kappa\left[C_{n,j}\langle x,\cdot\rangle\right](y)\right|h_\kappa^2(y)\,\mathrm{d}\sigma(y) \\
&\le cn^{d-2+2\gamma}2^{-j\left(\frac{d-2}{2}+\gamma\right)}\theta^{2\gamma+d-1}\left[\int_{c(\varpi,\theta)} h_\kappa^2(x)\,\mathrm{d}\sigma(x)\right]^{-1}\|f\|_{\kappa,1},
\end{aligned}$$

which is the desired inequality. □

9.3.1.3 L^2 Estimates

We prove the following estimate:

Lemma 9.3.4. *For every $f \in L^2(h_\kappa^2,\mathbb{S}^{d-1})$,*

$$\|Y_{n,j}(f)\|_{\kappa,2} \le cn^{-1}2^j\|f\|_{\kappa,2}.$$

Proof. For simplicity, we shall write $\xi_j = \xi_1$ for $j \geq 1$. Also let $\lambda = \lambda_\kappa$ in this proof. From Eq. (7.4.5) and the definition of $Y_{n,j}$ in Eq. (9.3.2), it follows that each $Y_{n,j}$ is a multiplier operator,

$$Y_{n,j}(f) = \sum_{k=0}^{\infty} m_{n,j}(k)\, \mathrm{proj}_k(h_\kappa^2; f),$$

where equality is understood in a distributional sense, and

$$m_{n,j}(k) := c_{n,k} \int_0^\pi C_n^\lambda(\cos t) C_k^\lambda(\cos t) \xi_j \left(\frac{n^2 \sin^2 t}{4^{j-1}} \right) \sin^{2\lambda} t \,\mathrm{d}t$$

with $|c_{n,k}| \leq cnk^{-2\lambda+1}$. Hence, it is enough to prove

$$\sup_k |m_{n,j}(k)| \leq c\, n^{-1} 2^j. \tag{9.3.5}$$

If $k \geq \frac{n}{4}$, then using the fact that $|(\sin\theta)^\lambda C_n^\lambda(\cos\theta)| \leq cn^{\lambda-1}$, a straightforward computation gives

$$\begin{aligned} |m_{n,j}(k)| &\leq |c_{n,k}| \int_0^\pi \left| C_n^\lambda(\cos t) C_k^\lambda(\cos t) \xi_j \left(\frac{n^2 \sin^2 t}{4^{j-1}} \right) \right| \sin^{2\lambda} t \,\mathrm{d}t \\ &\leq c \int_0^\pi \left| \xi_j \left(\frac{n^2 \sin^2 t}{4^{j-1}} \right) \right| \mathrm{d}t \leq c \frac{2^j}{n}, \end{aligned}$$

where the last step follows easily using the support of ξ_j.

For $k \leq \frac{n}{4}$, we shall use the following formula (cf. [5, p. 319, Theorem 6.8.2]):

$$C_k^\lambda(t) C_n^\lambda(t) = \sum_{i=0}^{\min\{k,n\}} a(i,k,n) C_{k+n-2i}^\lambda(t), \tag{9.3.6}$$

where

$$a(i,k,n) := \frac{(k+n+\lambda-2i)(\lambda)_i(\lambda)_{k-i}(\lambda)_{n-i}(2\lambda)_{k+n-i}}{(k+n+\lambda-i)i!(k-i)!(n-i)!(\lambda)_{k+n-i}} \frac{(k+n-2i)!}{(2\lambda)_{k+n-2i}}.$$

For $k \leq n/4$, it is easy to see that

$$\begin{aligned} |a(i,k,n)| &\sim \left(\frac{(i+1)(\min\{k,n\}-i+1)(k+n-i+1)}{k+n-2i+1} \right)^{\lambda-1} \\ &\sim (i+1)^{\lambda-1}(k-i+1)^{\lambda-1}. \end{aligned} \tag{9.3.7}$$

Consequently, it follows that for $k \le n/4$,

$$
\begin{aligned}
|m_{n,j}(k)| &\le cnk^{-2\lambda+1}\sum_{i=0}^{k}(i+1)^{\lambda-1} \\
&\quad \times (k-i+1)^{\lambda-1}\left|\int_0^{\pi} C_{k+n-2i}^{\lambda}(\cos t)\xi_j\left(\frac{n^2\sin^2 t}{4^{j-1}}\right)\sin^{2\lambda} t\,\mathrm{d}t\right| \\
&\le cn^{\lambda+\frac{1}{2}}\max_{3n/4\le m\le 5n/4}\left|\int_{-1}^{1} P_m^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(s)\xi_j\left(\frac{n^2(1-s^2)}{4^{j-1}}\right)(1-s^2)^{\lambda-\frac{1}{2}}\,\mathrm{d}s\right|.
\end{aligned}
$$

Then using the estimate (B.1.7), we obtain

$$
m_{n,0}(k) \le cn^{2\lambda}\int_{1-|s|\le cn^{-2}}(1-|s|)^{\lambda-\frac{1}{2}}\,\mathrm{d}s \le cn^{-1}.
$$

If $j \ge 1$, then for all $\ell \in \mathbb{N}$, it follows that

$$
\left|\frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell}\left(\xi_1\left(\frac{n^2(1-s^2)}{4^{j-1}}\right)(1-s^2)^{\lambda-\frac{1}{2}}\right)\right| \le c\left(\frac{2^j}{n}\right)^{2\lambda-1-2\ell},
$$

since $1-s^2 \sim (\frac{2^j}{n})^2$ in the support of ξ_1'; consequently, we obtain by integration by parts the properties of the Jacobi polynomials (B.1.5) and (B.1.7) that

$$
\begin{aligned}
&m_{n,j}(k) \le cn^{\lambda+\frac{1}{2}-\ell} \\
&\quad\times \max_{3n/4\le m\le 5n/4}\left|\int_{-1}^{1} P_{m+\ell}^{(\lambda-\frac{1}{2}-\ell,\lambda-\frac{1}{2}-\ell)}(s)\frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell}\left(\xi_1\left(\frac{n^2(1-s^2)}{4^{j-1}}\right)(1-s^2)^{\lambda-\frac{1}{2}}\right)\mathrm{d}s\right| \\
&\quad\le c2^{j(\lambda-\ell)}2^j n^{-1} \le c2^j n^{-1}
\end{aligned}
$$

on choosing $\ell \ge \lambda$ Thus in both cases, we get the desired estimate. □

9.3.1.4 Proof of Theorem 9.1.2, Case I: $\gamma < \sigma_\kappa - \frac{d-2}{2}$

Recall $\nu = \frac{2+2\sigma_\kappa}{2+\sigma_\kappa}$. We set, in this subsection,

$$
A := \int_{c(\varpi,\theta)} h_\kappa^2(y)\,\mathrm{d}\sigma(y).
$$

Recall the decomposition (9.3.2). For a generic f, we set

$$
T_{n,j}f := Y_{n,j}(f\chi_{c(\varpi,\theta)})\chi_{c(\varpi,\theta)}, \qquad 0 \le j \le N_n.
$$

Clearly, if f is supported in $c(\varpi,\theta)$ and $x \in c(\varpi,\theta)$, then $T_{n,j}f(x) = Y_{n,j}f(x)$. Using Lemmas 9.3.3 and 9.3.4, we have

$$\|T_{n,j}f\|_\infty \le cn^{2\gamma+d-2}2^{-j(\frac{d-2}{2}+\gamma)}\theta^{2\gamma+d-1}A^{-1}\|f\chi_{c(\varpi,\theta)}\|_{\kappa,1},$$
$$\|T_{n,j}f\|_{\kappa,2} \le cn^{-1}2^j\|f\chi_{c(\varpi,\theta)}\|_{\kappa,2}. \tag{9.3.8}$$

Hence, by the Riesz–Thorin convexity theorem, we obtain

$$\|T_{n,j}f\|_{\kappa,v'} \le cn^{-1}2^{j\left(1-(\frac{d}{2}+\gamma)\frac{1}{\sigma_\kappa+1}\right)}(n\theta)^{\frac{2\gamma+d-1}{\sigma_\kappa+1}}A^{1-\frac{2}{v}}\|f\|_{\kappa,v}. \tag{9.3.9}$$

On the other hand, using Eq. (9.3.8), Hölder's inequality, and Eq. (5.1.9), we obtain

$$\begin{aligned}\|T_{n,j}f\|_{\kappa,v'} &\le \|T_{n,j}f\|_\infty A^{1-\frac{1}{v}} \le cn^{2\gamma+d-2}2^{-j\left(\frac{d-2}{2}+\gamma\right)}\theta^{2\gamma+d-1}A^{-\frac{1}{v}}\|f\chi_{c(\varpi,\theta)}\|_{\kappa,1}\\ &\le cn^{-1}2^{-j\left(\frac{d-2}{2}+\gamma\right)}(n\theta)^{2\gamma+d-1}A^{1-\frac{2}{v}}\|f\|_{\kappa,v}.\end{aligned} \tag{9.3.10}$$

Now assume that f is supported in $c(\varpi,\theta)$ and $\frac{2^{j_0-1}}{n} \le \theta \le \frac{2^{j_0}}{n}$ for some $1 \le j_0 \le N_n$. Using Eq. (9.3.2) and Minkowski's inequality, we have

$$\left\|\mathrm{proj}_n(h_\kappa^2;f)\chi_{c(\varpi,\theta)}\right\|_{\kappa,v'} \le \sum_{j=0}^{2j_0}\|T_{n,j}f\|_{\kappa,v'} + \sum_{j=2j_0+1}^{N_n}\|T_{n,j}f\|_{\kappa,v'} =: \Sigma_1 + \Sigma_2.$$

For the first sum Σ_1, we use Eq. (9.3.9) to obtain

$$\begin{aligned}\Sigma_1 &\le cn^{-1}(n\theta)^{\frac{2\gamma+d-1}{\sigma_\kappa+1}}A^{1-\frac{2}{v}}\|f\|_{\kappa,v}\sum_{j=0}^{2j_0}2^{j\left(1-(\frac{d}{2}+\gamma)\frac{1}{\sigma_\kappa+1}\right)}\\ &\le cn^{\frac{\sigma_\kappa}{1+\sigma_\kappa}}\theta^{\frac{2\sigma_\kappa+1}{\sigma_\kappa+1}}A^{1-\frac{2}{v}}\|f\|_{\kappa,v},\end{aligned}$$

since $\gamma < \sigma_\kappa - \frac{d-2}{2}$ readily implies that $1-(\frac{d}{2}+\gamma)\frac{1}{\sigma_\kappa+1} > 0$. For the second sum Σ_2, we use Eq. (9.3.10) to obtain

$$\begin{aligned}\Sigma_2 &\le cn^{-1}(n\theta)^{2\gamma+d-1}A^{1-\frac{2}{v}}\|f\|_{\kappa,v}\sum_{j=2j_0+1}^{\infty}2^{-j(\frac{d-2}{2}+\gamma)}\\ &\le cn^{-1}(n\theta)A^{1-\frac{2}{v}}\|f\|_{\kappa,v} \le cn^{-1}(n\theta)^{\frac{2\sigma_\kappa+1}{\sigma_\kappa+1}}A^{1-\frac{2}{v}}\|f\|_{\kappa,v}\\ &= cn^{\frac{\sigma_\kappa}{1+\sigma_\kappa}}\theta^{\frac{2\sigma_\kappa+1}{\sigma_\kappa+1}}A^{1-\frac{2}{v}}\|f\|_{\kappa,v},\end{aligned}$$

where in the third inequality we have used the fact that $n\theta \ge 1$.

Putting the above together proves Theorem 9.1.2 in the case $\gamma < \sigma_\kappa - \frac{d-2}{2}$. □

9.3.2 Proof of Theorem 9.1.2, Case II: $\gamma = \sigma_\kappa - \frac{d-2}{2}$

Recall that $|\varpi_j| \ge 4\theta$ for $1 \le j \le v$, $|\varpi_j| < 4\theta$ for $v+1 \le j \le d$, and $\gamma = \gamma_\varpi = \sum_{j=v+1}^{d} \kappa_j$. In this case, either $v = 1$ and $|\varpi_1| = \max_{1\le j\le d} |\varpi_j| \ge \frac{1}{\sqrt{d}}$, or $v \ge 2$ and $\kappa_1 = \cdots = \kappa_v = 0$. Therefore, by Eq. (5.1.9), we have

$$\int_{c(\varpi,\theta)} h_\kappa^2(x)\,\mathrm{d}\omega(x) \sim \theta^{d-1}\left(\prod_{j=1}^{v} |\varpi_j|^{2\kappa_j}\right)\theta^{2\gamma} \sim \theta^{2\sigma_\kappa+1}.$$

Hence, Theorem 9.1.2 in this case is equivalent to the following proposition.

Proposition 9.3.5. *Let f be supported in $c(\varpi,\theta)$ with $\theta \in (n^{-1}, 1/(8d)]$ and let $\nu := \frac{2\sigma_\kappa+2}{\sigma_\kappa+2}$ and $\nu' := \frac{\nu}{\nu-1}$. Then*

$$\left\|\mathrm{proj}_n(h_\kappa^2; f)\chi_{c(\varpi,\theta)}\right\|_{\kappa,\nu'} \le c n^{\frac{\sigma_\kappa}{1+\sigma_\kappa}} \|f\|_{\kappa,\nu}.$$

To prove Proposition 9.3.5, we use the method of analytic interpolation [159, p. 205]. For $z \in \mathbb{C}$, define

$$\mathscr{P}_n^z f(x) := (f *_\kappa G_n^z)(x) = a_\kappa \int_{\mathbb{S}^{d-1}} f(y) V_\kappa\left[G_n^z(\langle x,\cdot\rangle)\right](y) h_\kappa^2(y)\,\mathrm{d}\sigma(y) \quad (9.3.11)$$

for $x \in \mathbb{S}^{d-1}$, where

$$G_n^z(t) = (\sigma_\kappa+1)(1-z)\frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(t)(1-t^2+n^{-2})^{\frac{\sigma_\kappa-(\sigma_\kappa+1)z}{2}}. \quad (9.3.12)$$

From Eq. (7.4.4), it readily follows that

$$\mathscr{P}_n^{\frac{\sigma_\kappa}{1+\sigma_\kappa}} f = \mathrm{proj}_n(h_\kappa^2; f).$$

For the rest of this subsection, we shall use c_τ to denote a general constant satisfying $|c_\tau| \le c(1+|\tau|)^\ell$ for some inessential positive number ℓ.

9.3.2.1 Estimate for $z = 1 + \mathrm{i}\tau$

Lemma 9.3.6. *For $\tau \in \mathbb{R}$,*

$$\|\mathscr{P}_n^{1+\mathrm{i}\tau} f\|_{\kappa,2} \le c_\tau \|f\|_{\kappa,2}.$$

Proof. From Eqs. (9.3.11), (9.3.12), and (7.4.5), it follows that

$$\mathrm{proj}_k(h_\kappa^2; \mathscr{P}_n^{1+\mathrm{i}\tau} f) = J_n(k)\,\mathrm{proj}_k(h_\kappa^2; f), \qquad k = 0,1,\ldots,$$

where

$$J_n(k) := \mathscr{O}(1) n k^{-2\lambda_\kappa+1} \tau' \int_0^\pi C_k^{\lambda_\kappa}(\cos t) C_n^{\lambda_\kappa}(\cos t)(\sin^2 t + n^{-2})^{-\frac{1}{2}+i\tau'}(\sin t)^{2\lambda_\kappa}\,\mathrm{d}t$$

and $\tau' = -\frac{\sigma_\kappa+1}{2}\tau$. Therefore, it is sufficient to prove

$$|J_n(k)| \le c_\tau, \qquad \forall k,n \in \mathbb{N}. \tag{9.3.13}$$

For $k < \frac{n}{4}$, Eq. (9.3.13) can be established as in the proof of Lemma 9.3.4. In fact, using Eqs. (9.3.6) and (9.3.7), we obtain

$$\begin{aligned}|J_n(k)| &\le c|\tau| n^{\lambda_\kappa+\frac{1}{2}} \\ &\quad \times \max_{3n/4\le m\le 5n/4}\left|\int_{-1}^{1} P_m^{(\lambda_\kappa-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(s)(1-s^2+n^{-2})^{-\frac{1}{2}+i\tau'}(1-s^2)^{\lambda_\kappa-\frac{1}{2}}\,\mathrm{d}s\right|,\end{aligned}$$

which is dominated by

$$\begin{aligned}&c_\tau + c_\tau n^{\lambda_\kappa+\frac{1}{2}-\ell} \max_{3n/4\le m\le 5n/4}\int_{-1+n^{-2}}^{1-n^{-2}}\Big| P_{m+\ell}^{(\lambda_\kappa-\frac{1}{2}-\ell,\lambda_\kappa-\frac{1}{2}-\ell)}(s) \\ &\quad \times \frac{d^\ell}{\mathrm{d}s^\ell}\left((1-s^2+n^{-2})^{-\frac{1}{2}+i\tau'}(1-s^2)^{\lambda_\kappa-\frac{1}{2}}\right)\Big|\,\mathrm{d}s \le c'_\tau,\end{aligned}$$

using integration by parts $\ell > \lambda_\kappa$ times. This proves Eq. (9.3.13) for $k < \frac{n}{4}$.

For $k \ge \frac{n}{4}$, Eq. (9.3.13) is established as follows. Since $P_j^{(\alpha,\beta)}(-t) = (-1)^j P_j^{(\alpha,\beta)}(t)$ and $C_j^\lambda(t) = \mathscr{O}(1) j^{\lambda-\frac{1}{2}} P_j^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(t)$, we can write

$$\begin{aligned}J_n(k) &= \mathscr{O}(1) k^{-\lambda_\kappa+\frac{1}{2}} n^{\lambda_\kappa+\frac{1}{2}} \tau' \left[\int_0^{4n^{-1}} + \int_{4n^{-1}}^{\frac{\pi}{2}}\right] P_k^{(\lambda_\kappa-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(\cos t) \\ &\quad \times P_n^{(\lambda_\kappa-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(\cos t)(\sin^2 t + n^{-2})^{-\frac{1}{2}+i\tau'}(\sin t)^{2\lambda_\kappa}\,\mathrm{d}t \\ &=: J_{n,1}(k) + J_{n,2}(k).\end{aligned}$$

Since $|P_j^{(\alpha,\alpha)}(t)| \le c j^\alpha$, a straightforward calculation shows that $|J_{n,1}(k)| \le c_\tau$. To estimate $J_{n,2}(k)$, we need the asymptotics in Lemma B.1.1, which give

$$P_j^{(\alpha,\beta)}(\cos t) = \pi^{-\frac{1}{2}} j^{-\frac{1}{2}} \left(\sin\frac{t}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{t}{2}\right)^{-\beta-\frac{1}{2}} \left[\cos(N_j t + \tau_\alpha) + \mathscr{O}(1)(j\sin t)^{-1}\right]$$

for $j^{-1} \le t \le \pi - j^{-1}$, where $N_j = j + \frac{\alpha+\beta+1}{2}$ and $\tau_\alpha = -\frac{\pi}{2}(\alpha+\frac{1}{2})$. Applying this asymptotic formula with $\alpha = \beta = \lambda_\kappa - 1/2$, we obtain, for $k \ge \frac{n}{4}$ and $4n^{-1} \le t \le \frac{\pi}{2}$,

$$k^{-\lambda_\kappa+\frac{1}{2}} n^{\lambda_\kappa+\frac{1}{2}} P_k^{(\lambda_\kappa-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(\cos t) P_n^{(\lambda_\kappa-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(\cos t)(\sin t)^{2\lambda_\kappa}$$
$$= \mathscr{O}(1)\Big[\cos\big((k-n)t\big) + \cos\big((k+n+2\lambda_\kappa)t - \lambda_\kappa\pi\big)\Big] + \mathscr{O}\left(\frac{1}{nt}\right),$$

using the cosine addition formula. Also, note that

$$\left(\frac{1}{\sin^2 t + n^{-2}}\right)^{\frac{1}{2}-i\tau'} = t^{-1+2i\tau'} + \mathscr{O}(t) + \mathscr{O}\left(n^{-2}t^{-3}\right), \quad 4n^{-1} \le t \le \frac{\pi}{2}.$$

It follows that

$$|J_{n,2}(k)| \le c_\tau + c_\tau \sup_{\ell\in\mathbb{R}} \left|2\tau' \int_{4n^{-1}}^{\frac{\pi}{2}} t^{-1+2i\tau'} e^{i\ell t}\,\mathrm{d}t\right|$$
$$\le c_\tau + c_\tau \sup_{a<b} \left|\int_a^b e^{it}\,\mathrm{d}t^{2i\tau'}\right| \le c_\tau.$$

This proves the desired inequality (9.3.13) for $k \ge \frac{n}{4}$. □

9.3.2.2 Estimate for $\mathbf{z} = \mathrm{i}\boldsymbol{\tau}$

Lemma 9.3.7. *If $\tau \in \mathbb{R}$ and f is supported in $c(\varpi,\theta)$, then*

$$\sup_{x\in c(\varpi,\theta)} |\mathscr{P}_n^{\mathrm{i}\tau} f(x)| \le c_\tau n^{\sigma_\kappa} \|f\|_{1,\kappa}.$$

Proof. Since f is supported in $c(\varpi,\theta)$, we have

$$\sup_{x\in c(\varpi,\theta)} |\mathscr{P}_n^{\mathrm{i}\tau}(f)(x)| \le \sup_{x\in c(\varpi,\theta)} \int_{c(\varpi,\theta)} |f(y)| \left|V_\kappa\left[G_n^{\mathrm{i}\tau}(\langle x,\cdot\rangle)\right](y)\right| h_\kappa^2(y)\,\mathrm{d}\sigma(y)$$
$$\le \|f\|_{1,\kappa} \sup_{x,y\in c(\varpi,\theta)} \left|V_\kappa\left[G_n^{\mathrm{i}\tau}(\langle x,\cdot\rangle)\right](y)\right|.$$

Thus, it is sufficient to prove

$$\left|V_\kappa\left[G_n^{\mathrm{i}\tau}(\langle x,\cdot\rangle)\right](y)\right| \le c_\tau n^{\sigma_\kappa} \qquad \text{for all } x,y \in c(\varpi,\theta). \tag{9.3.14}$$

We note that Eq. (9.3.14) is trivial when $\kappa_{\min} = 0$, since in this case, $\|G_n^{i\tau}\|_\infty \le c_\tau n^{\lambda_\kappa} = c_\tau n^{\sigma_\kappa}$. So we shall assume $\kappa_{\min} > 0$ for the rest of the proof.

To prove Eq. (9.3.14), we claim that it is enough to prove that

$$\left|\int_{-1}^1 G_n^{\mathrm{i}\tau}(at+s)(1-t^2)^{\delta-1}(1+t)\,\mathrm{d}t\right| \le c_\tau n^{\sigma_\kappa}, \tag{9.3.15}$$

whenever $|a| \ge \varepsilon_d > 0$, $|a|+|s| \le 1$, $\delta \ge \kappa_{\min}$, where c_τ is independent of s.

To see this, let $x,y \in c(\varpi,\theta)$, and without loss of generality, assume $\varpi_1 = \max_{1\le j\le d}|\varpi_j|$. Then $\varpi_1 \ge 1/\sqrt{d}$, which implies that $|x_1|,|y_1| \ge 1/\sqrt{d}-\theta \ge 1/\sqrt{d}-1/(8d) > 0$, so that $|x_1y_1| \ge \varepsilon_d > 0$. Thus, invoking Eq. (9.3.15) with $a = x_1y_1$, $\delta = \kappa_1$, and $s = \sum_{j=2}^d t_jx_jy_j$ gives

$$\left|\int_{-1}^1 G_n^{\mathrm{i}\tau}\Big(\sum_{j=1}^d x_jy_jt_j\Big)(1-t_1^2)^{\kappa_1-1}(1+t_1)\,\mathrm{d}t_1\right| \le c_\tau n^{\sigma_\kappa}.$$

The desired inequality (9.3.14) then follows by Fubini's theorem and the integral representation of V_κ in Eq. (7.2.2). This proves the claim.

For the proof of Eq. (9.3.15), by symmetry, it is sufficient to prove

$$\left|\int_{-1}^1 G_n^{\mathrm{i}\tau}(at+s)(1-t)^{\delta-1}\xi(t)\,\mathrm{d}t\right| \le c_\tau n^{\sigma_\kappa}, \tag{9.3.16}$$

where ξ is a C^∞ function supported in $[-\frac{1}{2},1]$, whenever $|a| \ge \varepsilon_d > 0$, $|a|+|s| \le 1$, and $\delta \ge \kappa_{\min}$.

Let $\eta_0 \in C^\infty(\mathbb{R})$ be such that $\chi_{[-\frac{1}{2},\frac{1}{2}]} \le \eta_0 \le \chi_{[-1,1]}$, and let $\eta_1(t) := 1-\eta_0(t)$. Set, in this subsection,

$$B := \frac{n^{-1}+\sqrt{1-|a+s|}}{4n}.$$

We then split the integral in Eq. (9.3.16) into a sum $I_0(a,s)+I_1(a,s)$ with

$$I_j(a,s) := \int_{-1}^1 G_n^{\mathrm{i}\tau}(at+s)\eta_j\Big(\frac{1-t}{B}\Big)(1-t)^{\delta-1}\xi(t)\,\mathrm{d}t, \qquad j=0,1.$$

It is easy to verify that $1+n\sqrt{1-|at+s|} \sim 1+n\sqrt{1-|a+s|}$ whenever $t \in [1-B,1]\cap[-1,1]$. Therefore, for $1-B \le t \le 1$, using Eq. (B.1.7),

$$|G_n^{\mathrm{i}\tau}(at+s)| \le cn^{\lambda_\kappa}(n^{-1}+\sqrt{1-|at+s|})^{-\lambda_\kappa+\sigma_\kappa} \le c\,n^{\sigma_\kappa}B^{-\kappa_{\min}},$$

which implies that

$$|I_0(a,s)| \le c\int_{\max\{1-B,-\frac{1}{2}\}}^1 |G_n^{\mathrm{i}\tau}(at+s)|(1-t)^{\delta-1}\,\mathrm{d}t \le cn^{\sigma_\kappa}B^{\delta-\kappa_{\min}} \le cn^{\sigma_\kappa}.$$

To estimate $I_1(a,s)$, we write

$$G_n^{\mathrm{i}\tau}(at+s)\eta_1\Big(\frac{1-t}{B}\Big)(1-t)^{\delta-1}\xi(t) = c_nP_n^{(\lambda_\kappa-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(at+s)\varphi(t),$$

where $|c_n| \le c_\tau n^{\lambda_\kappa+\frac12}$ and

$$\varphi(t) := \left(1-(at+s)^2+n^{-2}\right)^{\frac{\sigma_\kappa}{2}-\frac{\sigma_\kappa+1}{2}\mathrm{i}\tau}\eta_1\left(\frac{1-t}{B}\right)\xi(t)(1-t)^{\delta-1}.$$

Recall that $|a| \ge \varepsilon_d > 0$. Using integration by parts ℓ times gives

$$|I_1(a,s)| \le cn^{\lambda_\kappa+\frac12-\ell}\int_{-1}^{1}\left|P_{n+\ell}^{(\lambda_\kappa-\frac12-\ell,\lambda_\kappa-\frac12-\ell)}(at+s)\right||\varphi^{(\ell)}(t)|\,\mathrm{d}t.$$

If $-\frac12 \le t \le 1-B/2$, then $1-|at+s| \ge 1-|a|-|s|+(1-|t|)|a| \ge c(1-t) \ge cB \ge cn^{-2}$, which implies, in particular, $\left(1-(at+s)^2+n^{-2}\right)^{-1} \le c(1-t)^{-1}$. Since φ is supported in $(-\frac12, 1-\frac{B}{2})$, which gives $B^{-1} \le (1-t)^{-1}$, it follows from Leibniz's rule that

$$|\varphi^{(\ell)}(t)| \le c_\tau(1-|at+s|)^{\frac{\sigma_\kappa}{2}}(1-t)^{\delta-\ell-1}.$$

Therefore, choosing $\ell > 2\delta$ and recalling that $\delta \ge \kappa_{\min}$, we have by Eq. (B.1.7) that

$$\begin{aligned}|I_1(a,s)| &\le c_\tau n^{\lambda_\kappa-\ell}\int_{-\frac12}^{1-\frac{B}{2}}(1-|at+s|)^{\frac{\sigma_\kappa+\ell}{2}-\frac{\lambda_\kappa}{2}}(1-t)^{\delta-1-\ell}\,\mathrm{d}t\\ &\le c_\tau n^{\lambda_\kappa-\ell}\int_{\frac{B|a|}{2}}^{\frac32|a|}(1-|a+s|+u)^{\frac{\sigma_\kappa+\ell}{2}-\frac{\lambda_\kappa}{2}}u^{\delta-1-\ell}\,\mathrm{d}u.\end{aligned}$$

Using the fact that $(1-|a+s|+u)^\alpha \le c((1-|a+s|)^\alpha+u^\alpha)$, we break the last integral into a sum J_1+J_2, where

$$\begin{aligned}J_1 &\le c_\tau n^{\lambda_\kappa-\ell}\int_{\frac{B|a|}{2}}^{\infty}(1-|a+s|)^{\frac{\sigma_\kappa+\ell}{2}-\frac{\lambda_\kappa}{2}}u^{\delta-1-\ell}\,\mathrm{d}u\\ &\le c_\tau n^{\lambda_\kappa-\ell}(1-|a+s|)^{\frac{\sigma_\kappa+\ell}{2}-\frac{\lambda_\kappa}{2}}B^{\delta-\ell} \le c_\tau n^{\lambda_\kappa-\ell}n^{\ell-\delta}(nB)^{\delta+\sigma_\kappa-\lambda_\kappa}\\ &= c_\tau n^{\lambda_\kappa-\delta}(nB)^{\delta-\kappa_{\min}} \le c_\tau n^{\lambda_\kappa-\delta} \le c_\tau n^{\sigma_k}\end{aligned}$$

and

$$\begin{aligned}J_2 &\le c_\tau n^{\lambda_\kappa-\ell}\int_{\frac{B|a|}{2}}^{\infty}u^{\frac{\sigma+\ell}{2}-\frac{\lambda_\kappa}{2}}u^{\delta-1-\ell}\,\mathrm{d}u \le c_\tau n^{\lambda_\kappa-\ell}B^{\frac{\sigma_\kappa-\ell}{2}-\frac{\lambda_\kappa}{2}+\delta}\\ &= c_\tau n^{\lambda_\kappa-\delta}(n^2B)^{\frac{\kappa_{\min}-\ell}{2}}(nB)^{\delta-\kappa_{\min}} \le c_\tau n^{\lambda_\kappa-\delta} \le c_\tau n^{\sigma_\kappa}.\end{aligned}$$

Putting the above together, we obtain the desired estimate (9.3.16) and complete the proof of Lemma 9.3.7. □

9.3.2.3 Proof of Proposition 9.3.5

Define

$$T^z f = n^{\sigma_\kappa(z-1)} \mathscr{P}_n^z(f\chi_{c(\varpi,\theta)})\chi_{c(\varpi,\theta)}, \quad 0 \le \operatorname{Re} z \le 1.$$

By Lemmas 9.3.6 and 9.3.7, we have

$$\|T^{1+i\tau} f\|_{\kappa,2} \le c_\tau \|f\|_{\kappa,2} \quad \text{and} \quad \|T^{i\tau} f\|_\infty \le c_\tau \|f\|_{1,\kappa}.$$

This allows us to apply Stein's interpolation theorem [157, p. 205] to the analytic family of operators T^z, which yields

$$\|T^{\frac{\sigma_\kappa}{1+\sigma_\kappa}} f\|_{\kappa,\nu'} \le c\|f\|_{\kappa,\nu}.$$

Consequently, using the fact that

$$T^{\frac{\sigma_\kappa}{1+\sigma_\kappa}} f = n^{-\frac{\sigma_\kappa}{1+\sigma_\kappa}} \mathscr{P}_n^{\frac{\sigma_\kappa}{1+\sigma_\kappa}}(f\chi_{c(\varpi,\theta)})\chi_{c(\varpi,\theta)} = n^{-\frac{\sigma_\kappa}{1+\sigma_\kappa}} \operatorname{proj}_n(h_\kappa^2; f\chi_{c(\varpi,\theta)})\chi_{c(\varpi,\theta)},$$

we have proved Proposition 9.3.5. □

9.4 Notes and Further Results

For ordinary spherical harmonics, it was observed by Bonami and Clerc in [18] that the boundedness of the projection operators in Theorem 9.1.1 is enough for establishing the boundedness of the Cesàro means in Theorem 9.2.1, and Sogge [155] proved that the approach indeed works. For the h-harmonics, the additional stronger local estimate in Theorem 9.1.3 is needed to establish Theorem 9.2.1.

The estimate in (ii) of Theorem 9.1.1 is expected to be sharp for $\kappa \ne 0$ as well, but the proof remains open. In place of $(x_1 + ix_2)^n$ used for proving the sharpness of (ii) when $\kappa = 0$, one may consider

$$F_n(x) := V_\kappa[(x_1 + ix_2)^n], \qquad x \in \mathbb{R}^d,$$

where V_κ is the intertwining operator associated with h_κ^2 and $\mathbb{Z}_2^d$. It is easy to see that F_n is an h-harmonic of degree n, and furthermore, it can be explicitly given in terms of generalized Gegenbauer polynomials on using [67, Prop. 5.6.10]. However, the evaluation of the norm of $\|F_n\|_{\kappa,p}$ indicates that it does not yield the sharpness of (ii) for $\kappa \ne 0$.

The proof of Theorem 9.2.1 follows essentially the approach of Sogge [156]. The estimates for h-harmonics, however, are far more involved than those of ordinary harmonics.

The proof of Theorem 9.2.2 follows the approach in [4], which can be traced back to [130].

Chapter 10
Weighted Best Approximation by Polynomials

The structure of spherical harmonics allows us to develop a theory of best polynomial approximation on the sphere based essentially on multipliers, which is, historically, the first approach in this direction. It turns out that the entire framework based on multipliers can be established more generally for h-spherical harmonics associated with reflection-invariant weight functions developed in Chap. 7, which leads to a theory of weighted best approximation by polynomials that we shall present in its full generality.

Throughout this chapter, the weight function h_κ will be associated with a reflection group, and we work with the weight space $L^p(h_\kappa^2) := L^p(\mathbb{S}^{d-1};\mathbb{S}^{d-1})$. In the first section, two different moduli of smoothness are defined in weighted L^p spaces via generalized translation operators, which are multiplier operators, and they are shown to satisfy many of the basic properties of classical moduli of smoothness. An equivalent K-functional will be defined in terms of the h-spherical Laplacian; the fractional power of the latter is studied in the third section, where a Bernstein-type inequality for the spherical h-Laplacian and a useful general multiplier theorem are proved. The matching weighted K-functional is defined in the third section, where several fundamental results, including the realizations and the direct and inverse theorems, are established. The equivalences between the two moduli of smoothness and the K-functional are more difficult to prove; the two equivalences are given in the fourth and the fifth sections, respectively. These equivalences, together with properties of K-functionals, allow us to establish deeper results for weighted moduli of smoothness, including the Jackson inequality and the inverse inequality, in the sixth section.

10.1 Moduli of Smoothness and Best Approximation

Let G be a finite reflection group and let h_κ be the weight function defined in Eq. (7.3.1), which is invariant under G. Then h_κ is a homogeneous function of degree $\sum_{v\in R_+} \kappa_v$. Recall that

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$$\lambda_\kappa = \sum_{v \in R_+} \kappa_v + \frac{d-2}{2}.$$

Associated with the weight h_κ^2, a generalized translation operator T_θ^κ is defined in Eq. (7.4.10) for all $\theta \in \mathbb{R}$, which we write as

$$\operatorname{proj}_n^\kappa(T_\theta^\kappa f) = R_n^{\lambda_\kappa}(\cos\theta)\operatorname{proj}_n^\kappa f, \quad \text{where} \quad R_n^\lambda(t) := \frac{C_n^\lambda(t)}{C_n^\lambda(1)}. \tag{10.1.1}$$

The operators T_θ^κ are uniformly bounded on $L^p(h_\kappa^2)$, as shown in Eq. (7.4.12),

$$\|T_\theta^\kappa f\|_{\kappa,p} \le \|f\|_{\kappa,p}, \quad 1 \le p \le \infty. \tag{10.1.2}$$

When $\kappa = 0$, it is the usual translation operator on $\mathbb{S}^{d-1}$, also called the spherical mean operator.

For $r > 0$ and $0 < \theta < \pi$, we define the rth-order difference operator

$$\triangle_{\theta,\kappa}^r := (I - T_\theta^\kappa)^{r/2} = \sum_{n=0}^{\infty} (-1)^n \binom{r/2}{n} (T_\theta^\kappa)^n, \tag{10.1.3}$$

in a distributional sense, by

$$\operatorname{proj}_n^\kappa \left[\triangle_{\theta,\kappa}^r f\right] = \left(1 - R_n^{\lambda_\kappa}(\cos\theta)\right)^{r/2} \operatorname{proj}_n^\kappa f, \quad n = 0,1,2,\ldots. \tag{10.1.4}$$

Definition 10.1.1. Let $r > 0$ and $0 < \theta < \pi$. For $f \in L^p(h_\kappa^2)$ and $1 \le p < \infty$ or $f \in C(\mathbb{S}^{d-1})$ and $p = \infty$, the weighted rth-order modulus of smoothness is defined by

$$\omega_r(f,t)_{\kappa,p} := \sup_{0<\theta\le t} \|\triangle_{\theta,\kappa}^r f\|_{\kappa,p}, \quad 0 < t < \pi. \tag{10.1.5}$$

This definition makes sense, since the next proposition shows that $\omega_r(f,t)_{\kappa,p}$ satisfies basic properties of the usual moduli.

Proposition 10.1.2. *Let $f \in L^p(h_\kappa^2)$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then*

1. $\omega_\kappa(f,t)_{\kappa,p} \le 2^{r+2}\|f\|_{\kappa,p}$;
2. $\omega_r(f,t)_{\kappa,p} \to 0$ *if* $t \to 0+$;
3. $\omega_r(f,t)_{\kappa,p}$ *is monotone nondecreasing on* $(0,\pi)$;
4. $\omega_r(f+g,t)_{\kappa,p} \le \omega_r(f,t)_{\kappa,p} + \omega_r(g,t)_{\kappa,p}$;
5. *for* $0 < s < r$,

$$\omega_r(f,t)_{\kappa,p} \le 2^{(r-s)+2}\omega_s(f,t)_{\kappa,p}.$$

Proof. Using Eqs. (10.1.2) and (10.1.3), we have

$$\|\triangle^r_{\theta,\kappa} f\|_{\kappa,p} \le \sum_{n=0}^{\infty} \left| \binom{r/2}{n} \right| \|(T^\kappa_\theta)^n f\|_{\kappa,p} \le 2^{r+2} \|f\|_{\kappa,p}, \tag{10.1.6}$$

which proves the first assertion. If f is a spherical polynomial, then the second assertion can be verified directly from the definition (10.1.4). The assertion for a general $f \in L^p(h^2_\kappa)$ follows from Eq. (10.1.6) and a density argument. The remaining assertions are easy consequences of Eq. (10.1.6) and the definition (10.1.5). □

More properties of $\omega_r(f,t)_{\kappa,p}$ will be given after we prove its equivalence to a K-functional. The definition of $\omega_r(f,t)_{\kappa,p}$ involves higher powers of the generalized translation operator T^κ_θ, which are rather difficult to compute, even in the unweighted case in which $T^\kappa_\theta = T_\theta$ is given explicitly by the integral of the spherical average (2.1.6). An alternative way of defining a modulus of smoothness that avoids high powers of T^κ_θ is to take the central difference with respect to the step θ. This leads to the definition of our second moduli of smoothness, which we define only for integer orders. Let $\hat{\triangle}^r_\theta$ denote the central difference operator defined by

$$\hat{\triangle}_\theta f(t) = f\left(t + \frac{\theta}{2}\right) - f\left(t - \frac{\theta}{2}\right), \quad \hat{\triangle}^r_\theta := \hat{\triangle}_\theta \hat{\triangle}^{r-1}_\theta, \quad r = 1, 2, \ldots.$$

From the definition, it follows immediately that for $f : \mathbb{R} \mapsto \mathbb{R}$,

$$\hat{\triangle}^r_\theta f(t) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j f\left(t + \frac{r\theta}{2} - j\theta\right).$$

Definition 10.1.3. Given a positive integer ℓ and $\theta \in \mathbb{R}$, we define the 2ℓth-order difference operator $\tilde{\triangle}^{2\ell}_{\theta,\kappa}$ by

$$\tilde{\triangle}^{2\ell}_{\theta,\kappa} f(x) := \frac{(-1)^\ell}{\binom{2\ell}{\ell}} \hat{\triangle}^{2\ell}_\theta G_{x,f}(0), \quad \text{where} \quad G_{x,f}(t) := T^\kappa_t f(x). \tag{10.1.7}$$

For $f \in L^p(h^2_\kappa)$ and $1 \le p < \infty$ or $f \in C(\mathbb{S}^{d-1})$ and $p = \infty$, define

$$\tilde{\omega}_{2\ell}(f,t)_{\kappa,p} := \sup_{0<\theta\le t} \|\tilde{\triangle}^{2\ell}_{\theta,\kappa} f\|_{\kappa,p}, \quad \theta \in (0,\pi). \tag{10.1.8}$$

Lemma 10.1.4. *For $\ell \in \mathbb{N}$ and $\theta \in \mathbb{R}$, we have*

$$\tilde{\triangle}^{2\ell}_{\theta,\kappa} f = f + \frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} T^\kappa_{j\theta} f. \tag{10.1.9}$$

Proof. By definition, one has

$$\begin{aligned}\hat{\triangle}_\theta^{2\ell} G_{x,f}(0) &= \sum_{j=0}^{2\ell} \binom{2\ell}{j} (-1)^j G_{x,f}((\ell-j)\theta) = \sum_{j=0}^{2\ell} \binom{2\ell}{j} (-1)^j T^\kappa_{(\ell-j)\theta} f(x) \\ &= (-1)^\ell \binom{2\ell}{\ell} f(x) + \sum_{i=1}^{\ell} \binom{2\ell}{\ell-i} (-1)^{i+\ell} T^\kappa_{i\theta} f(x) \\ &\quad + \sum_{i=1}^{\ell} \binom{2\ell}{i+\ell} (-1)^{i+\ell} T^\kappa_{-i\theta} f(x).\end{aligned}$$

Since $T^\kappa_\theta f(x)$ is an even function of $\theta \in \mathbb{R}$, it follows that

$$\hat{\triangle}_\theta^{2\ell} G_{x,f}(0) = (-1)^\ell \binom{2\ell}{\ell} \left[f(x) + \frac{2}{\binom{2\ell}{\ell}} \sum_{i=1}^{\ell} \binom{2\ell}{\ell-i} (-1)^i T^\kappa_{i\theta} f(x) \right].$$

The desired identity then follows. □

As will be shown later in this chapter, the above two moduli of smoothness are equivalent:

$$\omega_{2\ell}(f,t)_{\kappa,p} \sim \tilde{\omega}_{2\ell}(f,t)_{\kappa,p}, \quad 1 \le p \le \infty, \quad 0 < t < \pi. \tag{10.1.10}$$

An interesting point is that for $d \ge 3$, one can drop the supremum sign on the right-hand side of both Eqs. (10.1.5) and (10.1.8) without essential impact on the moduli. More precisely, we have

$$\omega_r(f,t)_{\kappa,p} \sim \|\triangle^r_{t,\kappa} f\|_{\kappa,p}, \quad 1 \le p \le \infty, \quad 0 < t < \pi/2, \tag{10.1.11}$$

and

$$\tilde{\omega}_{2\ell}(f,t)_{\kappa,p} \sim \|\tilde{\triangle}^{2\ell}_{t,\kappa} f\|_{\kappa,p}, \quad 1 \le p \le \infty, \quad 0 < t < \pi/2\ell. \tag{10.1.12}$$

The equivalences (10.1.10), (10.1.11), and (10.1.12) will follow from the equivalences of these moduli of smoothness with a K-functional, which are established later in Theorems 10.4.1 and 10.5.1.

Definition 10.1.5. For $f \in L^p(h_\kappa^2)$ and $1 \le p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ and $p = \infty$, the weighted best approximation of f by spherical polynomials of degree at most n is defined by

$$E_n(f)_{\kappa,p} := \inf_{P \in \Pi_n(\mathbb{S}^{d-1})} \|f - P\|_{\kappa,p}.$$

We will need the near-best-approximation operator defined via a cutoff function for h-spherical harmonic expansions.

Definition 10.1.6. Let η be a C^∞-function on $[0,\infty)$ such that $\eta(t) = 1$ for $0 \le t \le 1$ and $\eta(t) = 0$ for $t \ge 2$. For $f \in L(h_\kappa^2)$, we define

$$L_n^\kappa f := \sum_{j=0}^{2n} \eta\left(\frac{j}{n}\right) \operatorname{proj}_j^\kappa f, \quad n = 1, 2, \ldots. \tag{10.1.13}$$

The following proposition collects several useful results on the operator L_n^κ.

Proposition 10.1.7. *Let $f \in L^p(h_\kappa^2)$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then*

(1) $L_n^\kappa f \in \Pi_{2n-1}(\mathbb{S}^{d-1})$ *and* $L_n^\kappa f = f$ *for* $f \in \Pi_n(\mathbb{S}^{d-1})$.
(2) *For* $n \in \mathbb{N}$, $\|L_n^\kappa f\|_{\kappa,p} \le c\|f\|_{\kappa,p}$.
(3) *For* $n \in \mathbb{N}$,

$$\|f - L_n^\kappa f\|_{\kappa,p} \le (1+c) E_n(f)_{\kappa,p}.$$

The proof of this proposition is almost identical to that of Proposition 2.6.3, on using the boundedness of the Cesàro means of the *h*-harmonic expansion proved in Theorem 7.4.4.

10.2 Fractional Powers of the Spherical *h*-Laplacian

Recall that the space $\mathscr{H}_n^d(h_\kappa^2)$ of *h*-spherical harmonics on $\mathbb{S}^{d-1}$ of degree n is an eigenvector space of the *h*-Laplace–Beltrami operator $\Delta_{h,0}$; namely,

$$\mathscr{H}_n^d(h_\kappa^2) = \left\{ f \in C^2(\mathbb{S}^{d-1}) : \ \Delta_{h,0} f = -n(n+2\lambda_\kappa) f \right\}, \quad n = 0, 1, \ldots.$$

Accordingly, we can define fractional powers of $\Delta_{h,0}$ as follows.

Definition 10.2.1. For $r > 0$ and $1 \le p \le \infty$, a function $f \in L^p(h_\kappa^2)$ is said to belong to the weighted Sobolev space $\mathscr{W}_p^r(h_\kappa^2)$ if there exists a function $g \in L^p(h_\kappa^2)$, which we denote by $(-\Delta_{h,0})^{r/2} f$, such that

$$\operatorname{proj}_n^\kappa \left[(-\Delta_{h,0})^{r/2} f \right] = (n(n+2\lambda_\kappa))^{r/2} \operatorname{proj}_n^\kappa f, \quad n = 0, 1, \ldots, \tag{10.2.1}$$

where we assume $f, g \in C(\mathbb{S}^{d-1})$ when $p = \infty$. The fractional spherical *h*-Laplacian $(-\Delta_{h,0})^{r/2}$ is then a linear operator on the space $\mathscr{W}_1^r(h_\kappa^2)$ defined by Eq. (10.2.1).

The Bernstein inequality holds for the fractional spherical *h*-Laplacian.

Lemma 10.2.2. *If $1 \le p \le \infty$, $r > 0$, and $f \in \Pi_n(\mathbb{S}^{d-1})$, then*

$$\left\| (-\Delta_{h,0})^{r/2} f \right\|_{\kappa,p} \le c n^r \|f\|_{\kappa,p}.$$

Proof. If f belongs to $\Pi_n(\mathbb{S}^{d-1})$, then so does $(-\Delta_{h,0})^{r/2} f$. Thus, we may write

$$(-\Delta_{h,0})^{r/2} f = L_n^\kappa (-\Delta_{h,0})^{r/2} f = \sum_{j=0}^{2n} \eta\left(\frac{j}{n}\right)(j(j+2\lambda_\kappa))^{r/2} \operatorname{proj}_j^\kappa f.$$

Let $\ell = \lceil \lambda_\kappa \rceil$, the smallest integer greater than λ_κ, and $\mu_j := \eta(\frac{j}{n})(j(j+2\lambda_\kappa))^{r/2}$. Summation by parts $\ell+1$ times gives then

$$(-\Delta_{h,0})^{r/2} f = \sum_{j=0}^{2n} \triangle^{\ell+1} \mu_j A_j^\ell S_j^\ell(h_\kappa^2; f),$$

where $A_j^\delta := \binom{j+\delta}{j} \sim j^\delta$ and S_j^ℓ are the Cesàro (C,ℓ) means of the h-harmonic series. A straightforward computation, using Eq. (A.3.3), shows that

$$\left|\triangle^{\ell+1} \mu_j\right| \le c(j+1)^{r-\ell-1}, \quad j \ge 0.$$

Thus, recalling the uniform boundedness of the (C,δ) means of h-harmonic expansions, we deduce

$$\left\|(-\Delta_{h,0})^{r/2} f\right\|_{\kappa,p} \le c \sum_{j=0}^{2n} (j+1)^{r-\ell-1} j^\ell \|f\|_{\kappa,p} \le c n^r \|f\|_{\kappa,p},$$

which proves the desired Bernstein inequality. □

The above proof relies only on the boundedness of the Cesàro operators of h-spherical harmonic expansions. Following the same idea, we can prove a more general result, which will be used repeatedly in the following sections.

Proposition 10.2.3. *Let $\{u_k\}_{k=0}^\infty$ be a sequence of real numbers satisfying*

$$\sup_{n\ge 0} |u_n| + \sum_{n=0}^\infty \left|\triangle^{\ell+1} u_n\right| A_n^\ell \le M \tag{10.2.2}$$

for some positive integer $\ell > \lambda_\kappa$. Let $f \in L^p(h_\kappa^2)$ for $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ for $p = \infty$. Then the limit $u := \lim_{n\to\infty} u_n$ exists, the series

$$Tf = \sum_{n=0}^\infty \left(\triangle^{\ell+1} u_n\right) A_n^\ell S_n^\ell(h_\kappa^2; f)$$

converges in $\|\cdot\|_{\kappa,p}$ norm, and the operator $T : L^p(h_\kappa^2) \to L^p(h_\kappa^2)$ satisfies

$$\operatorname{proj}_n^\kappa(Tf) = (u_n - u) \operatorname{proj}_n^\kappa(f), \quad n = 0, 1, \ldots, \tag{10.2.3}$$

and

$$\|Tf\|_{\kappa,p} \le cM\|f\|_{\kappa,p}, \quad 1 \le p \le \infty. \tag{10.2.4}$$

If, in addition, $u = \lim_{n\to\infty} u_n = 0$*, then*

$$\left\|\sum_{n=0}^{\infty} u_n \operatorname{proj}_n^{\kappa} f\right\|_{\kappa,p} \le c\left(\sum_{n=0}^{\infty} |\triangle^{\ell+1} u_n| n^{\ell}\right)\|f\|_{\kappa,p}, \tag{10.2.5}$$

where the series on the left converges in the norm of $L^p(h_\kappa^2)$.

Proof. That limit $\lim_{n\to\infty} u_n = u$ follows directly from Lemma A.4.1. For simplicity, we write $S_n^\ell f = S_n^\ell(h_\kappa^2; f)$ in this proof. Using Eq. (10.2.2) and the uniform boundedness of $S_n^\ell(h_\kappa^2)$, it is easily seen that the series $\sum_{k=0}^{\infty} \left(\triangle^{\ell+1} u_k\right) A_k^\ell S_k^\ell f$ converges in $L^p(h_\kappa^2)$ norm and that the inequality (10.2.4) holds.

We now prove Eq. (10.2.3). Without loss of generality, we may assume that $u = 0$. Since each projection operator $\operatorname{proj}_n^\kappa$ is bounded on $L^p(h_\kappa^2)$, it follows that

$$\operatorname{proj}_n^\kappa(Tf) = \sum_{j=0}^{\infty} \left(\triangle^{\ell+1} u_j\right) A_j^\ell \operatorname{proj}_n^\kappa \left(S_j^\ell f\right) = \left(\sum_{j=n}^{\infty} (\triangle^{\ell+1} u_j) A_{j-n}^\ell\right) \operatorname{proj}_n^\kappa f,$$

since $\operatorname{proj}_n^\kappa \left(S_j^\ell f\right)$ is equal to 0 if $j < n$ and is equal to A_{j-n}^ℓ / A_j^ℓ if $j \ge n$ by the definition of S_j^ℓ. Therefore, in order to prove Eq. (10.2.3), it is sufficient to show that

$$\sum_{j=n}^{\infty} (\triangle^{\ell+1} u_j) A_{j-n}^\ell = u_n,$$

which, however, is an easy consequence of the identity

$$\sum_{k=0}^{\infty} u_k r^k = \sum_{n=0}^{\infty} \left(\sum_{j=n}^{\infty} \left(\triangle^{\ell+1} u_j\right) A_{j-n}^\ell\right) r^n, \quad 0 < r < 1.$$

Finally, by Eq. (10.2.3), it is readily seen that Eq. (10.2.5) is a special case of Eq. (10.2.4). □

Lemma 10.2.4. *If* $r > 0$*,* $1 \le p \le \infty$*, and* $f \in \mathscr{W}_p^r(h_\kappa^2)$*, then*

$$\|f - L_n^\kappa f\|_{\kappa,p} \le c n^{-r} \left\|(-\Delta_{h,0})^{r/2} f\right\|_{\kappa,p}, \quad n = 1, 2, \ldots.$$

Proof. Let ℓ be a positive integer greater than λ_κ. Without loss of generality, we may assume that $n > \ell + 1$. Since $\eta(x) = 1$ for $x \in [0,1]$, we may write

$$f - L_n^\kappa f = \sum_{j=n}^{\infty} \left(1 - \eta\left(\frac{j}{n}\right)\right) \left(j(j + 2\lambda_\kappa)\right)^{-r/2} \operatorname{proj}_j^\kappa \left((-\Delta_{h,0})^{r/2} f\right).$$

Thus, applying Eq. (10.2.5), where $\mu_j = (1-\eta(\frac{j}{n}))(j(j+2\lambda_\kappa))^{-r/2}$, we deduce that

$$\|f - L_n^\kappa f\|_{\kappa,p} \le c\left(\sum_{k=n-\ell-1}^{\infty} |\triangle^{\ell+1}\mu_j| k^\ell\right) \left\|(-\Delta_{h,0})^{r/2} f\right\|_{\kappa,p}.$$

Since $\eta \in C^\infty[0,\infty)$ satisfies $\chi_{[0,1]}(x) \le \eta(x) \le \chi_{[0,2]}(x)$, it is easily seen that $|\triangle^{\ell+1}\mu_j| \le c j^{-r-\ell-1}$ for all $j \ge 1$. Hence,

$$\begin{aligned}\|f - L_n^\kappa f\|_{\kappa,p} &\le c\left(\sum_{j=n-\ell-1}^{\infty} j^{-r-\ell-1} j^\ell\right) \left\|(-\Delta_{h,0})^{r/2} f\right\|_{\kappa,p} \\ &\le c n^{-r} \left\|(-\Delta_{h,0})^{r/2} f\right\|_{\kappa,p}.\end{aligned}$$

The proof is complete. □

10.3 *K*-Functionals and Best Approximation

For $f \in L^p(h_\kappa^2)$, we define its K-functional in terms of the h-spherical Laplacian.

Definition 10.3.1. Given $r > 0$, the rth K-functional of $f \in L^p(h_\kappa^2)$ is defined by

$$K_r(f,t)_{\kappa,p} := \inf_{g \in \mathscr{W}_p^r(h_\kappa^2)} \left\{ \|f-g\|_{\kappa,p} + t^r \left\|(-\Delta_{h,0})^{r/2} g\right\|_{\kappa,p} \right\}. \tag{10.3.6}$$

We will show in later sections that this K-functional is equivalent to the two moduli of smoothness defined in the first section. Our main tool is the following realization of the K-functionals.

Theorem 10.3.2. *Let $f \in L^p(h_\kappa^2)$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. If $t \in (0,1)$ and n is a positive integer such that $n \sim t^{-1}$, then*

$$K_r(f,t)_{\kappa,p} \sim \|f - L_n^\kappa f\|_{p,\kappa} + n^{-r} \left\|(-\Delta_{h,0})^{r/2} L_n^\kappa f\right\|_{\kappa,p}. \tag{10.3.7}$$

Proof. The upper estimate of Eq. (10.3.7) follows directly from setting $g = L_n^\kappa f$ in the definition (10.3.6). To prove the lower estimate, choose $g \in \mathscr{W}_p^r(h_\kappa^2)$ such that

$$\|f-g\|_{\kappa,p} + t^r \left\|(-\Delta_{h,0})^{r/2} g\right\|_{\kappa,p} \le 2K_r(f,t)_{\kappa,p}.$$

By Lemma 10.2.4 and the uniform boundedness of the operator L_n^κ on $L^p(h_\kappa^2)$,

$$\begin{aligned}\|f - L_n^\kappa f\|_{\kappa,p} &\le \|f-g\|_{\kappa,p} + \|g - L_n^\kappa g\|_{\kappa,p} + \|L_n^\kappa (f-g)\|_{\kappa,p} \\ &\le c\|f-g\|_{\kappa,p} + c t^r \left\|(-\Delta_{h,0})^{r/2} g\right\|_{\kappa,p} \le c K_r(f,t)_{\kappa,p}.\end{aligned}$$

Furthermore, using the fact that $L_n^\kappa(-\Delta_{h,0})^{r/2} = (-\Delta_{h,0})^{r/2} L_n^\kappa$ and the Bernstein inequality in Lemma 10.2.2, we obtain

$$\begin{aligned} n^{-r} & \left\| (-\Delta_{h,0})^{r/2} L_n^\kappa f \right\|_{\kappa,p} \\ & \le n^{-r} \left\| (-\Delta_{h,0})^{r/2} L_n^\kappa (f-g) \right\|_{\kappa,p} + n^{-r} \left\| (-\Delta_{h,0})^{r/2} (L_n^\kappa g) \right\|_{\kappa,p} \\ & \le c \|L_n^\kappa (f-g)\|_{\kappa,p} + c t^r \left\| L_n^\kappa (-\Delta_{h,0})^{r/2} g \right\|_{\kappa,p} \\ & \le c \|f-g\|_{\kappa,p} + c t^r \left\| (-\Delta_{h,0})^{r/2} g \right\|_{\kappa,p} \le c K_r(f,t)_{\kappa,p}, \end{aligned}$$

on using the boundedness of L_n^κ. Together, the last two displayed inequalities give the desired lower estimate of Eq. (10.3.7). □

As a consequence of Theorem 10.3.2, we can establish the following direct and inverse inequalities for the best approximation in terms of K-functionals.

Theorem 10.3.3. *Let $f \in L^p(h_\kappa^2)$ if $1 \le p < \infty$, and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then for every $r > 0$ and $n \in \mathbb{N}$,*

$$E_n(f)_{\kappa,p} \le c\, K_r(f; n^{-1})_{\kappa,p} \tag{10.3.8}$$

and

$$K_r(f, n^{-1})_{\kappa,p} \le c\, n^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{\kappa,p}. \tag{10.3.9}$$

Proof. The Jackson inequality (10.3.8) follows directly from Proposition 10.1.7, the realization (10.3.7), and the monotonicity of the K-functional. To prove the reverse inequality (10.3.9), we assume that $2^{m-1} \le n < 2^m$, and for convenience, we set $L_{2^{-1}} f = 0$. By the definition of $K_r(f,t)_{\kappa,p}$ and the properties of $L_n^\kappa f$, we have

$$\begin{aligned} K_r(f, n^{-1})_{\kappa,p} & \le \|f - L_{2^m}^\kappa f\|_{\kappa,p} + 2^r 2^{-mr} \left\| (-\Delta_{h,0})^{r/2} L_{2^m}^\kappa f \right\|_{\kappa,p} \\ & \le c E_{2^m}(f)_{\kappa,p} + 2^r 2^{-mr} \sum_{j=0}^{m} \left\| (-\Delta_{h,0})^{r/2} [L_{2^j}^\kappa f - L_{2^{j-1}}^\kappa f] \right\|_{\kappa,p} \\ & \le c E_{2^m}(f)_{\kappa,p} + c 2^{-mr} \sum_{j=0}^{m} 2^{jr} \|L_{2^j}^\kappa f - L_{2^{j-1}}^\kappa f\|_{\kappa,p}, \end{aligned}$$

on using the Bernstein inequality in Lemma 10.2.2. Consequently, by the triangle inequality,

$$K_r(f,n^{-1})_{\kappa,p} \le c2^{-mr}\sum_{j=0}^{m} 2^{jr} E_{2^{j-1}}(f)_{\kappa,p} \le cn^{-r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{\kappa,p},$$

since $2^{m-1} \le n < 2^m$. The proof is complete. □

10.4 Equivalence of the First Modulus and the K-Functional

The main goal in this section is to prove the following theorem, which asserts the equivalence of the weighted moduli of smoothness $\omega_r(f,t)_{\kappa,p}$ and the K-functional.

Theorem 10.4.1. *Let $f \in L^p(h_\kappa^2)$ if $1 \le p < \infty$ and let $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. If $\theta \in (0, \frac{\pi}{2})$ and $r > 0$, then*

$$\omega_r(f,\theta)_{\kappa,p} \sim \left\| (I - T_\theta^\kappa)^{r/2} f \right\|_{\kappa,p} \sim K_r(f,\theta)_{\kappa,p}. \tag{10.4.1}$$

As a consequence of the equivalence between the modulus of smoothness and the K-functional and Theorem 10.3.3, we immediately deduce the characterization of the best approximation by modulus of smoothness.

Theorem 10.4.2. *Let $f \in L^p(h_\kappa^2)$ if $1 \le p < \infty$, and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then for every $r > 0$ and $n \in \mathbb{N}$,*

$$E_n(f)_{\kappa,p} \le c\,\omega_r(f;n^{-1})_{\kappa,p} \tag{10.4.2}$$

and

$$\omega_r(f,n^{-1})_{\kappa,p} \le c n^{-r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{\kappa,p}. \tag{10.4.3}$$

The proof of Theorem 10.4.1 relies on the following two lemmas.

Lemma 10.4.3. *For $r > 0$, $\theta \in (0, 3n^{-1}]$, and any $\ell \in \mathbb{N}$,*

$$\sum_{j=0}^{2n}\left| \triangle^{\ell+1}\left[\left(\frac{1 - R_j^{\lambda_\kappa}(\cos\theta)}{j(j+2\lambda_\kappa)\theta^2} \right)^r \eta\left(\frac{j}{n}\right) \right] \right| (j+1)^\ell \le c \tag{10.4.4}$$

and

$$\sum_{j=0}^{2n}\left| \triangle^{\ell+1}\left[\left(\frac{j(j+2\lambda_\kappa)\theta^2}{1 - R_j^{\lambda_\kappa}(\cos\theta)} \right)^r \eta\left(\frac{j}{n}\right) \right] \right| (j+1)^\ell \le c, \tag{10.4.5}$$

where the difference $\triangle^{\ell+1}$ is acting on j, and c depends only on κ, ℓ, r.

Proof. In terms of the normalized Jacobi polynomials, $R_j^{\lambda_\kappa} = R_j^{(\lambda_\kappa - \frac{1}{2}, \lambda_\kappa - \frac{1}{2})}$. Hence, using Eq. (B.5.6) and Lemma A.3.3, we deduce that if $i = 0, 1, \dots, \ell + 1$ and $j\theta \le C$, then

$$\left| \triangle^i \left(\frac{1 - R_j^{\lambda_\kappa}(\cos\theta)}{j(j+2\lambda_\kappa)\theta^2} \right)^r \right| \le C_{\kappa,r,\ell} \triangle_*^i \left(\frac{1 - R_j^{\lambda_\kappa}(\cos\theta)}{j(j+2\lambda_\kappa)\theta^2} \right),$$

where $\triangle_*^i$ is as defined in Eq. (A.3.6). However, Eqs. (A.3.6) and (B.5.7) imply that for $\theta \in (0, cn^{-1}]$ and $0 \le j \le 2n$,

$$\triangle_*^i \left(\frac{1 - R_j^{\lambda_\kappa}(\cos\theta)}{j(j+2\lambda_\kappa)\theta^2} \right) \le cn^{-i}, \quad i = 0, 1, \dots \ell + 1,$$

so that an application of Eqs. (A.3.3) and (A.3.4) yields that for $0 \le j \le 2n$,

$$\left| \triangle^{\ell+1} \left(\left(\frac{1 - R_j^{\lambda_\kappa}(\cos\theta)}{j(j+2\lambda_\kappa)\theta^2} \right)^r \eta\left(\frac{j}{n}\right) \right) \right| \le cn^{-\ell-1},$$

which in turn implies Eq. (10.4.4).

The proof of Eq. (10.4.5) is almost identical to that of Eq. (10.4.4), and we therefore omit the details. □

Lemma 10.4.4. *Let* $\ell_\kappa = \lceil \lambda_k \rceil$. *If* $r > 0$, $k^{-1} \le \theta \le \frac{\pi}{2}$, $j \in \mathbb{N}_0$, *and* $0 \le j \le \ell_\kappa + 1$, *then for every* $m \in \mathbb{N}$,

$$\left| \triangle^j \left(\frac{(1 - (1 - R_k^{\lambda_\kappa}(\cos\theta))^r)^{m+\ell_\kappa+1}}{(1 - R_k^{\lambda_\kappa}(\cos\theta))^r} \right) \right| \le c_{\kappa,m,r} (k\theta)^{-m\lambda_\kappa} \theta^j. \tag{10.4.6}$$

Proof. First, we prove that for $\theta \in [k^{-1}, \pi/2]$,

$$\left| \triangle^i \left((1 - R_k^{\lambda_\kappa}(\cos\theta))^{-r} \right) \right| \le c_{\kappa,r} \theta^i, \quad i = 0, 1, \dots, \ell_\kappa + 1. \tag{10.4.7}$$

Indeed, by Lemma B.5.3, there exists a constant $\gamma_\kappa \in (0,1)$ such that

$$0 < 1 - \gamma_\kappa \le 1 - R_k^{\lambda_\kappa}(\cos\theta) \le 1 + \gamma_\kappa, \quad \theta \in [k^{-1}, \pi/2].$$

Hence, using Lemma A.3.3, we conclude that for $\theta \in [k^{-1}, \pi/2]$,

$$\left| \triangle^i \left((1 - R_k^{\lambda_\kappa}(\cos\theta))^{-r} \right) \right| \le c_{\kappa,r} \triangle_*^i \left(1 - R_k^{\lambda_\kappa}(\cos\theta) \right),$$

which, by Lemma (B.5.1), is dominated by $c_{\kappa,r,i}\theta^i$, proving Eq. (10.4.7).

Next, we claim that for $\theta \in [k^{-1}, \pi/2]$ and $i = 0, 1, \ldots, \ell_\kappa + 1$,

$$\left| \triangle^i \left[\left(1 - \left(1 - R_k^{\lambda_\kappa}(\cos\theta) \right)^r \right)^{m+1+\ell_\kappa} \right] \right| \le c_{\kappa,r,m} \theta^i (k\theta)^{-m\lambda_\kappa}. \tag{10.4.8}$$

Indeed, setting $a_k(\theta) := 1 - (1 - R_k^{\lambda_\kappa}(\cos\theta))^r$, then using Eq. (B.5.5) with $j = 0$, we obtain

$$|a_k(\theta)| \le c_r \left| R_k^{\lambda_\kappa}(\cos\theta) \right| \le c_{\kappa,r} (k\theta)^{-\lambda_\kappa}, \tag{10.4.9}$$

which proves Eq. (10.4.8) for $i = 0$. For $1 \le i \le \ell_\kappa + 1$, using Eq. (A.3.7), we obtain

$$\begin{aligned} \left| \triangle^i (a_k(\theta))^{m+1+\ell_\kappa} \right| &\le c_{\kappa,m} \triangle_*^i (a_k(\theta)) \max_{k \le j \le k+i+1} (a_j(\theta))^{m+1+\ell_\kappa - i} \\ &\le c_{\kappa,m} \triangle_*^i \left(\left(1 - R_k^{\lambda_\kappa}(\cos\theta) \right)^r \right) (k\theta)^{-m\lambda_\kappa} \end{aligned}$$

by Eq. (10.4.9), which implies, by Eq. (10.4.7) and the definition of Δ_*^i, the inequality (10.4.8).

Finally, combining Eq. (10.4.7) with Eq. (10.4.8), we deduce the desired estimate (10.4.6) from Eq. (A.3.3). □

We are now in a position to prove Theorem 10.4.1.

Proof of Theorem 10.4.1. Let $n \in \mathbb{N}$ be such that $\theta^{-1} \le n \le 3\theta^{-1}$. First, we show that for every $g \in \Pi_n(\mathbb{S}^{d-1})$,

$$\left\| (I - T_\theta^\kappa)^{r/2} g \right\|_{\kappa,p} \le c_{p,r} n^{-r} \left\| (-\Delta_{h,0})^{r/2} g \right\|_{\kappa,p}. \tag{10.4.10}$$

For this, we use Eq. (10.1.4) and write

$$\begin{aligned} (I - T_\theta^\kappa)^{r/2} g &= \sum_{j=0}^{n} \left(1 - R_j^{\lambda_\kappa}(\cos\theta) \right)^{r/2} \operatorname{proj}_j^\kappa g \\ &= \theta^r \sum_{j=0}^{2n} \left(\frac{1 - R_j^{\lambda_\kappa}(\cos\theta)}{j(j+2\lambda_\kappa)\theta^2} \right)^{r/2} \eta\left(\frac{j}{n}\right) \operatorname{proj}_j^\kappa \left(-\Delta_{h,0} \right)^{r/2} g \Big). \end{aligned}$$

Invoking Proposition 10.2.3 with $\ell = \ell_\kappa$, we then deduce

$$\begin{aligned} \left\| (I - T_\theta^\kappa)^{r/2} g \right\|_{\kappa,p} \le c_\kappa \theta^r & \left(\sum_{j=0}^{2n} \left| \triangle^{\ell_\kappa+1} \left(\frac{1 - R_j^{\lambda_\kappa}(\cos\theta)}{j(j+2\lambda_\kappa)\theta^2} \right)^{r/2} \eta\left(\frac{j}{n}\right) \right| j^{\ell_\kappa} \right) \\ & \left\| (-\Delta_{h,0})^{r/2} g \right\|_{\kappa,p}, \end{aligned}$$

which, using Lemma 10.4.3, yields the estimate (10.4.10). With the help of Eq. (10.4.10), we obtain

$$\left\|(I-T_\theta^\kappa)^{r/2}f\right\|_{\kappa,p} \le c_{\kappa,r}\|f-L_{n/2}^\kappa f\|_{\kappa,p} + \left\|(I-T_\theta^\kappa)^{r/2}L_{n/2}^\kappa f\right\|_{\kappa,p}$$
$$\le c_{\kappa,r}\left(\|f-L_{n/2}^\kappa f\|_{\kappa,p} + n^{-r}\left\|(-\Delta_{h,0})^{r/2}L_{n/2}^\kappa g\right\|_{\kappa,p}\right),$$

which, together with Theorem 10.3.2, implies the upper estimate

$$\left\|(I-T_\theta^\kappa)^{r/2}f\right\|_{\kappa,p} \le c_{\kappa,r}K_r(f,\theta)_{\kappa,p}. \tag{10.4.11}$$

Next we prove the matching lower estimate that reverses the inequality of Eq. (10.4.11). By the definition of $K_r(f,t)_{\kappa,p}$, it suffices to prove both

$$n^{-r}\left\|(-\Delta_{h,0})^{r/2}(L_n^\kappa f)\right\|_{\kappa,p} \le c_{p,r}\left\|(I-T_\theta^\kappa)^{r/2}f\right\|_{\kappa,p} \tag{10.4.12}$$

and

$$\|f-L_n^\kappa f\|_{\kappa,p} \le c_{p,r}\left\|(I-T_\theta^\kappa)^{r/2}f\right\|_{\kappa,p}. \tag{10.4.13}$$

For the first inequality, we use Eq. (10.1.4) and the eigenvalues of $\Delta_{h,0}$ to write

$$\theta^{2r}\left(-\Delta_{h,0}\right)^{r/2}(L_n^\kappa f) = \sum_{j=0}^{2n}\left(\frac{j(j+2\lambda_\kappa)\theta^2}{1-R_j^{\lambda_\kappa}(\cos\theta)}\right)^{r/2}\eta\left(\frac{j}{n}\right)\operatorname{proj}_j^\kappa\left[(I-T_\theta^\kappa)^{r/2}f\right],$$

so that the desired estimate (10.4.12) is again a direct consequence of Proposition 10.2.3 and Lemma 10.4.3. To prove the second inequality (10.4.13), we temporarily set $a_j(\theta) = 1-(1-R_j^{\lambda_\kappa}(\cos\theta))^{r/2}$. Let m be the smallest integer such that $m\lambda_\kappa \ge \ell_\kappa + 2$. Using the fact that $(1-r)^{-1} = \sum_{i=0}^M r^i + r^{M+1}(1-r)^{-1}$ for $r<1$, we obtain

$$\left(1-R_j^{\lambda_\kappa}(\cos\theta)\right)^{-r/2} = (1-a_j(\theta))^{-1} = \sum_{i=0}^{m+\ell_\kappa}(a_j(\theta))^i + \frac{(a_j(\theta))^{m+1+\ell_\kappa}}{(1-R_j^{\lambda_\kappa}(\cos\theta))^{r/2}}.$$

Thus, setting $h := (I-T_\theta^\kappa)^{r/2}f$, we deduce

$$f-L_n^\kappa f = \sum_{j=n}^\infty \frac{1-\eta(\frac{j}{n})}{(1-R_j^{\lambda_\kappa}(\cos\theta))^{r/2}}\operatorname{proj}_j^\kappa h$$
$$= \sum_{i=0}^{m+\ell_\kappa}(I-L_n^\kappa)\left(I-(I-T_\theta^\kappa)^{r/2}\right)^i h$$
$$+ \sum_{j=n}^\infty\left(1-\eta\left(\frac{j}{n}\right)\right)\frac{(a_j(\theta))^{m+\ell_\kappa+1}}{(1-R_j^{\lambda_\kappa}(\cos\theta))^{r/2}}\operatorname{proj}_j^\kappa h,$$

where I denotes the identity operator. Since the operators L_n^κ and $(I - T_\theta^\kappa)^{r/2}$ are uniformly bounded on $L^p(h_\kappa^2)$, it follows that

$$\left\| \sum_{j=0}^{m+\ell_\kappa} (I - L_n^\kappa) \left(I - (I - T_\theta^\kappa)^{r/2} \right)^j h \right\|_{\kappa,p} \le c_{\kappa,r} \|h\|_{\kappa,p}.$$

Moreover, using Proposition 10.2.3 and Lemma 10.4.4, we obtain

$$\begin{aligned} &\left\| \sum_{k=0}^{\infty} \left(1 - \eta\left(\frac{k}{n}\right)\right) \frac{(a_k(\theta))^{m+\ell_\kappa+1}}{(1 - R_k^{\lambda_\kappa}(\cos\theta))^{r/2}} \operatorname{proj}_k^\kappa h \right\|_{\kappa,p} \\ &\quad \le c_\kappa \left(\sum_{k=n-\ell_\kappa-1}^{\infty} \left| \triangle^{\ell_\kappa+1} \left(\frac{\left(1-\eta\left(\frac{k}{n}\right)\right)(a_k(\theta))^{m+\ell_\kappa+1}}{(1 - R_k^{\lambda_\kappa}(\cos\theta))^{r/2}} \right) \right| (k+1)^{\ell_\kappa} \right) \|h\|_{\kappa,p} \\ &\quad \le c_{\kappa,r} \|h\|_{\kappa,p}. \end{aligned}$$

Putting these together, we have established Eq. (10.4.13), and as a result, proved that

$$\left\| \triangle_{\theta,\kappa}^r f \right\|_{\kappa,p} = \left\| (I - T_\theta^\kappa)^{r/2} f \right\|_{\kappa,p} \sim K_r(f,\theta)_{\kappa,p}.$$

This shows that $\omega_r(f,t)_{\kappa,p} \ge cK_r(f,t)_{\kappa,p}$, and furthermore, since $K_r(f,t)_{\kappa,p}$ is evidently an increasing function in t, taking the supremum over $0 \le \theta \le t$ shows that $\omega_r(f,t)_{\kappa,p} \le cK_r(f,t)_{\kappa,p}$. The proof is complete. □

10.5 Equivalence of the Second Modulus and the *K*-Functional

The main result in this section is the following equivalence theorem.

Theorem 10.5.1. *Let $f \in L^p(h_\kappa^2)$ if $1 \le p < \infty$ and let $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. If $\ell \in \mathbb{N}$ and $\theta \in (0, \frac{\pi}{2\ell})$, then*

$$\tilde{\omega}_{2\ell}(f,\theta)_{\kappa,p} \sim \left\| \tilde{\triangle}_{\theta,\kappa}^{2\ell} f \right\|_{\kappa,p} \sim K_{2\ell}(f,\theta)_{\kappa,p}. \tag{10.5.1}$$

As a result of this theorem and Theorem 10.4.1, the two moduli are equivalent.

Corollary 10.5.2. *Under the assumption of Theorem 10.5.1,*

$$\tilde{\omega}_{2\ell}(f,\theta)_{\kappa,p} \sim \omega_{2\ell}(f,\theta)_{\kappa,p}, \quad 1 \le p \le \infty.$$

For convenience, we make the following definition:

$$T_{\theta,\ell}^\kappa := I - \tilde{\triangle}_{\theta,\kappa}^{2\ell} = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} T_{j\theta}^\kappa, \tag{10.5.2}$$

where I denotes the identity operator, or $\tilde{\triangle}^{2\ell}_{\theta,\kappa} = I - T^{\kappa}_{\theta,\ell}$. By the definition of $\tilde{\triangle}^{2\ell}_{\theta,\kappa}$,

$$\operatorname{proj}_n^{\kappa}(T^{\kappa}_{\theta,\ell} f) = a_\ell(n,\theta)\operatorname{proj}_n^{\kappa} f, \quad n \in \mathbb{N}_0, \tag{10.5.3}$$

where

$$a_\ell(k,\theta) := \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} R_k^{\lambda_\kappa}(\cos j\theta). \tag{10.5.4}$$

For the proof of Theorem 10.5.1, we need the following lemma.

Lemma 10.5.3. *Let* $\theta \in (0, \frac{\pi}{2\ell}]$. *For a given constant* $\tau > 0$, *the following hold:*

(i) *For* $0 < k\theta \le \tau$,

$$0 < c_1 \le \frac{1 - a_\ell(k,\theta)}{(k(k+2\lambda_\kappa)\theta^2)^\ell} \le c_2 < \infty,$$

where c_1, c_2 *depend only on* κ, ℓ, *and* τ.

(ii) *For* $k\theta \ge \tau$, *there exists a number* $\gamma \in (0,1)$ *depending only on* κ, ℓ, *and* τ *such that*

$$a_\ell(k,\theta) \le \gamma < 1.$$

(iii) *For* $0 < k\theta \le \tau$ *and* $j \ge 1$,

$$\left| \triangle^j \left(\frac{1 - a_\ell(k,\theta)}{(k(k+2\lambda_\kappa)\theta^2)^\ell} \right) \right| \le c\left((k+1)^{1-j}\theta + (k+1)^{-j-1} \right),$$

where $c > 0$ *depends only on* κ, ℓ, *and* τ.

(iv) *For* $j \in \mathbb{N}_0$,

$$\left| \triangle^j a_\ell(k,\theta) \right| \le c_{\ell,j} \min\{\theta^j,\ (k\theta)^{-\lambda_\kappa}\theta^j\}.$$

Proof. Considering (i) and (ii) together, we need to establish these two assertions only for the case $\tau = \frac{1}{2\ell}$.

For the proof of (i) with $0 \le k\theta \le \frac{1}{2\ell}$, we set $f_k(u) := R_k^{\lambda_\kappa}(\cos u)$. Since $1 - a_\ell(k,\theta) = \frac{(-1)^\ell}{\binom{2\ell}{\ell}} \hat{\triangle}^{2\ell}_\theta f_k(0)$ and $\hat{\triangle}^m_\theta f_k(t) = (-1)^m \Delta^m_\theta f_k(t - m\theta/2)$, it follows from Eq. (A.3.4) that

$$1 - a_\ell(k,\theta) = \frac{(-1)^\ell}{\binom{2\ell}{\ell}} \int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} \cdots \int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} f_k^{(2\ell)}(u_1 + \cdots + u_{2\ell})\, \mathrm{d}u_1 \cdots \mathrm{d}u_{2\ell}. \tag{10.5.5}$$

Using Eq. (B.2.11), we can write

$$f_k(u) = R_k^{\lambda_\kappa}(\cos u) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \alpha(k,j,\lambda_\kappa) \cos(k-2j)u, \tag{10.5.6}$$

where the constants $\alpha(k,j,\lambda_\kappa)$ satisfy

$$0 < \alpha(k,j,\lambda_\kappa) \sim k^{-\lambda_\kappa} j^{\lambda_\kappa - 1}. \tag{10.5.7}$$

Taking derivatives, it follows that

$$(-1)^\ell f_k^{(2\ell)}(u) = \sum_{j=0}^{k} j^{2\ell} \alpha(k,j,\lambda_\kappa) \cos ju.$$

Since $\cos ju \sim 1$ if $0 \le j \le k$ and $|ku| \le 1$, it follows that for $|ku| \le 1$,

$$(-1)^\ell f_k^{(2\ell)}(u) \sim \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} j^{2\ell} k^{-\lambda_\kappa} j^{\lambda_\kappa - 1} \sim k^{2\ell},$$

which implies, together with Eq. (10.5.5), assertion (i) for $0 \le k\theta \le \frac{1}{2\ell}$.

We now prove (ii) with $k\theta \ge \frac{1}{2\ell}$. Since $|R_k^{\lambda_\kappa}(\cos\theta)| \le c_\kappa (k\theta)^{-\lambda_\kappa}$ by Eqs. (B.1.3) and (B.1.7), there exists a constant $c_* > 0$, depending only on ℓ and κ, such that $|a_\ell(k,\theta)| \le \frac{1}{2}$ whenever $|k\theta| \ge c_*$. To prove (ii) for the remaining case $\frac{1}{2\ell} \le |k\theta| \le c_*$, we use Eqs. (10.5.4) and (10.5.6) to obtain

$$\begin{aligned} a_\ell(k,\theta) &= \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \alpha(k,j,\lambda_\kappa) \left(\frac{-2}{\binom{2\ell}{\ell}} \sum_{i=1}^{\ell} (-1)^i \binom{2\ell}{\ell - i} \cos i(k-2j)\theta \right) \\ &= \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \alpha(k,j,\lambda_\kappa) \left(1 - \frac{4^\ell}{\binom{2\ell}{\ell}} \left(\sin \frac{(k-2j)\theta}{2} \right)^{2\ell} \right), \end{aligned}$$

which implies, since setting $u = 0$ in Eq. (10.5.6) shows that $\alpha(k,j,\lambda_\kappa)$ sums to 1,

$$a_\ell(k,\theta) = 1 - \frac{4^\ell}{\binom{2\ell}{\ell}} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \alpha(k,j,\lambda_\kappa) \left(\sin \frac{(k-2j)\theta}{2} \right)^{2\ell}. \tag{10.5.8}$$

Let us assume, temporarily, that $c_* > 1$. Let $\gamma = \lfloor \frac{1}{2}(1 - \frac{1}{c_*}) \rfloor$. It is easy to see that if $j \ge \gamma + 1$, then $(k-2j) \le k/c^*$, so that $(k-2j)\theta/2 \le k\theta/(2c_*) \le 1$ for $|k\theta| \le c^*$. Hence, by Eq. (10.5.7), we obtain that for $\frac{1}{2\ell} \le k\theta \le c_*$,

$$\begin{aligned} \frac{4^\ell}{\binom{2\ell}{\ell}} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \alpha(k,j,\lambda_\kappa) \left(\sin \frac{(k-2j)\theta}{2} \right)^{2\ell} &\ge \frac{4^\ell}{\binom{2\ell}{\ell}} \sum_{j=\gamma k+1}^{\lfloor \frac{k}{2} \rfloor} \alpha(k,j,\lambda_\kappa) \left(\frac{(k-2j)\theta}{\pi} \right)^{2\ell} \\ &\ge c \frac{1}{k} \sum_{j=\gamma k+1}^{\lfloor \frac{k}{2} \rfloor} (k-2j)^{2\ell} \theta^{2\ell} \ge c(k\theta)^{2\ell} \ge c > 0, \end{aligned}$$

which, combined with Eq. (10.5.8), implies (ii). If $c_* \le 1$, then $(k-2j)\theta \le k\theta \le c_* < 1$ for all j in the sum, and the above proof carries over with obvious modifications.

To prove (iii), we set, for simplicity, $g_k(x) = R_k^{\lambda_\kappa}(x)$, so that $f_k(t) = g_k(\cos t)$. Using induction on ℓ, it is easy to see that

$$f_k^{(2\ell)}(t) = \sum_{j=1}^{\min\{2\ell,k\}} g_k^{(j)}(\cos t) \sum_{\max\{j-\ell,0\} \le i \le \lfloor \frac{j}{2} \rfloor} c_{\ell,j,i} (\sin t)^{2i} (\cos t)^{j-2i}, \qquad (10.5.9)$$

where $c_{\ell,j,i}$ are constants independent of k. However, using Eq. (B.5.3), we have

$$g_k^{(j)}(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j R_k^{(\lambda_\kappa - \frac{1}{2}, \lambda_\kappa - \frac{1}{2})}(x) = A_{j,\kappa} \varphi_j(k) R_{k-j}^{(\lambda_\kappa + j - \frac{1}{2}, \lambda_\kappa + j - \frac{1}{2})}(x), \qquad (10.5.10)$$

where

$$A_{j,\kappa} = 2^j \left(\lambda_\kappa + \frac{1}{2}\right)_j \quad \text{and} \quad \varphi_j(k) = (-1)^j (-k)_j \left(\lambda_\kappa + \frac{k+1}{2}\right)_j.$$

Since $\varphi_j(k)$ is a polynomial of degree $2j$ in k, we have that for $v \in \mathbb{N}$,

$$\left| \triangle^v \left(\frac{\varphi_j(k)}{(k(k+2\lambda_\kappa))^\ell} \right) \right| \le c_{\ell,v} \begin{cases} (k+1)^{2j-2\ell-v}, & \text{if } j \ne \ell, \\ (k+1)^{-v-1}, & \text{if } j = \ell \text{ and } v > 0, \end{cases} \qquad (10.5.11)$$

which follows, when $j \ne \ell$, from $\triangle^v f(t) = f^{(v)}(\xi)/v!$ with some ξ between $[t, t+v]$ and a simple computation, whereas a cancellation of the highest term occurs when $j = \ell$ and $v > 0$. Furthermore, by Eq. (B.5.5), for $0 < kt \le \ell\tau$ and $v \in \mathbb{N}_0$,

$$\left| \triangle^v \left(R_{k-j}^{(\lambda_\kappa + j - \frac{1}{2}, \lambda_\kappa + j - \frac{1}{2})}(\cos t) \right) \right| \le c t^v,$$

which, together with Eq. (10.5.11), implies, on using Eq. (A.3.3), that for $0 < kt \le \ell\tau$ and $v \in \mathbb{N}_0$,

$$\begin{aligned} \left| \triangle^v \left(\frac{g_k^{(j)}(\cos t)}{(k(k+2\lambda_\kappa))^\ell} \right) \right| &= A_{j,\kappa} \left| \triangle^v \left(\frac{\varphi_j(k)}{(k(k+2\lambda_\kappa))^\ell} R_{k-j}^{(\lambda_\kappa + j - \frac{1}{2}, \lambda_\kappa + j - \frac{1}{2})}(\cos t) \right) \right| \\ &\le c_{\ell,v,\tau} \begin{cases} \left((k+1)^{-v-1} + t^v\right), & \text{if } 1 \le j \le \ell, \\ (k+1)^{2j-2\ell-v}, & \text{if } \ell+1 \le j \le 2\ell. \end{cases} \end{aligned}$$

Consequently, we then deduce from Eq. (10.5.9) that

$$\left|\triangle^{v}\left(\frac{f_k^{(2\ell)}(t)}{(k(k+2\lambda_\kappa))^\ell}\right)\right| \le c_{\ell,v,\tau}\left(\sum_{j=1}^{\ell}\left((k+1)^{-v-1}+t^v\right)+\sum_{j=\ell+1}^{2\ell}(k+1)^{2j-2\ell-v}t^{2(j-\ell)}\right)$$
$$\le c_{\ell,v,\tau}\left((k+1)^{-v-1}+t^v+(k+1)^{-v+2}t^2\right), \qquad (10.5.12)$$

where $v \in \mathbb{N}_0$ and $0 < kt \le \ell\tau$. Now, by Eq. (10.5.5), we obtain that for $j \ge 1$,

$$\triangle^{j}\left[\frac{1-a_\ell(k,\theta)}{(k(k+2\lambda_\kappa)\theta^2)^\ell}\right] = \frac{(-1)^\ell}{\binom{2\ell}{\ell}}\int_{-\frac{\theta}{2}}^{\frac{\theta}{2}}\cdots\int_{-\frac{\theta}{2}}^{\frac{\theta}{2}}\triangle^{j}\left[\frac{f_k^{(2\ell)}(u_1+\cdots+u_{2\ell})}{(k(k+2\lambda_\kappa)\theta^2)^\ell}\right]\mathrm{d}u_1\cdots\mathrm{d}u_{2\ell}.$$

Since $f_k^{(2\ell)}$ is an even function, we deduce from Eq. (10.5.12) that for $0 < k\theta \le \tau$ and $j \ge 1$,

$$\left|\triangle^{j}\left(\frac{1-a_\ell(k,\theta)}{(k(k+2\lambda_\kappa)\theta^2)^\ell}\right)\right| \le c_{\ell,j,\tau}\left((k+1)^{-j-1}+\theta^j+(k+1)^{-j+2}\theta^2\right)$$
$$\le c_{\ell,j,\tau}\left((k+1)^{-j-1}+(k+1)^{-j+1}\theta\right).$$

This completes the proof of (iii).

Finally, (iv) is a direct consequence of Eq. (10.5.4) and Lemma B.5.1. □

We are now in a position to prove Theorem 10.5.1.

Proof of Theorem 10.5.1. The proof runs along the same lines as that of Theorem 10.4.1. Let $\ell_\kappa = \lceil \lambda_\kappa \rceil$ and let $n = \lceil \theta^{-1} + 4\ell_\kappa + 4 \rceil$. First, we show that for $g \in \Pi_n(\mathbb{S}^{d-1})$,

$$\left\|\tilde{\triangle}_{\theta,\kappa}^{2\ell} g\right\|_{\kappa,p} = \left\|g - T_{\theta,\ell}^{\kappa} g\right\|_{\kappa,p} \le c_{\kappa,\ell} n^{-2\ell}\left\|\Delta_{h,0}^{\ell} g\right\|_{\kappa,p}. \qquad (10.5.13)$$

Since $g \in \Pi_n(\mathbb{S}^{d-1})$ implies that $T_{\theta,\ell}^{\kappa} g \in \Pi_n(\mathbb{S}^{d-1})$ by Eq. (10.5.2), using Eq. (10.5.3) and the eigenvalues of $\Delta_{h,0}$, we obtain

$$g - T_{\theta,\ell}^{\kappa} g = L_n^{\kappa}\left(g - T_{\theta,\ell}^{\kappa}\right) = \sum_{k=0}^{2n}\eta\left(\frac{k}{n}\right)\left(1-a_\ell(k,\theta)\right)\operatorname{proj}_k^{\kappa} f$$
$$= \theta^{2\ell}\sum_{k=0}^{2n}\eta\left(\frac{k}{n}\right)\frac{1-a_\ell(k,\theta)}{(k(k+2\lambda_\kappa)\theta^2)^\ell}\operatorname{proj}_k^{\kappa}\left((-\Delta_{h,0})^\ell g\right). \qquad (10.5.14)$$

However, it follows by Lemma 10.5.3 that for $0 \le k \le 2n$,

$$\left|\triangle^{\ell_\kappa+1}\left(\eta\left(\frac{k}{n}\right)\frac{1-a_\ell(k,\theta)}{(k(k+2\lambda_\kappa)\theta^2)^\ell}\right)\right| \le c_{\kappa,\eta}\left(k^{-\ell_\kappa}n^{-1}+(k+1)^{-\ell_\kappa-2}\right),$$

which implies immediately that

$$\sum_{k=0}^{2n}\left|\triangle^{\ell_\kappa+1}\left(\eta\left(\frac{k}{n}\right)\frac{1-a_\ell(k,\theta)}{(k(k+2\lambda_\kappa)\theta^2)^\ell}\right)\right|(k+1)^{\ell_\kappa}\le c_{\kappa,\eta}.$$

Using this inequality and Eq. (10.5.14), we deduce Eq. (10.5.13) from Proposition 10.2.3. Now, applying Eq. (10.5.13) with $g=L^\kappa_{\lfloor n/2\rfloor}f$ and the boundedness of $T^k_{\theta,\ell}$, which follows from the boundedness of T^κ_θ, we deduce that

$$\begin{aligned}\left\|f-T^\kappa_{\theta,\ell}f\right\|_{\kappa,p}&\le\|f-g\|_{\kappa,p}+\left\|g-T^\kappa_{\theta,\ell}g\right\|_{\kappa,p}+\left\|T^\kappa_{\theta,\ell}(g-f)\right\|_{\kappa,p}\\&\le c\|f-g\|_{\kappa,p}+cn^{-2\ell}\left\|\Delta^\ell_{h,0}g\right\|_{\kappa,p}\sim K_{2\ell}(f,\theta)\end{aligned}$$

by Eq. (10.3.7), which gives the desired upper estimate.

To establish the matching lower estimate, it suffices to show, by the definition of the K-functional, that

$$n^{-2\ell}\left\|(-\Delta_{h,0})^\ell(L^\kappa_nf)\right\|_{\kappa,p}\le c_{\kappa,\ell}\left\|f-T^\kappa_{\theta,\ell}f\right\|_{\kappa,p}\tag{10.5.15}$$

and

$$\|f-L^\kappa_nf\|_{\kappa,p}\le c_{\kappa,\ell}\left\|f-T^\kappa_{\theta,\ell}f\right\|_{\kappa,p}.\tag{10.5.16}$$

For Eq. (10.5.15), we use Eq. (10.5.3) and the eigenvalues of $\Delta_{h,0}$ to write

$$\theta^{2\ell}(-\Delta_{h,0})^\ell(L^\kappa_nf)=\sum_{k=0}^{2n}\left(\frac{(k(k+2\lambda_\kappa)\theta^2)^\ell}{1-a_\ell(k,\theta)}\eta\left(\frac{k}{n}\right)\right)\operatorname{proj}^\kappa_k\left(f-T^\kappa_{\theta,\ell}f\right).\tag{10.5.17}$$

Using Lemma 10.5.3 and Eq. (A.3.9), we have that for $0\le k\le 2n$ and $0\le v\le\ell_\kappa+1$,

$$\begin{aligned}\left|\triangle^v\left(\frac{(k(k+2\lambda_\kappa)\theta^2)^\ell}{1-a_\ell(k,\theta)}\right)\right|&\le c_\kappa\triangle^v_*\left(\frac{1-a_\ell(k,\theta)}{(k(k+2\lambda_\kappa)\theta^2)^\ell}\right)\\&\le c_\kappa\left((k+1)^{-v+1}\theta+(k+1)^{-v-1}\right)\end{aligned}$$

if $1\le v\le\ell_\kappa$ and the replaced by a constant c_κ if $v=0$, which in turn implies

$$\left|\triangle^{\ell_\kappa+1}\left(\frac{(k(k+2\lambda_\kappa)\theta^2)^\ell}{1-a_\ell(k,\theta)}\eta\left(\frac{k}{n}\right)\right)\right|\le c_\kappa\left((k+1)^{-v_\kappa}n^{-1}+(k+1)^{-v_\kappa-2}\right),$$

and consequently,

$$\sum_{k=0}^{2n}\left|\triangle^{\ell_\kappa+1}\left(\frac{(k(k+2\lambda_\kappa)\theta^2)^\ell}{1-a_\ell(k,\theta)}\eta\left(\frac{k}{n}\right)\right)\right|(k+1)^{\ell_\kappa}\le c_\kappa.$$

Using this inequality and Eq. (10.5.17), we see that Eq. (10.5.15) follows from Proposition 10.2.3.

It remains to prove Eq. (10.5.16). Since the operators L_n^κ and $T_{\theta,\ell}^\kappa$ are uniformly bounded on $L^p(h_\kappa^2)$, it suffices to show that

$$\left\| f - L_n^\kappa f - \sum_{j=0}^{4} (I - L_n^\kappa) \left(T_{\theta,\ell}^\kappa\right)^j \left(f - T_{\theta,\ell}^\kappa f\right) \right\|_{\kappa,p} \le c_{\kappa,\ell} \left\| f - T_{\theta,\ell}^\kappa f \right\|_{\kappa,p}. \quad (10.5.18)$$

The reason for choosing the above operator lies in its expression as a multiplier operator. Indeed, we can write

$$f - L_n^\kappa f - \sum_{j=0}^{4} (I - L_n^\kappa)(T_{\theta,\ell}^\kappa)^j \left(f - T_{\theta,\ell}^\kappa f\right) = \sum_{k=0}^{\infty} b_\ell(k,\theta) \operatorname{proj}_k^\kappa \left(f - T_{\theta,\ell}^\kappa f\right), \quad (10.5.19)$$

where, using Eq. (10.5.3), the multiplier $b_\ell(k,\theta)$ is given by

$$b_\ell(k,\theta) = \left(1 - \eta\left(\frac{k}{n}\right)\right)\left(\frac{1}{1 - a_\ell(k,\theta)} - \sum_{j=0}^{4} (a_\ell(k,\theta))^j\right) = \left(1 - \eta\left(\frac{k}{n}\right)\right)\frac{(a_\ell(k,\theta))^5}{1 - a_\ell(k,\theta)},$$

which has better behavior than that of $f - L_n^\kappa f$ itself, since by Lemma 10.5.3,

$$a_\ell(k,\theta) \le \gamma_{\kappa,\ell} < 1, \quad \text{if } \theta > 0.$$

Using (iv) in Lemma 10.5.3 and Eq. (A.3.5), we deduce that for $k\theta \ge 1$ and $v \in \mathbb{N}_0$,

$$\left| \triangle^v \left(\frac{1}{1 - a_\ell(k,\theta)} \right) \right| \le c_{v,\ell} \triangle_*^v (a_\ell(k,\theta)) \le c_{v,\ell} \theta^v (k\theta)^{-\lambda_\kappa}.$$

Using Eq. (A.3.3), Lemma A.3.2, and (iv) in Lemma 10.5.3, it follows that

$$\left| \triangle^v \left(\frac{(a_\ell(k,\theta))^5}{1 - a_\ell(k,\theta)} \right) \right| \le c_{v,\ell} (k\theta)^{-5\lambda_\kappa} \theta^v.$$

This further implies that

$$\left| \triangle^{\ell_\kappa+1} b_\ell(k,\theta) \right| \le c_{\kappa,\ell} \begin{cases} n^{-\ell_\kappa-1}, & \text{if } n - \ell_\kappa - 1 \le k \le 2n, \\ n^{-\ell_\kappa-1}(k\theta)^{-5\lambda_\kappa}, & \text{if } k \ge 2n+1, \end{cases}$$

whereas $\triangle^{\ell_\kappa+1} b_\ell(k,\theta) = 0$ if $0 \le k \le n - \ell_\kappa - 2$, since $1 - \eta(\frac{k}{n}) = 0$ if $0 \le k \le n$. Consequently,

$$\sum_{k=0}^{\infty} \left| \triangle^{\ell_\kappa+1} (b_\ell(k,\theta)) \right| (k+1)^{\ell_\kappa} \le c_{\kappa,\ell}.$$

The estimate (10.5.18) then follows from Eq. (10.5.19) and Proposition 10.2.3. Consequently, we have established

$$\left\| \tilde{\triangle}^{2\ell}_{\theta,\kappa} f \right\|_{\kappa,p} = \left\| f - T^{\kappa}_{\theta,\ell} f \right\|_{\kappa,p} \sim K_{2\ell}(f,\theta)_{\kappa,p}.$$

As in the proof of Theorem 10.4.1, this also shows that $\hat{\omega}_{2\ell}(f,t)_{\kappa,p} \sim K_{2\ell}(f,t)_{\kappa,p}$. The proof is complete. □

10.6 Further Properties of Moduli of Smoothness

As an immediate consequence of Theorem 10.4.2, we have the following Marchaud-type inequality:

Theorem 10.6.1. *If* $r > 0$, $\alpha > 0$, $t \in (0,1)$, *and* $1 \le p \le \infty$, *then*

$$\omega_r(f,t)_{\kappa,p} \le ct^r \int_t^1 \frac{\omega_{r+\alpha}(f,u)_{\kappa,p}}{u^{r+1}} \,\mathrm{d}u. \tag{10.6.1}$$

Proof. Assume that $t \sim 2^{-m}$ for some $m \in \mathbb{N}$. Then by the reverse inequality (10.4.3) and the Jackson inequality (10.4.2), we obtain

$$\omega_r(f,t)_{\kappa,p} \le c2^{-mr} \sum_{j=0}^{m} 2^{jr} E_{2^j}(f)_{\kappa,p} \le c2^{-mr} \sum_{j=0}^{m} 2^{jr} \omega_{r+\alpha}(f,2^{-j})_{\kappa,p}.$$

Applying the monotonicity of $\omega_r(f,\cdot)_{\kappa,p}$, we further deduce that

$$\omega_r(f,t)_{\kappa,p} \le c2^{-mr} \sum_{j=0}^{m} \int_{2^{-j}}^{2^{-j+1}} \frac{\omega_{r+\alpha}(f,u)_{\kappa,p}}{u^{s+1}} \,\mathrm{d}u \le ct^r \int_t^1 \frac{\omega_{r+\alpha}(f,u)_{\kappa,p}}{u^{s+1}} \,\mathrm{d}u,$$

which proves Marchaud's inequality. □

Proposition 10.6.2. *The following statements are true for all* $1 \le p \le \infty$*:*

1. If $\ell > 0, r > 0$, *and* $t \in (0, \pi/\ell)$, *then*

$$\omega_r(f,\ell t)_{\kappa,p} \le c_r(\ell+1)^r \omega_r(f,t)_{\kappa,p}.$$

2. If $\alpha > 0$ *and* $f \in \mathscr{W}^{\alpha}_p(h^2_\kappa)$, *then for* $r > \alpha$,

$$\omega_r(f,t)_{\kappa,p} \le ct^{\alpha} \omega_{r-\alpha}\big((-\Delta_{h,0})^{\alpha/2} f, t\big)_{\kappa,p}.$$

3. If $f \in \Pi_n(\mathbb{S}^{d-1})$ *and* $\theta \sim n^{-1}$*, then*

$$n^{-r}\left\|(-\Delta_{h,0})^{r/2} f\right\|_{\kappa,p} \sim \left\|\triangle_{\theta,\kappa}^r f\right\|_{\kappa,p}.$$

Proof. Statement (i) is a direct consequence of Theorem 10.4.1 and the definition (10.3.6) of K-functionals. To prove (ii), let $n \in \mathbb{N}$ be such that $t \sim n^{-1}$. We then use Theorem 10.4.1 and Lemma 10.3.7 to obtain

$$\omega_r(f,t)_{\kappa,p} \sim K_r(f,n^{-1})_{\kappa,p} \sim \|f - L_n^\kappa f\|_{\kappa,p} + n^{-r}\left\|(-\Delta_{h,0})^{r/2} L_n^\kappa f\right\|_{\kappa,p}. \quad (10.6.2)$$

Set $g_n = f - L_n^\kappa f$, and observe that $L_{\lfloor n/2 \rfloor}^\kappa g_n = 0$. It follows from Lemma 10.3.7 that

$$\|f - L_n^\kappa f\|_{\kappa,p} = \left\|g_n - L_{\lfloor n/2 \rfloor}^\kappa g_n\right\|_{\kappa,p} \le c n^{-\alpha}\left\|(-\Delta_{h,0})^{\alpha/2} g_n\right\|_{\kappa,p}.$$

By the definition of g_n, it then follows from Proposition 10.1.7 and the Jackson inequality (10.4.2) that

$$\begin{aligned}\|f - L_n^\kappa f\|_{\kappa,p} &\le c n^{-\alpha}\left\|(-\Delta_{h,0})^{\alpha/2} f - L_n^\kappa\left[(-\Delta_{h,0})^{\alpha/2} f\right]\right\|_{\kappa,p} \\ &\le c n^{-\alpha} E_n\left((-\Delta_{h,0})^{\alpha/2} f\right)_{\kappa,p} \\ &\le c n^{-\alpha} \omega_{r-\alpha}\left((-\Delta_{h,0})^{\alpha/2} f, n^{-1}\right)_{\kappa,p}.\end{aligned}$$

Furthermore, since L_n^κ and $(-\Delta_{h,0})^{\frac{r}{2}}$ commute, we deduce by Theorems 10.4.1 and Eq. (10.3.7) that

$$\begin{aligned}n^{-r}\left\|(-\Delta_{h,0})^{r/2} L_n^\kappa f\right\|_{\kappa,p} &= n^{-\alpha} n^{-(r-\alpha)}\left\|(-\Delta_{h,0})^{\frac{r-\alpha}{2}}\left[L_n^\kappa(-\Delta_{h,0})^{\frac{\alpha}{2}} f\right]\right\|_{\kappa,p} \\ &\le c n^{-\alpha} \omega_{r-\alpha}\left((-\Delta_{h,0})^{\alpha/2} f, n^{-1}\right)_{\kappa,p}.\end{aligned}$$

A combination of the last two inequalities and Eq. (10.6.2) yields the desired inequality in (ii). Finally, to prove (iii), we use Theorem 10.4.1 to obtain

$$\left\|\triangle_{\theta,\kappa}^r f\right\|_{\kappa,p} \le c K_r(f,n^{-1})_{\kappa,p} \le c n^{-r}\left\|(-\Delta_{h,0})^{r/2} f\right\|_{\kappa,p},$$

where the last step uses the definition of the K-functional. □

10.7 Notes and Further Results

In the unweighted case, a brief account of the history of approximation on the sphere is given in the notes at the end of Chap. 4. With the main theorems spelled out in the present chapter, which covers the unweighted results as special cases, we can go into greater detail.

Many authors made contributions to the proof of the Jackson inequality (10.4.2) and its reverse (10.4.3), making use of the moduli of smoothness $\omega_r(f,t)_p$ on the sphere $\mathbb{S}^{d-1}$ in the unweighted case. This effort can be traced back to the work of Kushnirenko [103, 1958], who proved the Jackson inequality (10.4.2) on the sphere $\mathbb{S}^2$ for the case $r=2$ and $p=\infty$. In all dimensions, the inequality (10.4.2) on $\mathbb{S}^{d-1}$ was later proved by Butzer and Jansche [26, 1971] and Pawelke [135, 1972] for $r=2$ and $1 \le p \le \infty$, by Lizorkin and Nikolskii [132, 1987] for $r \in \mathbb{N}$ and $p=2$, and by Kalyabin [94, 1987] for $r \in \mathbb{N}$ and $1 < p < \infty$. Rustamov [146, 1991], [148, 1993] studied moduli of smoothness $\omega_r(f,t)_p$ for all $r > 0$ and presented a proof of Eq. (10.4.2) for the full range of $1 \le p \le \infty$. However, his proof, based on the fact that the unit ball of the Sobolev space W_p^r is weakly compact in L^p for $1 < p < \infty$, did not seem to work for $p = 1$ and ∞. A detailed proof that works for the full range of $1 \le p \le \infty$ was given in [174, Chap. 5]. The proof was later simplified in [13].

The weighted moduli of smoothness Eq. (10.1.5) and K-functionals (10.3.6) were defined and studied in [190], where the direct and inverse theorems, namely Theorem 10.4.2, the equivalence $\omega_r(f,t)_{\kappa,p} \sim K_r(f,t)_{\kappa,p}$, as well as several other useful properties of $\omega_r(f,t)_{\kappa,p}$ were established; see also [189].

Most of the results in Sect. 10.2 for K-functionals were proved by Ditzian [55] in a more general setting, where only Cesàro summability was assumed.

In the case of $1 < p < \infty$, both the Jackson inequality (10.4.2) and the Stechkin-type inverse Eq. (10.4.3) can be sharpened as follows: For $r > 0$ and $1 < p < \infty$,

$$n^{-r}\Big(\sum_{k=1}^{n} k^{-rs-1} E_k(f)_{\kappa,p}^s\Big)^{\frac{1}{s}} \le c_{\kappa,p}\,\omega_r(f,n^{-1})_{\kappa,p}, \quad s = \max\{p,2\}, \tag{10.7.1}$$

and

$$\omega_r(f,n^{-1})_{\kappa,p} \le C_{\kappa,p} n^{-r}\Big(\sum_{k=0}^{n} k^{-rq-1} E_k(f)_{\kappa,p}^q\Big)^{\frac{1}{q}}, \quad q = \min\{p,2\}. \tag{10.7.2}$$

In particular, in the case $p = 2$,

$$n^{-r}\left(\sum_{k=1}^{n} k^{-2r-1} E_k(f)_{2,\kappa}^2\right)^{\frac{1}{2}} \sim \omega_r(f,n^{-1})_{2,\kappa}.$$

In the unweighted case, for $d=2$ and 2π-periodic functions, both Eqs. (10.7.1) and (10.7.2) were established by Timan [165,166], and they were proven, for $d > 2$, in [41,43], respectively. The proofs of [41,43] work equally well for the weighted case discussed in this chapter. A different but more elegant proof of the sharp Jackson inequality (10.7.1) was given recently in [60], using semigroups and convex properties of L_p-spaces. The method used there also works for more general Banach spaces.

The unweighted case of Theorem 10.5.1 was proved in [40]. The proofs of Theorems 10.4.1 and 10.5.1 follow along the same lines as those of [13,40].

Chapter 11
Harmonic Analysis on the Unit Ball

Unlike the unit sphere, the unit ball is a domain that has a boundary. The boundary usually makes analysis on the domain more difficult. It turns out, however, that analysis on the unit ball is closely related to analysis on the unit sphere. Indeed, a large portion of harmonic analysis on the unit ball can be deduced from its counterparts on the sphere. What is needed for the sphere, however, is weighted results, even when we work in the unweighted setting on the unit ball. The weight functions that we consider are mostly invariant under $\mathbb{Z}_2^d$.

The connection between the ball and the sphere and its implications for orthogonal structure are discussed in the first section, followed by a convolution on the unit ball and a first discussion of orthogonal expansions in the second section. The connection is strong enough to yield satisfactory results for maximal functions and a multiplier theorem in the third section and sharp results for Cesàro means, including boundedness in L^p spaces, in the fourth section. The near-best-approximation operators on the ball that have highly localized kernels when the weight function is radial are addressed in the fifth section. Cubature formulas on the ball are studied in the sixth section. Finally, a further connection between analysis on the ball and analysis on higher-dimensional spheres is discussed in the seventh section.

11.1 Orthogonal Structure on the Unit Ball

We consider orthogonal polynomials with respect to a weight function on the unit ball $\mathbb{B}^d$. The classical weight function on the unit ball is

$$W_\mu(x) := (1 - \|x\|^2)^{\mu - 1/2}, \qquad \mu \geq 0, \tag{11.1.1}$$

which is a special case of a more general weight function

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$$W_\kappa(x) := h_\kappa^2(x)(1-\|x\|^2)^{\kappa_{d+1}-1/2}, \quad h_\kappa(x) = \prod_{i=1}^{d} |x_i|^{\kappa_i}, \quad \kappa_i \ge 0. \tag{11.1.2}$$

Moreover, the weight function W_κ is a special case of

$$W_{\kappa,\mu}(x) := h_\kappa^2(x)(1-\|x\|^2)^{\mu-1/2}, \quad \mu > -1/2, \tag{11.1.3}$$

where h_κ is the weight function (7.3.1) invariant under the reflection group. To keep the notation simple, we shall stick mostly with W_κ below, for which we set $\lambda_\kappa := |\kappa| + \frac{d-1}{2}$, which is the same as Eq. (7.1.11) but with d replaced by $d+1$.

Definition 11.1.1. For $n \in \mathbb{N}_0$, let $\mathscr{V}_n^d(W_\kappa)$ denote the space of orthogonal polynomials of degree exactly n with respect to the inner product

$$\langle f, g \rangle_{W_\kappa} := a_\kappa \int_{\mathbb{B}^d} f(x) g(x) W_\kappa(x) \mathrm{d}x,$$

where a_κ is the normalization constant of W_κ, $a_\kappa := 1/\int_{\mathbb{B}^d} W_\kappa(x)\mathrm{d}x$.

From the Gram–Schmidt process applied to monomials, it follows that

$$\dim \mathscr{V}_n^d(W_\kappa) = \binom{n+d-1}{n}, \quad n = 0, 1, 2, \ldots.$$

The orthogonal structure on the ball is closely related to the corresponding structure on the unit sphere. We start with a simple relation on polynomials over these two domains.

Let $\mathbb{S}_+^{d+1}$ denote the upper hemisphere $\mathbb{S}_+^d := \{x \in \mathbb{S}^d : x_{d+1} \ge 0\}$. A simple-minded relation between the ball and the sphere is

$$x \in \mathbb{B}^d \iff (x, x_{d+1}) \in \mathbb{S}_+^d, \quad x_{d+1} = \sqrt{1-\|x\|^2}. \tag{11.1.4}$$

The domain $\mathbb{S}_+^d$ induces a symmetry in the polynomial space. Let $\Pi_n^+(\mathbb{S}^d)$ denote the subspace of elements in $\Pi_n(\mathbb{S}^d)$ that are even in the $(d+1)$th coordinate. The mapping Eq. (11.1.4) leads immediately to the following basic result.

Lemma 11.1.2. *For each $n \ge 0$, the equation*

$$\Pi_n(\mathbb{S}^d) = \Pi_n^d \cup x_{d+1} \Pi_{n-1}^d \tag{11.1.5}$$

holds in the sense that for each $P \in \Pi_n(\mathbb{S}^d)$, there exist unique elements $p \in \Pi_n^d$ and $q \in \Pi_{n-1}^d$ such that

$$P(x, x_{d+1}) = p(x) + x_{d+1} q(x), \qquad (x, x_{d+1}) \in \mathbb{S}^d.$$

In particular, there is a one-to-one correspondence between Π_n^d and $\Pi_n^+(\mathbb{S}^d)$.

Proof. Let $P \in \Pi_n(\mathbb{S}^d)$. We can write $P(x, x_{d+1}) = \sum p_j(x) x_{d+1}^j$ for some $p_j \in \Pi_{n-j}^d$. Using the fact that $x_{d+1}^2 = 1 - \|x\|^2$, we have $P(x, x_{d+1}) = p(x) + x_{d+1} q(x)$, where $p \in \Pi_n^d$ and $q \in \Pi_{n-1}^d$. In particular, if $P \in \prod_n^+(\mathbb{S}^d)$, then $P(x, x_{d+1}) = p(x)$. The uniqueness of p and q is evident. □

One immediate consequence of the relation (11.1.4) is the following lemma, proved in Lemma A.5.4.

Lemma 11.1.3. *For every integrable function on* $\mathbb{S}^d$,

$$\int_{\mathbb{S}^d} f(y) \mathrm{d}\sigma(y) = \int_{\mathbb{B}^d} \left[f\left(x, \sqrt{1 - \|x\|^2}\right) + f\left(x, -\sqrt{1 - \|x\|^2}\right) \right] \frac{\mathrm{d}x}{\sqrt{1 - \|x\|^2}}. \tag{11.1.6}$$

The weight function W_κ is closely related to the product weight function $h_\kappa^2(x) = \prod_{i=1}^{d+1} |x_i|^{2\kappa_i}$ for $x \in \mathbb{S}^d$, which is as defined in Eq. (7.1.5) but with d replaced by $d+1$. Indeed, by Eq. (11.1.4),

$$W_\kappa(x) = h_\kappa^2\left(x, \sqrt{1 - \|x\|^2}\right), \quad x \in \mathbb{B}^d. \tag{11.1.7}$$

In particular, by Eq. (11.1.6), the normalization constant a_κ of W_κ satisfies $a_\kappa = 2/\omega_{d+1}^\kappa$, where ω_{d+1}^κ is defined in Eq. (7.1.6).

Theorem 11.1.4. *Let* $\kappa = (\kappa', \kappa_{d+1})$ *and* $\kappa' = (\kappa_1, \ldots, \kappa_d)$. *Then*

$$\mathscr{H}_n^{d+1}\left(h_\kappa^2\right) = \mathscr{V}_n^d\left(W_\kappa\right) \oplus x_{d+1} \mathscr{V}_{n-1}^d\left(W_{\kappa', \kappa_{d+1}+1}\right). \tag{11.1.8}$$

Proof. Let $Y \in \mathscr{H}_n^{d+1}(h_\kappa^2)$. Using $\xi_{d+1}^2 = 1 - \xi_1 - \cdots - \xi_d^2$, we can write $Y(\xi) = P(\xi') + \xi_{d+1} Q(\xi')$. In particular, $P(\xi') = (Y(\xi) + Y(\xi', -\xi_{d+1}))/2$ and $Q(\xi') = (Y(\xi) - Y(\xi', -\xi_{d+1}))/(2\xi_{d+1})$. Consequently, by Eq. (11.1.6),

$$\int_{\mathbb{B}^d} P(x) q(x) W_\kappa(x) \mathrm{d}x = \int_{\mathbb{S}^d} Y(\xi) q(\xi') h_\kappa^2(\xi) \mathrm{d}\sigma = 0, \quad \forall q \in \Pi_{n-1}^d,$$

so that $P \in \mathscr{V}_n^d(W_\kappa)$. A similar consideration also shows $Q \in \mathscr{V}_{n-1}^d(W_{\kappa', \kappa_{d+1}+1})$, since $W_{\kappa', \kappa_{d+1}+1}(x) = W_\kappa(x)|x_{d+1}|$. Thus, the left-hand side of Eq. (11.1.8) is a subset of the right-hand side.

On the other hand, if $\{P_\alpha^n\}$ is a basis of $\mathscr{V}_n^d(W_\kappa)$ and $\{Q_\alpha^{n-1}\}$ is a basis of $\mathscr{V}_{n-1}^d(W_{\kappa', \kappa_{d+1}+1})$, then since W_κ is invariant under $\mathbb{Z}_2^d$, one can show by induction that the polynomials P_α^n and Q_α^n are sums of monomials of even degree when n is even and sums of monomials of odd degree when n is odd. Consequently, since $r^2 = x_1^2 + \cdots + x_{d+1}^2$, it follows that $r^n P_\alpha^n(\xi')$ and $r^n \xi_{d+1} Q_\alpha^{n-1}(\xi')$ are homogeneous polynomials of degree n. Furthermore, by Eq. (11.1.6), it is easy to see that $P_\alpha^n(\xi')$ and $\xi_{d+1} Q_\alpha^{n-1}(\xi')$ are orthogonal polynomials with respect to h_κ^2 on $\mathbb{S}^d$, so that they are elements of $\mathscr{H}_n^{d+1}(h_\kappa^2)$. This proves the other side of the inclusion of Eq. (11.1.8). □

As an immediate consequence of this theorem, the space $\mathscr{V}_n^d(W_\kappa)$ can be identified with the subspace of those elements in $\mathscr{H}_n^{d+1}(h_\kappa^2)$ that are even in x_{d+1},

$$\mathscr{V}_n^d(W_\kappa) = \operatorname{span}\left\{Y \in \mathscr{H}_n^{d+1}(h_\kappa^2) : Y(x,x_{d+1}) = Y(x,-x_{d+1})\right\}. \tag{11.1.9}$$

It is of interest to note that the space $\mathscr{H}_n^d$ of classical spherical harmonics, that is, $\kappa = 0$, corresponds to the space $\mathscr{V}_n^d(W_0)$, where $W_0(x) = 1/\sqrt{1-\|x\|^2}$ is the analogue of the Chebyshev weight of one variable. The space $\mathscr{V}_n^d(W_{1/2})$ of orthogonal polynomials with respect to the unit weight $W_{1/2}(x) = 1$, that is, the unweighted case, corresponds to $\mathscr{H}_n^d(h_\kappa^2)$ with $h_\kappa^2(x) = |x_{d+1}|$. Thus, even if we want to work only with the unweighted case on the unit ball, we still need to understand the weighted $\mathscr{H}_n^d(h_\kappa^2)$.

Using the relation in Theorem 11.1.4, we can deduce from Eq. (7.1.14), which shows that elements in $\mathscr{H}_n^{d+1}$ are eigenfunctions of $\Delta_{h,0}$, that the elements in $\mathscr{V}_n^d(W_\kappa)$ are the eigenfunctions of a differential–difference operator.

Theorem 11.1.5. *The orthogonal polynomials in $\mathscr{V}_n^d(W_\kappa)$ satisfy*

$$\mathscr{D}_{\kappa,\mathbb{B}} u = -n(n+2|\kappa|+d-1)u, \tag{11.1.10}$$

where, with Δ_h as defined in Eq. (7.1.2)*,*

$$\mathscr{D}_{\kappa,\mathbb{B}} := \Delta_h - \langle x,\nabla\rangle^2 - (2|\kappa|+d-1)\langle x,\nabla\rangle. \tag{11.1.11}$$

Furthermore, $\mathscr{D}_{\kappa,\mathbb{B}}$ is invariant under $\mathbb{Z}_2^d$. In particular, for the classical weight function W_μ, $\mathscr{D}_{\kappa,\mathbb{B}}$ is a second-order differential operator with Δ_h replaced by Δ, invariant under the rotation group $O(d)$.

Proof. The proof comes down to showing that $\Delta_{h,0}$ becomes, in terms of the coordinates $r,\xi_1,\dots,\xi_d$ of $y = r\xi$ with $\xi = (\xi_1,\dots,\xi_{d+1}) \in \mathbb{S}^d$ on the upper space $\{y \in \mathbb{R}^{d+1} : y_{d+1} \ge 0\}$,

$$\Delta_{h,0} = \Delta_h^{(\xi)} - \langle \xi,\nabla_\xi\rangle^2 - (2|\kappa|+d-1)\langle \xi,\nabla_\xi\rangle,$$

where $\Delta_h^{(\xi)}$ and ∇_ξ are acting on $(\xi_1,\dots,\xi_d) \in \mathbb{B}^d$. The proof is a tedious but standard exercise, and we omit the details. That $\mathscr{D}_{\kappa,\mathbb{B}}$ is invariant under $\mathbb{Z}_2^d$ follows from this connection, and the case W_μ follows from a simple computation. □

From Eq. (11.1.9), we can easily deduce an orthogonal basis of $\mathscr{V}_n^d(W_\kappa)$ from the basis of $\mathscr{H}_n^{d+1}(h_\kappa^2)$, say from Eq. (7.1.10). We shall not need an explicit basis but will need a formula for the reproducing kernel.

Let $P_n(W_\kappa;\cdot,\cdot)$ denote the reproducing kernel of $\mathscr{V}_n^d(W_\kappa)$. It is uniquely determined by the reproducing property

$$\langle P_n(W_\kappa;x,\cdot),q\rangle_{W_\kappa} = q(x), \quad \forall q \in \mathscr{V}_n^d(W_\kappa).$$

Let $\mathrm{proj}_n(W_\kappa; f)$ denote the projection operator from $L^2(W_\kappa, \mathbb{B}^d)$ to $\mathscr{V}_n^d(W_\kappa)$. Then

$$\mathrm{proj}_n(W_\kappa; f, x) = a_\kappa \int_{\mathbb{B}^d} f(y) P_n(W_\kappa; x, y) W_\kappa(y) \mathrm{d}y. \tag{11.1.12}$$

Let Z_n^κ be the reproducing kernel of $\mathscr{H}_n^{d+1}(h_\kappa^2)$, defined in Eq. (7.2.8) but with d replaced by $d+1$. By Eqs. (7.2.2) and (7.2.10), the reproducing kernel $Z_n^\kappa(\cdot,\cdot)$ has a closed formula, from which a closed formula of $P_n(W\kappa;\cdot,\cdot)$ follows readily. To state the result, we give the following definition.

Definition 11.1.6. Let V_κ denote the intertwining operator for $h_\kappa^2(x) = \prod_{i=1}^{d+1} |x_i|^{2\kappa_i}$ as given in Eq. (7.2.2) but with d replaced by $d+1$. Define

$$V_\kappa^{\mathbb{B}} f(x, x_{d+1}) := \frac{1}{2}\left[V_\kappa f(x, x_{d+1}) + V_\kappa f(x, -x_{d+1})\right], \quad x \in \mathbb{R}^d. \tag{11.1.13}$$

Theorem 11.1.7. *The reproducing kernel $P_n(W_\kappa; x, y)$ satisfies*

$$\begin{aligned} P_n(W_\kappa; x, y) &= \frac{1}{2}\left[Z_n^\kappa\left((x, x_{d+1}), (y, y_{d+1})\right) + Z_n^\kappa\left((x, x_{d+1}), (y, -y_{d+1})\right)\right] \\ &= \frac{n+\lambda_k}{\lambda_k} V_\kappa^{\mathbb{B}}\left[C_n^{\lambda_\kappa}\left(\langle (y, y_{d+1}), \cdot \rangle\right)\right](x, x_{d+1}), \end{aligned} \tag{11.1.14}$$

where $x_{d+1} = \sqrt{1-\|x\|^2}$ and $y_{d+1} = \sqrt{1-\|y\|^2}$. In particular,

$$\begin{aligned} P_n(W_\kappa; x, y) = c_\kappa \frac{n+\lambda_\kappa}{\lambda_\kappa} \int_{[-1,1]^{d+1}} & C_n^{\lambda_\kappa}(x_1 y_1 t_1 + \cdots + x_{d+1} y_{d+1} t_{d+1}) \\ & \times \left[\prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1}\right] (1-t_{d+1}^2)^{\kappa_{d+1}-1} \mathrm{d}t. \end{aligned} \tag{11.1.15}$$

Proof. Denote the right-hand side of Eq. (11.1.14) by $Q_n(x,y)$ temporarily. Let $\langle \cdot, \cdot \rangle_\kappa$ be the inner product defined by Eq. (7.1.4) with $\mathbb{S}^{d-1}$ replaced by $\mathbb{S}^d$. For $p \in \Pi_n^d$, by Eq. (11.1.6), we have then

$$\langle Q_n(x, \cdot), p \rangle_{W_\kappa} = \frac{1}{2}\left[\langle Z_n^\kappa((x, x_{d+1}), \cdot), \tilde{p} \rangle_\kappa + \langle Z_n^\kappa((x, -x_{d+1}), \cdot), \tilde{p} \rangle_\kappa\right],$$

where $\tilde{p}(x, x_{d+1}) = p(x)$. Since $\mathscr{V}_n^d(W_\kappa) \subset \mathscr{H}_n^d(h_\kappa^2)$ by Eq. (11.1.8), it follows from the reproducing property of Z_n^κ that Q_n is the reproducing kernel of $V_n^d(W_\kappa)$, which proves the first equality of Eq. (11.1.14). The second equality of Eq. (11.1.14) is an immediate consequence of Eqs. (7.2.2) and (7.2.10). Finally, Eq. (11.1.15) follows from Eq. (11.1.14) and the explicit formula of V_κ in Eq. (7.2.2). □

The formula in the case of the classical weight function W_μ is as follows:

Corollary 11.1.8. *For $\mu \ge 0$, let $\lambda_\mu = \mu + \frac{d-1}{2}$. If $\mu > 0$, then*

$$P_n(W_\mu; x, y) = c_\mu \frac{n+\lambda_\mu}{\lambda_\mu} \int_{-1}^{1} C_n^{\lambda_\mu}\left(\langle x, y\rangle + x_{d+1} y_{d+1} t\right)(1-t^2)^{\mu-1} \mathrm{d}t, \qquad (11.1.16)$$

where $x_{d+1} = \sqrt{1-\|x\|^2}$, $y_{d+1} = \sqrt{1-\|y\|^2}$, and c_μ is as in Eq. (7.2.3)*, from which the expression for $\mu = 0$ becomes*

$$P_n(W_0; x, y) = \frac{n+\lambda_0}{2\lambda_0}\left[C_n^{\lambda_o}\left(\langle x, y\rangle + x_{d+1} y_{d+1}\right) + C_n^{\lambda_o}\left(\langle x, y\rangle - x_{d+1} y_{d+1}\right)\right].$$

By Eqs. (11.1.6) and (11.1.13), we can deduce a Funk–Hecke formula for functions on the unit ball. The most interesting case appears to be the one for the classical weight function W_μ.

Theorem 11.1.9. *Let $\lambda = \mu + \frac{d-1}{2}$ and let f be an integrable function such that $\int_{-1}^{1} |f(t)|(1-t^2)^{\lambda-1/2}\mathrm{d}t$ is finite. Then for $P_n \in \mathcal{V}_n^d(W_\mu)$,*

$$a_\mu \int_{\mathbb{B}^d} f(\langle x, y\rangle) P_n(y) W_\kappa(y) \mathrm{d}y = \Lambda_n(f) P_n(x), \quad x \in \mathbb{B}^d, \qquad (11.1.17)$$

where $\Lambda_n(f)$ is a constant defined by

$$\Lambda_n(f) = c_\lambda \int_{-1}^{1} f(t) \frac{C_n^\lambda(t)}{C_n^\lambda(1)} (1-t^2)^{\lambda-\frac{1}{2}} \mathrm{d}t,$$

where a_μ is the normalization constant of W_μ and c_λ is a constant such that $\Lambda_0(1) = 1$.

Proof. We apply the Funk–Hecke formula in Theorem 7.2.7 with d replaced by $d+1$ to the function $f(\langle (x,0), \cdot\rangle)(y, y_{d+1})$ with the parameters $\kappa_1 = \cdots = \kappa_d = 0$ and $\kappa_{d+1} = \mu$. By Eq. (7.2.2), we have then

$$V_\kappa f(\langle (x,0), \cdot\rangle)(y, y_{d+1}) = c_\mu \int_{-1}^{1} f(\langle x, y\rangle)(1+t_{d+1})(1-t_{d+1}^2)\mathrm{d}t_{d+1} = f(\langle x, y\rangle),$$

so that Eq. (7.2.11) becomes, under such specifications,

$$\frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} f(\langle x, y\rangle) Y_n^h(y) |y_{d+1}|^{2\mu} \mathrm{d}\sigma(y) = \Lambda_n(f) Y_n^h(x,0), \quad (x,0) \in \mathbb{S}^{d-1},$$

which implies, by Eqs. (11.1.6) and (11.1.8), that Eq. (11.1.17) holds for $x \in \mathbb{B}^d$. □

An application of the above theorem is the following formula for the Gegenbauer polynomials:

Corollary 11.1.10. *Let $\lambda = \mu + \frac{d-1}{2}$. Then for $\eta, \xi \in \mathbb{S}^{d-1}$,*

$$a_\mu \int_{\mathbb{B}^d} C_n^\lambda(\langle x, \xi\rangle) C_n^\lambda(\langle x, \eta\rangle) W_\mu(x) \mathrm{d}x = \frac{\lambda}{n+\lambda} C_n^\lambda(\langle \eta, \xi\rangle). \qquad (11.1.18)$$

Proof. Setting $y = \eta \in \mathbb{S}^{d-1}$ in Eq. (11.1.16) shows that $C_n^\lambda(\langle \cdot, \eta\rangle) \in \mathscr{V}_n^d(W_\mu)$. Setting $f(t) = C_n^\lambda(t)$ and $P_n = C_n^\lambda(\langle \cdot, \eta\rangle)$ in Eq. (11.1.17) proves Eq. (11.1.18). □

In the case of $d = 2$ and $\mu = 1/2$, we have $\lambda = 1$, and $C_n^1 = U_n$ is the Chebyshev polynomial of the second kind, $U_n(x) = \sin(n+1)\theta/\sin\theta$ with $x = \cos\theta$. In this case, Eq. (11.1.18) leads to the following theorem.

Theorem 11.1.11. *An orthonormal basis for $\mathscr{V}_n^2(W_{1/2})$, the space of orthogonal polynomials with respect to the measure $\mathrm{d}x/\pi$ on $\mathbb{B}^2$, is given by*

$$\left\{ U_n\left(x\cos\frac{k\pi}{n+1} + y\sin\frac{k\pi}{n+1} \right) : \ 0 \le k \le n \right\}. \tag{11.1.19}$$

Proof. The zeros of the polynomial U_n are $\cos\frac{k\pi}{n+1}$, $1 \le k \le n$, and $U_n(1) = (n+1)$. If $\xi = (\cos\frac{k\pi}{n+1}, \sin\frac{k\pi}{n+1})$ and $\eta = (\cos\frac{j\pi}{n+1}, \sin\frac{j\pi}{n+1})$, then $U_n(\langle \xi, \eta\rangle) = U_n(\cos\frac{(k-j)\pi}{n+1}) = (n+1)\delta_{k,j}$. Applying Eq. (11.1.18) with these ξ and η gives the stated result. □

For $\mathscr{V}_n^d(W_\mu)$ in general, we do not have an orthonormal basis in such simplicity. We can, however, say the following:

Theorem 11.1.12. *Let $\lambda = \mu + \frac{d-1}{2}$. The space $\mathscr{V}_n^d(W_\mu)$ has a basis consisting of functions $C_n^\lambda(\langle x, \xi_i\rangle)$ with $\xi_i \in \mathbb{S}^{d-1}$ and $1 \le i \le \binom{n+d-1}{n}$.*

Proof. We need to show that there exist $\xi_i \in \mathbb{S}^{d-1}$, $1 \le i \le r_n^d = \binom{n+d-1}{n}$, such that $\{C_n^\lambda(\langle x, \xi_i\rangle) : 1 \le i \le r_n^d\}$ is a set of linearly independent polynomials, that is, if $\sum_i c_i C_n^\lambda(\langle x, \xi_i\rangle) = 0$, then $c_i = 0$ for all i. On setting $x = \xi_j$, we see that it is sufficient to show that the matrix $A_\mu := [C_n^\lambda(\langle \xi_i, \xi_j\rangle)]_{i,j=1}^{r_n^d}$ is nonsingular. Treating $\xi_1, \ldots, \xi_{r_n^d}$ as variables, it follows that the determinant of A_μ is a polynomial in these variables, so that it is nonzero for almost all choices of $\xi_1, \ldots, \xi_{r_n^d}$. This completes the proof. □

Using Eq. (11.1.16) and the fact that $P_n(W_\mu; x, y) = \sum_k P_k(x)P_k(y)$ for an orthonormal basis $\{P_k\}$ of $\mathscr{V}_n^d(W_\mu)$, it is easy to see that the matrix A_μ in the proof is always nonnegative definite. Thus, if A_μ is positive definite, then $C_n^\lambda(\langle x, \xi_i\rangle)$ consists of a basis for $\mathscr{V}_n^d(W_\mu)$. For $d = 2$ and $\mu > 1/2$, we can choose $\xi_i = (\cos\frac{i\pi}{n+1}, \sin\frac{i\pi}{n+1})$, using the fact that C_n^λ can be written as a sum of C_n^1 with positive coefficients. For $d > 2$, it is not clear how the points ξ_i should be chosen.

We end this section with another connection between orthogonal polynomials for W_μ on the ball and those on the sphere. Let $a_m^d := \dim \mathscr{H}_m^d$.

Proposition 11.1.13. *For $n \in \mathbb{N}$, $0 \le j \le n/2$, and $1 \le k \le a_{n-2j}^d$, define*

$$P_{j,k}^n(x) = P_j^{(\mu-\frac{1}{2}, n-2j+\frac{d-2}{2})}(2\|x\|^2 - 1) Y_{k,n-2j}(x),$$

where $Y_{k,n-2j}$ are harmonic polynomials such that $\{Y_{k,n-2j} : 1 \le k \le a^d_{n-2j}\}$ is a mutually orthogonal basis of $\mathscr{H}^d_{n-2j}$. Then $\{P^n_{j,k} : 0 \le j \le n/2, 1 \le k \le a^d_{n-2j}\}$ is a mutually orthogonal basis of the space $\mathscr{V}^d_n(W_\mu)$.

Proof. This basis comes from spherical–polar coordinates. Indeed, using the fact that $Y_{k,n-2j}$ is a homogeneous polynomial of degree $n-2j$, we obtain

$$\langle P^n_{j,k}, P^m_{j',k'}\rangle_{W_\mu} = a_\mu \int_0^1 P_j^{(\mu-\frac{1}{2}, n-2j+\frac{d-2}{2})}(2r^2-1) P_{j'}^{(\mu-\frac{1}{2}, m-2j'+\frac{d-2}{2})}(2r^2-1) \times r^{n+m-2j-2j'+d-1}(1-r^2)^{\mu-\frac{1}{2}}\mathrm{d}r \int_{\mathbb{S}^{d-1}} Y_{k,n-2j}(\xi) S_{k',m-2j'}(\xi)\mathrm{d}\sigma(\xi).$$

Since the $Y_{k,n-2j}$ are mutually orthogonal, changing variables $2r^2-1=t$ shows that the orthogonality of $P^n_{j,k}$ and $P^m_{j',k'}$ follows from that of the Jacobi polynomials. □

11.2 Convolution and Orthogonal Expansions

We denote by $\|\cdot\|_{W_\kappa,p}$ the norm of the space $L^p(W_\kappa, \mathbb{B}^d)$,

$$\|f\|_{W_\kappa,p} := \left(a_\kappa \int_{\mathbb{B}^d} |f(x)|^p W_\kappa(x)\mathrm{d}x\right)^{1/p}, \qquad 1 \le p < \infty,$$

and as usual, consider $C(\mathbb{B}^d)$ with $\|f\|_{W_\kappa,\infty} = \|f\|_\infty$ for $p=\infty$.

The operator $V_\kappa^{\mathbb{B}}$ can be used to define a convolution structure $*_{\kappa,\mathbb{B}}$. Recall that $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$ and $\lambda_k = |\kappa| + \frac{d-1}{2}$.

Definition 11.2.1. For $f \in L^1(W_\kappa, \mathbb{B}^d)$ and $g \in L^1(w_{\lambda_\kappa}, [-1,1])$,

$$(f *_{\kappa,\mathbb{B}} g)(x) := a_\kappa \int_{\mathbb{B}^d} f(y) \left(V_\kappa^{\mathbb{B}}[g(\langle \cdot, (x, x_{d+1})\rangle)]\right)(y, y_{d+1}) W_\kappa(y)\mathrm{d}y, \qquad (11.2.1)$$

where $x_{d+1} = \sqrt{1-\|x\|^2}$ and $y_{d+1} = \sqrt{1-\|y\|^2}$.

By Eqs. (11.1.12) and (11.1.14), the projection operator $\mathrm{proj}_n(W_\kappa; f)$ is a convolution

$$\mathrm{proj}_n(W_\kappa; f) = f *_{\kappa,\mathbb{B}} Z^\kappa_n, \qquad Z^\kappa_n(t) := \frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(t), \qquad (11.2.2)$$

which is an analogue of Eq. (7.4.4). In fact, this convolution structure is related to the convolution structure $*_\kappa$ on the sphere $\mathbb{S}^d$.

Theorem 11.2.2. *Let F be defined by $F(x,x_{d+1}) := f(x)$. Then*

$$\left(f *_{\kappa,\mathbb{B}} g\right)(x) = (F *_\kappa g)\left(x, \pm\sqrt{1-\|x\|^2}\right). \tag{11.2.3}$$

In particular, for $f \in L^p(W_\kappa, \mathbb{B}^d)$, $1 \le p < \infty$, or $f \in C(\mathbb{B}^d)$, $p = \infty$,

$$\|f *_{\kappa,\mathbb{B}} g\|_{W_\kappa,p} = \|F *_\kappa g\|_{\kappa,p}, \quad 1 \le p \le \infty. \tag{11.2.4}$$

Proof. From Eqs. (11.1.13) and (11.1.6) it follows that

$$\left(f *_{\kappa,\mathbb{B}} g\right)(x) = \frac{1}{\omega_{d+1}^\kappa} \int_{\mathbb{S}^d} f(y) V_\kappa[g(\langle \cdot, (x,x_{d+1})\rangle)](\hat{y}) h_\kappa^2(\hat{y}) \mathrm{d}\sigma(\hat{y}),$$

where $\hat{y} = (y, y_{d+1})$, which proves Eq. (11.2.3) with $x_{d+1} = \sqrt{1-\|x\|^2}$. Since

$$V_\kappa[g(\langle \cdot, (x,x_{d+1})\rangle)](y, -y_{d+1}) = V_\kappa[g(\langle \cdot, (x,-x_{d+1})\rangle)](y, y_{d+1}),$$

a change of variable $y_{d+1} \mapsto -y_{d+1}$ in the last integral proves Eq. (11.2.3) with $x_{d+1} = -\sqrt{1-\|x\|^2}$. Equation (11.2.4) follows from Eqs. (11.2.3) and (11.1.6). □

As a consequence of the relation (11.2.3), Young's inequality holds for the convolution $*_{\kappa,\mathbb{B}}$: for $p,q,r \ge 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$, $f \in L^q(W_\kappa, \mathbb{B}^d)$, and $g \in L^r(w_{\lambda_\kappa}; [-1,1])$,

$$\|f *_{\kappa,\mathbb{B}} g\|_{W_\kappa,p} \le \|f\|_{W_\kappa,q} \|g\|_{\lambda_\kappa,r}. \tag{11.2.5}$$

Furthermore, an analogue of Theorem 7.4.3 holds, further justifying the definition of $*_{\kappa,\mathbb{B}}$ as a convolution. By Eqs. (11.2.2) and (11.2.3), we immediately deduce that

$$\operatorname{proj}_n(W_\kappa; f, x) = \operatorname{proj}_n^\kappa F\left(x, \pm\sqrt{1-\|x\|^2}\right), \quad F(x,x_{d+1}) = f(x). \tag{11.2.6}$$

The Fourier orthogonal series with respect to W_κ on the ball $\mathbb{B}^d$ are defined in terms of $\mathscr{V}_n^d(W_\kappa)$. For $f \in L^2(W_\kappa, \mathbb{B}^d)$,

$$f(x) = \sum_{n=0}^\infty \operatorname{proj}_n(W_\kappa; f, x), \tag{11.2.7}$$

and an analogue of Eq. (2.2.2) follows from the usual Hilbert space theory. For convergence of the series (11.2.7) beyond the L^2 setting, we again consider summability methods. We denote by $S_n^\delta(W_\kappa; f)$ the Cesàro (C, δ) means of the series (11.2.7),

$$S_n^\delta(W_\kappa; f) := \frac{1}{A_n^\delta} \sum_{j=0}^{n} A_{n-j}^\delta \operatorname{proj}_j(W_\kappa; f) = f *_{\kappa,\mathbb{B}} K_n^\delta(W_\kappa), \tag{11.2.8}$$

where $K_n^\delta(W;t) = k_n^\delta\left(w_{\lambda_\kappa}; 1, t\right)$, just as in Eq. (7.4.8). Then we deduce immediately from Theorems 11.2.2 and 7.4.4 the following result.

Theorem 11.2.3. *The Cesàro means of the orthogonal expansions with respect to W_κ on $\mathbb{B}^d$ satisfy the following conditions:*

1. *If $\delta \geq 2\lambda_\kappa + 1$, then $S_n^\delta(W_\kappa)$ is a nonnegative operator.*
2. *If $\delta > \lambda_\kappa$, then $S_n^\delta(W_\kappa; f)$ converges to f in $L^p(W_\kappa; \mathbb{B}^d)$ for $1 \leq p \leq \infty$.*

We can also define a translation operator $T_\theta(W_\kappa; f)$.

Definition 11.2.4. For $0 \leq \theta \leq \pi$, the translation operator $T_\theta(W_\kappa)$ is defined by

$$\operatorname{proj}_n\left(W_\kappa; T_\theta(W_\kappa; f)\right) = \frac{C_n^{\lambda_\kappa}(\cos\theta)}{C_n^{\lambda_\kappa}(1)} \operatorname{proj}_n(W_\kappa; f), \quad n = 0, 1, \ldots. \tag{11.2.9}$$

This operator is closely related to the translation operator $T_\theta^\kappa f$.

Proposition 11.2.5. *The translation operator $T_\theta(W_\kappa)$ is well defined for all $f \in L^1(W_\kappa, \mathbb{B}^d)$, and it has the following properties:*

(i) *Let $F(x, x_{d+1}) = f(x)$. Then $T_\theta(W_\kappa; f) = T_\theta^\kappa F(x, \sqrt{1 - \|x\|^2})$.*
(ii) *For $f \in L^2(W_\kappa, \mathbb{B}^d)$ and $g \in L^1(w_{\lambda_\kappa}, [-1, 1])$,*

$$(f *_{\kappa,\mathbb{B}} g)(x) = c_{\lambda_\kappa} \int_0^\pi T_\theta(W_\kappa; f, x) g(\cos\theta)(\sin\theta)^{2\lambda_\kappa} d\theta. \tag{11.2.10}$$

(iii) *$T_\theta(W_\kappa; f)$ preserves positivity, i.e., $T_\theta(W_\kappa; f) \geq 0$ if $f \geq 0$.*
(iv) *For $f \in L^p(W_\kappa, \mathbb{B}^d)$, $1 \leq p < \infty$, or $f \in C(\mathbb{B}^d)$,*

$$\|T_\theta(W_\kappa, f)\|_{W_\kappa,p} \leq \|f\|_{W_\kappa,p} \quad \textit{and} \quad \lim_{\theta \to 0^+} \|T_\theta(W_\kappa, f) - f\|_{W_\kappa,p} = 0. \tag{11.2.11}$$

Proof. The first item follows form the definition and Eq. (11.2.6). Using (i) and Eq. (11.2.6), all other properties follow from the corresponding ones in Proposition 7.4.7. □

In the case of the classical weight function W_μ, the translation operator $T_\theta(W_\mu)$ can be expressed as an integral operator. Let I denote the identity matrix and

$$A(x) := (1 - \|x\|^2) I + x^T x, \quad x = (x_1, \ldots, x_d) \in \mathbb{B}^d.$$

Theorem 11.2.6. *For W_μ on $\mathbb{B}^d$, the generalized translation operator is given by*

$$T_\theta(W_\mu; f, x) = b_\mu (1-\|x\|^2)^{\frac{d-1}{2}} \int_\Omega f\left(\cos\theta x + \sin\theta \sqrt{1-\|x\|^2}\, u\right) \times \left(1 - uA(x)u^T\right)^{\mu-1} \mathrm{d}u, \tag{11.2.12}$$

where Ω is the ellipsoid $\Omega = \{u : uA(x)u^T \le 1\}$ in $\mathbb{R}^d$ and b_μ is the normalization constant of W_μ.

Proof. Let $T_\theta^* f$ denote the right-hand side of Eq. (11.2.12). By Eq. (11.2.9), we need to show that $T_\theta^* f = f$ for all $f \in \mathscr{V}_n^d(W_\mu)$. By Theorem 11.1.12, it is sufficient to show that with $\lambda = \mu + \frac{d-1}{2}$,

$$T_\theta^* C_n^\lambda(\langle x, y\rangle) = \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} C_n^\lambda(\langle x, y\rangle), \quad y \in \mathbb{S}^d. \tag{11.2.13}$$

The matrix $A(x)$ has two distinguished eigenvalues; one is $r = 1$ with an eigenvector x, and the other is $r = \sqrt{1-\|x\|}$, repeated $d-1$ times, with eigenspace $\{y : \langle x, y\rangle = 0\}$. Let $U(x)$ denote the unitary matrix determined by its first column $x/\|x\|$. Then

$$A(x) = U(x)\Lambda(x)U(x)^T, \quad \Lambda(x) = \mathrm{diag}\left\{1, \sqrt{1-\|x\|^2}, \ldots, \sqrt{1-\|x\|^2}\right\}.$$

Changing variables $u \mapsto v = uU(x)$, the quadratic form becomes

$$uA(x)u^T = v\Lambda(x)v^T = v_1^2 + \sqrt{1-\|x\|^2}(v_2^2 + \cdots + v_d^2),$$

which suggests one more change of variables $v \mapsto s = \sqrt{1-\|x\|^2}\, vD^{-1}(x)$, in which

$$D(x) = \mathrm{diag}\left\{\sqrt{1-\|x\|^2}, 1, \ldots 1\right\},$$

so that the quadratic form becomes $uA(x)u^T = ss^T = \|x\|$ and the domain $uA(x)u^T \le 1$ becomes $\mathbb{B}^d$ in variables s. Since $U(x)$ is unitary, $\mathrm{d}u = \mathrm{d}v = \mathrm{d}s/(1-\|x\|^2)^{(d-1)/2}$. Consequently, we obtain

$$T_\theta^* C_n^\lambda(\langle x, y\rangle) = a_\kappa \int_{\mathbb{B}^d} C_n^\lambda\left(\cos\theta\langle x, y\rangle + \sin\theta\langle s, yU(x)D(x)\rangle\right)(1-\|s\|^2)^{\mu-1}\mathrm{d}s,$$

where we have used $\langle sD(x)U(x)^T, y\rangle = \langle s, yU(x)D(x)\rangle$. Since the first column of $U(x)$ is $x/\|x\|$ and U is unitary, the norm of the vector $yU(x)D(x)$ is

$$\|yU(x)D(x)\|^2 = yU(x)D^2(x)U^T(x)y^T = yy^T - yU(I - D(x)^2)U^T y^T = 1 - \langle x, y\rangle^2,$$

since $\|y\| = 1$ and $I - D^2 = \mathrm{diag}\{\|x\|^2, 0, \ldots, 0\}$. Hence, using the formula (A.5.2), we conclude that

$$T_\theta^* C_n^\lambda(\langle x,y\rangle) = c_\lambda \int_{-1}^1 C_n^\lambda\left(\cos\theta\langle x,y\rangle + \sin\theta\sqrt{1-\langle x,y\rangle^2}\,t\right)(1-t^2)^{\lambda-1}\mathrm{d}t.$$

The product formula (B.2.9) for the Gegenbauer polynomials then establishes Eq. (11.2.13) and completes the proof. □

11.3 Maximal Functions and a Multiplier Theorem

In analogy to the Definition 7.5.1, we define a maximal function on the unit ball.

Definition 11.3.1. For $f \in L^1(W_\kappa, \mathbb{B}^d)$, the maximal function $\mathscr{M}_\kappa^{\mathbb{B}} f$ is defined by

$$\mathscr{M}_\kappa^{\mathbb{B}} f(x) = \sup_{0<\theta\le\pi} \frac{\int_0^\theta T_\phi(W_\kappa; |f|, x)(\sin\phi)^{2\lambda_\kappa}\mathrm{d}\phi}{\int_0^\theta (\sin\phi)^{2\lambda_\kappa}\mathrm{d}\phi}. \tag{11.3.1}$$

Since $T_\phi(W_\kappa; f)$ is related to $T_\theta^\kappa F$, the maximal function $\mathscr{M}_\kappa^{\mathbb{B}} f$ is related to $\mathscr{M}_\kappa f$ defined in Definition 7.5.1 with d replaced by $d+1$.

Proposition 11.3.2. *For $f \in L^1(W_\kappa, \mathbb{B}^d)$, define $F(x, x_{d+1}) = f(x)$. Then*

$$\mathscr{M}_\kappa^{\mathbb{B}} f(x) = \mathscr{M}_\kappa F\left(x, \sqrt{1-\|x\|^2}\right). \tag{11.3.2}$$

Furthermore, define

$$\mathsf{e}(x,\theta) := \left\{(y, y_{d+1}) : \langle x,y\rangle + \sqrt{1-\|x\|^2}\,y_{d+1} \ge \cos\theta, \quad y_{d+1} \ge 0\right\}.$$

Then an alternative formula for $\mathscr{M}_\kappa^{\mathbb{B}} f$ is

$$\mathscr{M}_\kappa^{\mathbb{B}} f(x) = \sup_{0\le\theta\le\pi} \frac{\int_{\mathbb{B}^d} |f(y)| V_\kappa^{\mathbb{B}}\left[\chi_{\mathsf{e}(x,\theta)}\right]\left(y, \sqrt{1-\|y\|^2}\right) W_\kappa(y)\mathrm{d}y}{\int_{\mathbb{B}^d} V_\kappa^{\mathbb{B}}\left[\chi_{\mathsf{e}(x,\theta)}\right]\left(y, \sqrt{1-\|y\|^2}\right) W_\kappa(y)\mathrm{d}y}. \tag{11.3.3}$$

Proof. The first equation is a direct consequence of (i) in Proposition 11.2.5 and the definitions of the two maximal functions. To prove the second equation, by Eqs. (11.3.2) and (7.5.3), it suffices to show that

$$\int_{\mathbb{S}^d} |F(u)| V_\kappa\left[\chi_{\mathsf{b}(X,\theta)}\right](u) h_\kappa^2(u)\mathrm{d}\sigma = 2\int_{\mathbb{B}^d} |f(y)| V_\kappa^{\mathbb{B}}\left[\chi_{\mathsf{e}(x,\theta)}\right](Y) W_\kappa(y)\mathrm{d}y,$$

where $X = (x, \sqrt{1-\|x\|^2})$ and $Y = (y, \sqrt{1-\|y\|^2})$. By Eqs. (11.1.13) and (11.1.6), it suffices to show that

$$V_\kappa^{\mathbb{B}}\left[\chi_{\mathtt{b}(X,\theta)}\right]\left(y,\sqrt{1-\|y\|^2}\right)=2V_\kappa^{\mathbb{B}}\left[\chi_{\mathtt{e}(x,\theta)}\right]\left(y,\sqrt{1-\|y\|^2}\right).$$

Let $\mathtt{e}_-(x,\theta)=\{(y,y_{d+1}:(y,-y_{d+1})\in\mathtt{e}(x,\theta)\}$. By the definition of $V_\kappa^{\mathbb{B}}$, it is easy to see that

$$V_\kappa^{\mathbb{B}}\left[\chi_{\mathtt{e}_-(x,\theta)}\right](y,y_{d+1})=V_\kappa^{\mathbb{B}}\left[\chi_{\mathtt{e}(x,\theta)}\right](y,-y_{d+1}).$$

Since $\mathtt{b}(X,\theta)=\mathtt{e}(x,\theta)\cup\mathtt{e}_-(x,\theta)$ and $\mathtt{e}(x,\theta)\cap\mathtt{e}_-(x,\theta)$ has measure zero, the desired equation follows from the fact that $\mathscr{M}_\kappa^{\mathbb{B}}f(x,x_{d+1})$ is an even function in x_{d+1}. □

As we will show later in this section, $\mathscr{M}_\kappa^{\mathbb{B}}f$ is closely related to a weighted Hardy–Littlewood maximal function. The following result shows that this maximal function is handy for dealing with the convolution operators on the ball.

Theorem 11.3.3. *Assume that $g\in L^1([-1,1],w_{\lambda_\kappa})$ and $|g(\cos\theta)|\le k(\theta)$ for all θ, where $k(\theta)$ is a continuous, nonnegative, and decreasing function on $[0,\pi]$. Then for $f\in L^1(W_\kappa,\mathbb{B}^d)$,*

$$|(f*_{\kappa,\mathbb{B}}g)(x)|\le c\mathscr{M}_\kappa^{\mathbb{B}}(|f|)(x),\quad x\in\mathbb{B}^d,$$

where $c=\int_0^\pi k(\theta)(\sin\theta)^{2\lambda_\kappa}\mathrm{d}\theta$.

Proof. Because of Eqs. (11.2.3) and (11.3.2), this is a consequence of Theorem 7.5.6. □

The maximal function is of weak type $(1,1)$, for which we need to define

$$\operatorname{meas}_\kappa^{\mathbb{B}}E:=\int_E W_\kappa(x)\mathrm{d}x,\qquad E\subset\mathbb{B}^d.$$

Theorem 11.3.4. *If $f\in L^1(W_\kappa;\mathbb{B}^d)$, then $\mathscr{M}_k^{\mathbb{B}}$ satisfies*

$$\operatorname{meas}_\kappa^{\mathbb{B}}\left\{x\in\mathbb{B}^d:\mathscr{M}_\kappa^{\mathbb{B}}f(x)\ge\alpha\right\}\le c\frac{\|f\|_{W_\kappa,1}}{\alpha},\ \ \forall\alpha>0.$$

Furthermore, if $f\in L^p(W_\kappa;\mathbb{B}^d)$ for $1<p\le\infty$, then $\|\mathscr{M}_\kappa^{\mathbb{B}}f\|_{W_\kappa,p}\le c\|f\|_{W_\kappa,p}$.

Proof. Since $\mathscr{M}_\kappa^{\mathbb{B}}f(x)=\mathscr{M}_\kappa F(x,\sqrt{1-\|x\|^2})$, it follows from Eq. (11.1.6) that

$$\begin{aligned}\operatorname{meas}_\kappa^{\mathbb{B}}\left\{x\in\mathbb{B}^d:\mathscr{M}_\kappa^{\mathbb{B}}f(x)\ge\alpha\right\}&=\int_{\mathbb{B}^d}\chi_{\{\mathscr{M}_\kappa^{\mathbb{B}}f(x)\ge\alpha\}}(x)W_\kappa(x)\mathrm{d}x\\&=\int_{\mathbb{S}_+^d}\chi_{\{\mathscr{M}_\kappa F(y)\ge\alpha\}}(y)h_\kappa^2(y)\mathrm{d}\sigma(y).\end{aligned}$$

Enlarging the domain of the last integral to all of $\mathbb{S}^d$ shows that

$$\mathrm{meas}_\kappa^{\mathbb{B}}\left\{x\in\mathbb{B}^d : \mathscr{M}_\kappa^{\mathbb{B}} f(x)\ge\alpha\right\}\le \mathrm{meas}_\kappa\left\{y\in\mathbb{S}^d : \mathscr{M}_\kappa F(y)\ge\alpha\right\}.$$

Consequently, by Theorem 7.5.3, we obtain

$$\mathrm{meas}_\kappa^{\mathbb{B}}\left\{x\in\mathbb{B}^d : \mathscr{M}_\kappa^{\mathbb{B}} f(x)\ge\alpha\right\}\le c\frac{\|F\|_{\kappa,1}}{\alpha},$$

from which the weak $(1,1)$ inequality follows from $\|F\|_{\kappa,1}=\|f\|_{W_\kappa,1}$. Since $\mathscr{M}_\kappa^{\mathbb{B}} f$ is evidently of strong type (∞,∞) by Eq. (11.3.3), this completes the proof. □

The connection (11.2.6) also allows us to deduce a multiplier theorem for orthogonal expansions with respect to W_κ from Theorem 7.5.10.

Theorem 11.3.5. *Let $\{\mu_j\}_{j=0}^\infty$ be a sequence of real numbers that satisfies*

1. $\sup_j|\mu_j|\le c<\infty$,
2. $\sup_j 2^{j(k-1)}\sum_{l=2^j}^{2^{j+1}}|\Delta^k u_l|\le c<\infty$,

where k is the least integer greater than or equal to $\lambda_\kappa+1$, and $\lambda_\kappa=\frac{d-1}{2}+\sum_{j=1}^{d+1}\kappa_j$. Then $\{\mu_j\}$ defines an $L^p(W_\kappa;\mathbb{B}^d)$, $1<p<\infty$, multiplier; that is,

$$\left\|\sum_{j=0}^\infty \mu_j\,\mathrm{proj}_j^\kappa f\right\|_{W_\kappa,p}\le c\|f\|_{W_\kappa,p},\qquad 1<p<\infty,$$

where c is independent of $\{\mu_j\}$ and f.

We state the above results for W_κ defined in Eq. (11.1.2). The proof shows, however, that they remain valid for the more general weight functions (11.1.3) associated with any finite reflection group.

In the case of W_κ in Eq. (11.1.2), we can also define a weighted Hardy–Littlewood maximal function. For this, we need to consider an appropriate distance on $\mathbb{B}^d$, which should take into consideration the distinction between interior points and boundary points of the ball. The distance is defined by

$$\mathtt{d}_{\mathbb{B}}(x,y)=\arccos\left(\langle x,y\rangle+\sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\right),\ \ x,y\in\mathbb{B}^d,$$

which is the projection of the geodesic distance of $\mathbb{S}_+^d$ on $\mathbb{B}^d$. Thus, one can define the weighted Hardy–Littlewood maximal function as

$$M_\kappa^{\mathbb{B}} f(x):=\sup_{0<\theta\le\pi}\frac{\int_{\mathtt{d}_{\mathbb{B}}(x,y)\le\theta}|f(y)|W_\kappa(y)\mathrm{d}y}{\int_{\mathtt{d}_{\mathbb{B}}(x,y)\le\theta}W_\kappa(y)\mathrm{d}y},\ \ x\in\mathbb{B}^d.$$

To handle the denominator of this maximal function, we need the following result, which shows, in particular, that $W_\kappa(y)$ is a doubling weight on $\mathbb{B}^d$.

Lemma 11.3.6. *If $\tau = (\tau_1, \ldots, \tau_{d+1}) > -\frac{1}{2}\mathbb{1}$, then for every $x \in \mathbb{B}^d$ and $0 \le \theta \le \pi$,*

$$\int_{\mathtt{d}_{\mathbb{B}}(y,x) \le \theta} W_\tau(y) \mathrm{d}y \sim \theta^d \prod_{j=1}^{d+1} (|x_j| + \theta)^{2\tau_j},$$

where $x_{d+1} = \sqrt{1 - \|x\|^2}$ and $W_\tau(y)$ is defined as in Eq. (11.1.2), but with $\tau_i > -\frac{1}{2}$.

Proof. Let $X = (x, x_{d+1})$. Recall that $\mathtt{c}(X,\theta) = \{y \in \mathbb{S}^d : \mathtt{d}(X,y) \le \theta\}$. Set

$$\mathtt{c}_+(X,\theta) := \{(y_1, \ldots, y_{d+1}) \in \mathtt{c}(X,\theta) : \ y_{d+1} \ge 0\}.$$

From Eq. (11.1.6), it follows that

$$\int_{\mathtt{d}_{\mathbb{B}}(y,x) \le \theta} W_\tau(y) \, \mathrm{d}y = \int_{\mathtt{c}_+(X,\theta)} h_\tau^2(z) \, \mathrm{d}\sigma(z), \tag{11.3.4}$$

which, together with Eq. (5.1.9), implies the desired upper estimate

$$\int_{\mathtt{d}_{\mathbb{B}}(y,x) \le \theta} W_\tau(y) \, \mathrm{d}y \le \int_{\mathtt{c}(X,\theta)} h_\tau^2(z) \, \mathrm{d}\sigma(z) \le c\, \theta^d \prod_{j=1}^{d+1} (|x_j| + \theta)^{2\tau_j}.$$

To prove the lower estimate, we choose a point $z = (z_1, \ldots, z_{d+1}) \in \mathtt{c}(X, \frac{\theta}{2})$ with $z_{d+1} \ge \varepsilon\theta$, where $\varepsilon > 0$ is a sufficiently small constant depending only on d. Clearly, $\mathtt{c}(z, \frac{\varepsilon\theta}{2}) \subset \mathtt{c}_+(X,\theta)$. Hence, by Eq. (11.3.4), we obtain

$$\int_{\mathtt{d}_{\mathbb{B}}(y,x) \le \theta} W_\tau(y) \, \mathrm{d}y \ge \int_{\mathtt{c}(z, \frac{\varepsilon\theta}{2})} h_\tau^2(y) \, \mathrm{d}\sigma(y) \sim \theta^d \prod_{j=1}^{d+1} (|z_j| + \theta)^{2\tau_j},$$

on using Eq. (5.1.9). Using the fact that $z \in \mathtt{c}(X,\theta)$, the last expression is easily seen to be $\sim \theta^d \prod_{j=1}^{d+1} (|x_j| + \theta)^{2\tau_j}$, which proves the desired lower estimate. □

We are in a position to prove an analogue of Theorem 7.6.4.

Theorem 11.3.7. *Let $f \in L^1(W_\kappa; \mathbb{B}^d)$. Then for every $x \in \mathbb{B}^d$,*

$$\mathscr{M}_\kappa^{\mathbb{B}} f(x) \le c \sum_{\varepsilon \in \mathbb{Z}_2^d} M_\kappa^{\mathbb{B}} f(x\varepsilon). \tag{11.3.5}$$

Proof. As shown in the proof of Proposition 11.3.2,

$$\begin{aligned} \int_{\mathbb{B}^d} V_\kappa^{\mathbb{B}}[\chi_{\mathtt{e}(x,\theta)}](Y) W_\kappa(y) \, \mathrm{d}y &= \frac{1}{2} \int_{\mathbb{S}^d} V_\kappa[\chi_{\mathtt{b}(X,\theta)}](y) h_\kappa^2(y) \mathrm{d}y \\ &\sim \int_0^\theta (\sin\phi)^{2\lambda_k} \mathrm{d}\phi \sim \theta^{2\lambda_\kappa + 1}. \end{aligned}$$

The proof of Eq. (11.3.5) follows almost exactly the proof of Theorem 7.6.4. The main effort lies in the proof of the following inequality:

$$V_\kappa^{\mathbb{B}}[\chi_{\mathsf{e}(x,\theta)}](Y) \le c \prod_{j=1}^{d+1} \frac{\theta^{2\kappa_j}}{(|x_j|+\theta)^{2\kappa_j}} \chi_{\{y\in\mathbb{B}^d : \mathsf{d}_{\mathbb{B}}(\bar{x},\,\bar{y})\le\theta\}}(y), \tag{11.3.6}$$

where $x_{d+1} = \sqrt{1-\|x\|^2}$ and $\bar{z} = (|z_1|,\dots,|z_d|)$ for $z=(z_1,\dots,z_d)\in\mathbb{B}^d$. However, using the explicit formula of V_κ and the fact that $y_{d+1} = \sqrt{1-\|y\|^2}$, we have

$$\begin{aligned} V_\kappa^{\mathbb{B}}[\chi_{\mathsf{e}(x,\theta)}](Y) &= \frac{1}{2}\left(V_\kappa[\chi_{\mathsf{e}(x,\theta)}](y,y_{d+1}) + V_\kappa[\chi_{e(x,\theta)}](y,-y_{d+1})\right) \\ &= c_\kappa \int_D \left[\prod_{j=1}^{d}(1+t_j)(1-t_j^2)^{\kappa_j-1}\right](1-t_{d+1}^2)^{\kappa_{d+1}-1}\,\mathrm{d}t, \end{aligned}$$

where the domain D is given by

$$D := \left\{(t_1,\dots,t_d,t_{d+1}) \in [-1,1]^d\times[0,1] : \sum_{j=1}^{d+1} t_j x_j y_j \ge \cos\theta\right\}.$$

This last integral can be estimated exactly as in the proof of Lemma 7.6.3, which yields the desired inequality (11.3.6). □

As a consequence of the above theorem, we have the following analogues of Theorems 7.6.5 and 7.6.7.

Corollary 11.3.8. *If* $-\frac{\mathbb{1}}{2} < \tau \le \kappa$ *and* $f \in L^1(W_\tau;\mathbb{B}^d)$, *then* $\mathscr{M}_\kappa f$ *satisfies*

$$\mathrm{meas}_\tau^{\mathbb{B}}\{x : \mathscr{M}_\kappa^{\mathbb{B}} f(x) \ge \alpha\} \le c\frac{\|f\|_{W_\tau,1}}{\alpha}, \qquad \forall\alpha>0.$$

Furthermore, if $1<p<\infty$, $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$ *and* $f\in L^p(W_\tau;\mathbb{B}^d)$, *then*

$$\left\|\mathscr{M}_k^{\mathbb{B}} f\right\|_{W_\tau,p} \le c\|f\|_{W_\tau,p}.$$

Corollary 11.3.9. *Let* $1<p<\infty$, $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$, *and let* $\{f_j\}_{j=1}^\infty$ *be a sequence of functions. Then*

$$\left\|\left(\sum_{j=1}^\infty (\mathscr{M}_\kappa^{\mathbb{B}} f_j)^2\right)^{1/2}\right\|_{W_\tau,p} \le c\left\|\left(\sum_{j=1}^\infty |f_j|^2\right)^{1/2}\right\|_{W_\tau,p}.$$

11.4 Projection Operators and Cesàro Means on the Ball

The Cesàro (C,δ) mean $S_n^\delta(W_\kappa;f)$ of the orthogonal expansions with respect to W_κ on $\mathbb{B}^d$ is again an integral operator,

$$S_n^\delta(W_\kappa;f,x) = a_\kappa \int_{\mathbb{B}^d} f(y) K_n^\delta(W_\kappa;x,y) W_\kappa(y) \mathrm{d}y.$$

The kernel $K_n^\delta(W_\kappa;\cdot,\cdot)$ is closely related to the kernel $K_n^\delta(h_\kappa^2;\cdot,\cdot)$ associated with h_κ^2. Indeed, by Eq. (11.1.14),

$$K_n^\delta(W_\kappa;x,y) = \frac{1}{2}\left[K_n^\delta(h_\kappa^2;X,(y,y_{d+1})) + K_n^\delta(h_\kappa^2;X,(y,-y_{d+1}))\right], \qquad (11.4.1)$$

where $X=(x,x_{d+1})$, $x_{d+1}=\sqrt{1-\|x\|^2}$, and $y_{d+1}=\sqrt{1-\|y\|^2}$. Most of the results on $S_n^\delta(W_\kappa;f)$ can be deduced from this relation. Recall that

$$\sigma_\kappa = \frac{d-1}{2} + |\kappa| - \kappa_{\min} \quad \text{with} \quad \kappa_{\min} = \min_{1\le i\le d+1} \kappa_i.$$

Theorem 11.4.1. *Let W_κ be defined as in Eq.* (11.1.2) *and let $\delta > -1$. For $p=1$ or ∞,*

$$\|S_n^\delta(W_\kappa)\|_{W_k,p} \sim \begin{cases} 1, & \delta > \sigma_\kappa, \\ \log n, & \delta = \sigma_\kappa, \\ n^{-\delta+\sigma_\kappa}, & -1 < \delta < \sigma_\kappa. \end{cases}$$

In particular, $S_n^\delta(W_\kappa;f)$ converges in $L^p(W_\kappa;\mathbb{B}^d)$ for all $1 \le p \le \infty$ if and only if $\delta > \sigma_\kappa$. Furthermore,

$$\|\operatorname{proj}_n(W_\kappa)\|_{W_k,p} \sim n^{\sigma_\kappa},$$

unless $\min_{1\le i\le d+1}\kappa_i = \kappa_{d+1}$ and n is odd, in which case the norm has an upper bound of $c\,n^{\sigma_\kappa}$.

Proof. For $p=1$ and ∞, the operator norm of $S_n^\delta(W_\kappa)$ is given by

$$\|S_n^\delta(W_\kappa)\|_{W_k,\infty} = \|S_n^\delta(W_\kappa)\|_{W_k,1} = \sup_{x\in\mathbb{B}^d} a_\kappa \int_{\mathbb{B}^d} \left|K_n^\delta(W_\kappa;x,y)\right| W_\kappa(y)\mathrm{d}y.$$

For the estimate from above, we use Eqs. (11.4.1) and (11.1.6) to deduce that

$$a_\kappa \int_{\mathbb{B}^d} \left|K_n^\delta(W_\kappa;x,y)\right| W_\kappa(y)\mathrm{d}y \le \frac{1}{2\omega_d^\kappa}\int_{\mathbb{S}^d} \left|K_n^\delta(h_\kappa^2;x,y)\right| h_\kappa^2(y)\mathrm{d}\sigma(y),$$

from which the upper estimate follows readily from Theorem 8.1.1. For the estimate from below, we need to consider two cases. In the first case, $\kappa_{\min} = \kappa_i$, $1 \le k \le d$, we may assume that $j = 1$ and choose $x = e_1$. Then $K_n^\delta(W_\kappa; e_1, y) = k_n^\delta(v_{\sigma_\kappa,\kappa_1}; 1, y_1)$ as in Eq. (8.4.5), so that the proof follows from that of Theorem 8.1.1. In the second case, $\kappa_{\min} = \kappa_{d+1}$, we choose $x = (0, \dots, 0)$. By Eqs. (11.1.15) and (B.3.5), it follows that

$$P_n(W_\kappa; 0, y) = \tilde{C}_n^{\mu,\lambda_\kappa-\mu}(0)\tilde{C}_n^{\mu,\lambda_\kappa-\mu}(\|y\|), \quad \mu = k_{\min}, \tag{11.4.2}$$

where $\tilde{C}_n^{\mu,\lambda}$ stands for the orthonormal generalized Gegenbauer polynomial, which implies that $K_n^\delta(W_\kappa; 0, y) = k_n^\delta(v_{\mu,\lambda_\kappa-\mu}; 0, \|y\|)$, so that

$$\begin{aligned} \|S_n^\delta(W_\kappa^2)\|_{\kappa,\infty} &\ge a_\kappa \int_{\mathbb{B}^d} \left| K_n^\delta(W_\kappa; 0, y) \right| W_\kappa(y) \mathrm{d}y \\ &= c \int_0^1 \left| k_n^\delta(v_{\kappa_1,\sigma_\kappa}; 0, r) \right| v_{\kappa_1,\sigma_\kappa}(r) \mathrm{d}r = c\, T_n^\delta(v_{\kappa_1,\sigma_\kappa}; 0), \end{aligned}$$

where $T_n^\delta(v_{\kappa_1,\sigma_\kappa}; 0)$ is defined in Eq. (8.5.1), from which the lower estimate for $\delta > -1$ follows from Proposition 8.5.1. For the projection operator, we use Eq. (B.3.1) to obtain that

$$P_{2n}(W_\kappa; 0, y) = (-1)^n \frac{2n+\lambda_k}{\lambda_k} \frac{(\lambda_k)_n}{(\kappa_{d+1}+\frac{1}{2})_n} P_n^{(\kappa_{d+1}-\frac{1}{2},\sigma_\kappa-\frac{1}{2})}(2\|y\|^2 - 1).$$

Using polar coordinates and then changing variables $2r^2 - 1 \mapsto t$, it follows that

$$\begin{aligned} &\int_{\mathbb{B}^d} |P_{2n}(W_\kappa; x, 0)|\, W_\kappa(x) \mathrm{d}x \\ &\quad \sim n^{\sigma_\kappa+\frac{1}{2}} \int_{-1}^1 \left| P_n^{(\kappa_{d+1}-\frac{1}{2},\sigma_k-\frac{1}{2})}(t) \right| w_{\kappa_{d+1}-\frac{1}{2},\sigma_k-\frac{1}{2}}(t) \mathrm{d}t \sim n^{\sigma_\kappa}, \end{aligned}$$

where the last step follows by Eq. (B.1.8). This completes the proof. □

By Eqs. (11.4.2) and (B.3.5), $P_{2n+1}(W_\kappa; 0, y) \equiv 0$, so that the above method fails when $\kappa_{\min} = \kappa_{d+1}$ and n is odd.

We can also state an analogue of Theorem 8.1.3, for which we define

$$\mathbb{B}^d_{\text{int}} := \mathbb{B}^d \setminus \bigcup_{i=1}^d \{x \in \mathbb{B}^d : x_i = 0\}.$$

Theorem 11.4.2. *Let W_κ be as in Eq. (11.1.2). Let f be continuous on $\mathbb{B}^d$. If $\delta > \frac{d-1}{2}$, then $S_n^\delta(W_\kappa; f)$ converge to f for every $x \in \mathbb{B}^d_{\text{int}}$, and the convergence is uniform over each compact set contained inside $\mathbb{B}^d_{\text{int}}$.*

This theorem follows directly from Eq. (11.4.1) and the proof of Theorem 8.1.3, which relies only on an upper estimate of the kernel.

Using the relations Eqs. (11.1.14) and (11.4.1), we can also deduce the boundedness of the projection operators and Cesàro means on $\mathbb{B}^d$ from the corresponding results in Chap. 9. We state the results below.

Theorem 11.4.3. *Let $d \ge 2$ and $n \in \mathbb{N}$. Then*

(i) for $1 \le p \le \frac{2(\sigma_k+1)}{\sigma_k+2}$,

$$\|\mathrm{proj}_n(W_\kappa; f)\|_{W_\kappa,2} \le c n^{\delta_\kappa(p)} \|f\|_{W_\kappa,p};$$

(ii) for $\frac{2(\sigma_k+1)}{\sigma_k+2} \le p \le 2$,

$$\|\mathrm{proj}_n(W_\kappa; f)\|_{W_\kappa,2} \le c n^{\sigma_k(\frac{1}{p}-\frac{1}{2})} \|f\|_{W_\kappa,p}.$$

Furthermore, the estimate in (i) is sharp.

This theorem is an analogue of, and deduced from, Theorem 9.1.1.

Theorem 11.4.4. *Suppose that $f \in L^p(W_\kappa; \mathbb{B}^d)$, $1 \le p \le \infty$, $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{2\sigma_\kappa+2}$, and*

$$\delta > \delta_\kappa(p) := \max\left\{(2\sigma_\kappa + 1)\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right\}.$$

Then $S_n^\delta(W_\kappa; f)$ converges to f in $L^p(W_\kappa; \mathbb{B}^d)$ and

$$\sup_{n\in\mathbb{N}} \|S_n^\delta(W_\kappa; f)\|_{W_\kappa,p} \le c\|f\|_{W_\kappa,p}.$$

Theorem 11.4.5. *Assume $1 \le p \le \infty$ and $0 < \delta \le \delta_\kappa(p)$. Then there exists a function $f \in L^p(W_\kappa; \mathbb{B}^d)$ such that $S_n^\delta(W_\kappa; f)$ diverges in $L^p(W_\kappa; \mathbb{B}^d)$.*

These two theorems are analogues of, and deduced from, Theorems 9.2.1 and 9.2.2.

11.5 Near-Best-Approximation Operators and Highly Localized Kernels

In analogy to Definition 2.6.2, we define near-best-approximation operators on the ball.

Definition 11.5.1. Let η be a C^∞-function on $[0,\infty)$ such that $\eta(t) = 1$ for $0 \le t \le 1$ and $\eta(t) = 0$ for $t \ge 2$. Define

$$L_n(W_\kappa; f, x) := f *_{\kappa,\mathbb{B}} L_n(x) = a_\kappa \int_{\mathbb{B}^d} f(y) L_n(W_\kappa; x, y) W_\kappa(y) \mathrm{d}y \qquad (11.5.1)$$

for $x \in \mathbb{B}^d$ and $n = 0, 1, 2, \ldots$, where

$$L_n(W_\kappa; x, y) := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) P_k(W_\kappa; x, y). \tag{11.5.2}$$

For $f \in L^p(W_\kappa, \mathbb{B}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$, the error of best approximation to f by polynomials of degree at most n is defined by

$$E_n(f)_{W_\kappa,p} := \inf_{g \in \Pi_n} \|f - g\|_{W_\kappa,p}, \qquad 1 \le p \le \infty. \tag{11.5.3}$$

The following theorem is an analogue of Theorem 2.6.3 with essentially the same proof.

Theorem 11.5.2. *Let $f \in L^p(W_\kappa; \mathbb{B}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Then*

(1) $L_n(W_\kappa; f) \in \Pi^d_{2n-1}$ and $L_n f = f$ for $f \in \Pi^d_n$.
(2) For $n \in \mathbb{N}$, $\|L_n(W_\kappa; f)\|_{W_\kappa;p} \le c\|f\|_{W_\kappa;p}$.
(3) For $n \in \mathbb{N}$,

$$\|f - L_n(W_\kappa; f)\|_{W_\kappa;p} \le (1 + c) E_n(f)_{W_\kappa;p}.$$

In the case of spherical harmonics, the kernel $L_n(\langle x, y\rangle)$ is highly localized, as shown in Theorem 2.6.5. For W_κ in Eq. (11.1.2), however, the kernel $L_n(W_\kappa, x, y)$ is not localized at a given point x but at all congruent points of $\bar{x} = (|x_1|, \ldots, |x_d|)$. However, for the classical weight function W_μ in Eq. (11.1.1), the kernel $K_n(W_\mu; x, y)$ is highly localized in the sense that it is highly localized around the main diagonal $x = y$ in $\mathbb{B}^d \times \mathbb{B}^d$.

Theorem 11.5.3. *Let $\mu \ge 0$ and let ℓ be a positive integer. There exists a constant c_ℓ depending only on ℓ, d, μ, and η such that*

$$|L_n(W_\mu; x, y)| \le c_\ell \frac{n^d}{\sqrt{\mathscr{W}_\mu(n; x)}\sqrt{\mathscr{W}_\mu(n; y)}(1 + n\,\mathtt{d}_{\mathbb{B}}(x, y))^\ell} \tag{11.5.4}$$

for $x, y \in \mathbb{B}^d$, where

$$\mathscr{W}_\mu(n; x) := \left(\sqrt{1 - \|x\|^2} + n^{-1}\right)^{2\mu}. \tag{11.5.5}$$

Proof. We give the proof for $\mu > 0$; the case $\mu = 0$ is easier. Applying the closed formula for $P_n(W_\mu; x, y)$ in Eq. (11.1.16), we see that

$$L_n(W_\mu; x, y) = c_\mu \int_{-1}^{1} L_n\left(\langle x, y\rangle + \sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2}\, t\right)(1 - t^2)^{\mu - 1} \mathrm{d}t,$$

where the kernel function is defined by

$$L_n(t) := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \frac{k+\lambda}{\lambda} C_k^{\lambda}(t), \quad \lambda = \mu + \frac{d-1}{2}, \quad t \in [-1,1]. \tag{11.5.6}$$

For $t = \cos\theta$, $0 \le \theta \le \pi$, we have $\theta/2 \sim \sin\theta/2 \sim \sqrt{1-t}$. Since $C_n^{\lambda}(t)$ is a constant multiple of the Jacobi polynomial $P_k^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(t)$, the estimate (2.6.8) with $j = 0$ implies that

$$|L_n(t)| \le c_\ell \frac{n^{2\lambda+1}}{(1+n\sqrt{1-t})^{\ell}}, \quad -1 \le t \le 1.$$

Consequently, introducing the notation

$$t(x,y;u) := \langle x,y\rangle + u\sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}, \tag{11.5.7}$$

we see that the kernel is bounded by

$$\left|L_n(W_\mu;x,y)\right| \le c_\ell n^{2\lambda+1} \int_{-1}^{1} \frac{(1-u^2)^{\mu-1}\mathrm{d}u}{(1+n\sqrt{1-t(x,y;u)})^{\ell}}.$$

Thus, the desired estimate (11.5.4) will follow once the following claim is established: for $\ell > 3\mu+1$ and $x,y \in \mathbb{B}^d$,

$$\int_{-1}^{1} \frac{(1-u^2)^{\mu-1}\mathrm{d}u}{(1+n\sqrt{1-t(x,y;u)})^{\ell}} \le c \frac{n^{-2\mu}}{\sqrt{\mathscr{W}(n,x)}\sqrt{\mathscr{W}(n,y)}(1+n\mathrm{d}_{\mathbb{B}}(x,y))^{\ell-3\mu-1}}, \tag{11.5.8}$$

where $c > 0$ depends only on μ, ℓ, and d.

Set $t := t(x,y;u)$ for short, and define $A(x,y) := \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}$. Then we can write $1-t = 1-\langle x,y\rangle - A(x,y) + (1-u)A(x,y)$, which implies

$$\begin{aligned} 1-t &\ge 1-\langle x,y\rangle - A(x,y) \\ &= 1-\cos \mathrm{d}_{\mathbb{B}}(x,y) = 2\sin^2\frac{\mathrm{d}_{\mathbb{B}}(x,y)}{2} \ge \frac{2}{\pi^2}\mathrm{d}_{\mathbb{B}}(x,y)^2 \end{aligned} \tag{11.5.9}$$

and

$$1-t \ge \frac{2}{\pi^2}\mathrm{d}_{\mathbb{B}}(x,y)^2 + (1-u)A(x,y) \ge (1-u)A(x,y). \tag{11.5.10}$$

The estimate (11.5.9) leads immediately to

$$\int_{-1}^{1} \frac{(1-u^2)^{\mu-1}\mathrm{d}u}{(1+n\sqrt{1-t})^{\ell}} \le \frac{c}{(1+n\mathrm{d}_{\mathbb{B}}(x,y))^{\ell}}. \tag{11.5.11}$$

Inequality (11.5.8) will follow from this and the estimate

$$\int_{-1}^{1} \frac{(1-u^2)^{\mu-1}\mathrm{d}u}{(1+n\sqrt{1-t})^{\ell}} \le \frac{cn^{-2\mu}}{A(x,y)^{\mu}(1+n\mathrm{d}_{\mathbb{B}}(x,y))^{\ell-2\mu-1}}. \tag{11.5.12}$$

To establish this last estimate, we split the integral over $[-1,1]$ into two integrals: one over $[-1,0]$ and the other over $[0,1]$. For the integral over $[-1,0]$, we write the factor $(1+n\sqrt{1-t})^{\ell}$ as the product of $(1+n\sqrt{1-t})^{k-2\mu}$ and $(1+n\sqrt{1-t})^{2\mu}$. Then we apply inequalities (11.5.9) and (11.5.10) to the first and the second terms, respectively. This gives

$$\begin{aligned}\int_{-1}^{0} \cdots &\le \frac{c}{(1+n\mathrm{d}_{\mathbb{B}}(x,y))^{\ell-2\mu}} \int_{-1}^{0} \frac{(1-u^2)^{\mu-1}}{[n^2A(x,y)(1-u)]^{\mu}}\mathrm{d}u \\ &\le \frac{cn^{-2\mu}}{A(x,y)^{\mu}(1+n\mathrm{d}_{\mathbb{B}}(x,y))^{\ell-2\mu}}.\end{aligned}$$

We now estimate the integral over $[0,1]$. Using Eq. (11.5.10) and applying the substitution $u \mapsto s = A(x,y)n^2(1-u)$, we obtain

$$\begin{aligned}\int_{0}^{1} \cdots &\le c\int_{0}^{1} \frac{(1-u^2)^{\mu-1}}{(1+n\sqrt{\mathrm{d}_{\mathbb{B}}(x,y)^2+A(x,y)(1-u)})^{\ell}}\mathrm{d}u \\ &\le \frac{c}{(A(x,y)n^2)^{\mu}} \int_{0}^{A(x,y)n^2} \frac{s^{\mu-1}}{(1+\sqrt{n^2\mathrm{d}_{\mathbb{B}}(x,y)^2+s})^{\ell}}\mathrm{d}s \\ &\le \frac{cn^{-2\mu}}{A(x,y)^{\mu}(1+n\mathrm{d}_{\mathbb{B}}(x,y))^{\ell-2\mu-1}} \int_{0}^{\infty} \frac{s^{\mu-1}\mathrm{d}s}{(1+\sqrt{n^2\mathrm{d}_{\mathbb{B}}(x,y)^2+s})^{2\mu+1}} \\ &\le \frac{cn^{-2\mu}}{A(x,y)^{\mu}(1+n\mathrm{d}_{\mathbb{B}}(x,y))^{\ell-2\mu-1}}.\end{aligned}$$

Putting these estimates together gives Eq. (11.5.12).

To complete the proof of Eq. (11.5.8), we need the following simple inequality:

$$(a+n^{-1})(b+n^{-1}) \le 3(ab+n^{-2})(1+n|a-b|), \quad a,b \ge 0,\, n \ge 1. \tag{11.5.13}$$

To prove this, we assume that $b \ge a$ and define $\gamma \ge 1$ from $b = \gamma a$. Assume first $\gamma \ge 3$. Then Eq. (11.5.13) will follow if we show that $(a+b)n^{-1} \le 3n^{-2}n|a-b|$, which is equivalent to $\gamma+1 \le 3(\gamma-1)$. But this holds because $\gamma \ge 3$. Let now $1 \le \gamma < 3$. Then it suffices to show that $(a+b)n^{-1} \le 2(ab+n^{-2})$. In turn, this inequality holds if $4an^{-1} \le 2(a^2+n^{-2})$, which is clearly true. Thus Eq. (11.5.13) is established. Inequalities (A.1.4) and (11.5.13) yield

$$\begin{aligned}&\left(\sqrt{1-\|x\|^2}+n^{-1}\right)\left(\sqrt{1-\|y\|^2}+n^{-1}\right) \\ &\quad \le 3\left(\sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}+n^{-2}\right)(1+n\mathrm{d}_{\mathbb{B}}(x,y)),\end{aligned}$$

which along with Eqs. (11.5.11) and (11.5.12) implies Eq. (11.5.8). □

11.6 Cubature Formulas on the Unit Ball

As in the case of cubature formulas on the sphere in Chap. 6, for a weight function W defined on a region $\Omega \subset \mathbb{R}^d$, a cubature formula of degree n is a finite sum such that

$$\int_{\Omega} f(x)W(x)\mathrm{d}x = \sum_{j=1}^{N} \lambda_j f(x_j) =: Q_n(f), \quad \forall f \in \Pi_n^d. \tag{11.6.1}$$

For a region Ω, we shall always assume that it has a positive measure in $\mathbb{R}^d$, and our main concern is $\Omega = \mathbb{B}^d$. We again assume that W is nonnegative. We usually consider positive cubature formulas for which all λ_j are greater than 0 and require that all x_k belong to Ω.

Theorem 11.6.1. *If a cubature formula on a region $\Omega \subset \mathbb{R}^d$ is of degree n, then its number of nodes N satisfies*

$$N \ge \dim \Pi_{\lfloor \frac{n}{2} \rfloor} = \binom{m+d-1}{m}, \quad m = \left\lfloor \frac{n}{2} \right\rfloor. \tag{11.6.2}$$

This theorem is classical and can be proved exactly like Theorem 6.1.2. The lower bound in Eq. (11.6.2) is not sharp in general, especially not for $\Omega = \mathbb{B}^d$.

Let G be a finite group. If both $W(x)$ and Ω are invariant under G and so is $Q_n(f)$, then Sobolev's theorem on invariant cubature holds.

Theorem 11.6.2. *Assume that W defined on Ω is invariant under a finite group G. If the cubature formula $Q_n(f)$ is invariant under G, then $Q_n(f)$ is of degree n if and only if Eq.* (11.6.1) *holds for all polynomials in Π_n that are invariant under G.*

This is an analogue of Theorem 6.1.7 and follows from the same proof.

11.6.1 Cubature Formulas on the Ball and on the Sphere

Our main concern below is the structure of cubature formulas on $\mathbb{B}^d$, which are closely related to cubature formulas on the sphere $\mathbb{S}^d$.

We need to make a distinction between weight functions on the sphere and on the ball. Let H be a weight function defined on $\mathbb{S}^d$. With respect to H, we define

$$W_H(x) := \frac{H(x, \sqrt{1-\|x\|})}{\sqrt{1-\|x\|^2}}, \quad x \in \mathbb{B}^d.$$

Theorem 11.6.3. *Let H defined on $\mathbb{S}^d$ be symmetric with respect to x_{d+1}.*

(i) If there is a cubature formula of degree n for W_H on $\mathbb{B}^d$,

$$\int_{\mathbb{B}^d} g(x) W_H(x) \frac{\mathrm{d}x}{\sqrt{1-\|x\|^2}} = \sum_{i=1}^N \lambda_i g(x_i), \quad g \in \Pi_n^d, \tag{11.6.3}$$

with all nodes in $\mathbb{B}^d$, then there is a cubature formula of degree n on $\mathbb{S}^d$,

$$\int_{\mathbb{S}^d} f(y) H(y) \mathrm{d}\sigma(y) = \sum_{i=1}^N \lambda_i \Big[f(x_i, \sqrt{1-\|x_i\|^2}) + f(x_i, -\sqrt{1-\|x_i\|^2}) \Big], \tag{11.6.4}$$

for all $f \in \Pi_n(\mathbb{S}^{d-1})$.

(ii) If there is a cubature formula of degree n for H with all nodes on $\mathbb{S}^d$,

$$\int_{\mathbb{S}^d} f(y) H(y) \mathrm{d}\sigma(y) = \sum_{i=1}^N \lambda_i f(y_i), \quad f \in \Pi_n(\mathbb{S}^d), \tag{11.6.5}$$

then there is a cubature formula of degree n for W_H on $\mathbb{B}^d$,

$$\int_{\mathbb{B}^d} g(x) W_H(x) \frac{\mathrm{d}x}{\sqrt{1-\|x\|^2}} = \frac{1}{2} \sum_{i=1}^N \lambda_i g(x_i), \quad g \in \Pi_n^d, \tag{11.6.6}$$

where $x_i \in \mathbb{B}^d$ are the first d components of y_i. Moreover, if Eq. (11.6.4) exists, then it implies Eq. (11.6.3).

Proof. (i) With Eq. (11.6.3) given, to prove Eq. (11.6.4) it suffices to prove, by Eq. (11.1.6), that

$$\int_{\mathbb{B}^d} \left[f\left(x, \sqrt{1-\|x\|^2}\right) + f\left(x, -\sqrt{1-\|x\|^2}\right) \right] W_H(x) \frac{\mathrm{d}x}{\sqrt{1-\|x\|^2}}$$
$$= \sum_{i=1}^N \lambda_i \left[f\left(x_i, \sqrt{1-\|x_i\|^2}\right) + f\left(x_i, -\sqrt{1-\|x_i\|^2}\right) \right], \quad \forall f \in \Pi_n^d. \tag{11.6.7}$$

By Eq. (11.1.5), $\Pi_n(\mathbb{S}^{d-1}) = \Pi_n^d + x_{d+1}\Pi_{n-1}^d$. If $f(x, x_{d+1}) = x_{d+1} g(x)$, then both sides of Eq. (11.6.7) are zero, so that equality holds. If $f(x, x_{d+1}) = g(x)$, then it is evident that Eq. (11.6.7) reduces to Eq. (11.6.3).

(ii) By Eq. (11.1.6), the cubature formula (11.6.5) is equivalent to

$$\int_{\mathbb{B}^d} \Big[f(x, \sqrt{1-\|x\|^2}) + f(x, -\sqrt{1-\|x\|^2}) \Big] W_H(x) \frac{\mathrm{d}x}{\sqrt{1-\|x\|^2}} = \sum_{i=1}^N \lambda_i f(y_i)$$

for all $f \in \Pi_n(\mathbb{S}^d)$. Since $\Pi_n(\mathbb{S}^{d-1}) = \Pi_n^d + x_{d+1}\Pi_{n-1}^d$, it holds for $f(x, x_{d+1}) = g(x)$, $\forall g \in \Pi_n^d$, which gives Eq. (11.6.7). □

Note that the cubature formulas for the surface measure $\mathrm{d}\sigma$ on $\mathbb{S}^d$ correspond to the cubature formulas for $\mathrm{d}x/\sqrt{1-\|x\|^2}$ on $\mathbb{B}^d$. And the cubature formulas for $\mathrm{d}x$ on $\mathbb{B}^d$ correspond to the cubature formulas for $|x_{d+1}|\mathrm{d}\sigma(x)$ on $\mathbb{S}^d$. In general, cubature formulas for the weight function W_κ in Eq. (11.1.2) on $\mathbb{B}^d$ correspond to cubature formulas for $h_\kappa^2(x) = \prod_{i=1}^{d+1} |x_i|^{2\kappa_i}$ on $\mathbb{S}^d$.

11.6.2 Positive Cubature Formulas and the MZ Inequality

The correspondence in Theorem 11.6.3 allows us to deduce the existence of positive cubature formulas for a maximal separated set of nodes. First we need a definition.

Definition 11.6.4. Let $\varepsilon > 0$. A subset Λ of $\mathbb{B}^d$ is called ε-separated if $\mathsf{d}_{\mathbb{B}}(x,y) \ge \varepsilon$ for any two distinct points $x, y \in \Lambda$. An ε-separated subset Λ of $\mathbb{B}^d$ is called maximal if $\mathbb{B}^d = \bigcup_{y\in\Lambda} \mathsf{c}_{\mathbb{B}}(y,\varepsilon)$, where

$$\mathsf{c}_{\mathbb{B}}(y,\varepsilon) := \left\{ x \in \mathbb{B}^d : \mathsf{d}_{\mathbb{B}}(x,y) \le \varepsilon \right\}.$$

A subset Λ of $\mathbb{B}^d$ is said to be extended maximal ε-separated if

$$1 \le \sum_{\eta\in\Lambda} \chi_{\mathsf{c}_{\mathbb{B}}(\eta,\varepsilon)}(x) \le c_d \qquad \forall x \in \mathbb{B}^d.$$

Theorem 11.6.5. *Given an extended maximal $\frac{\delta}{n}$-separated subset $\Lambda \subset \mathbb{B}^d$ with $\delta \in (0, \delta_0)$ for some small $\delta_0 > 0$, there exist positive numbers λ_y, $y \in \Lambda$ such that $\lambda_y \sim \mathrm{meas}_\kappa^{\mathbb{B}}\left(\mathsf{c}_{\mathbb{B}}(y, \frac{\delta}{n})\right)$ for all $y \in \Lambda$ and*

$$\int_{\mathbb{B}^d} f(x) W_\kappa(x) \mathrm{d}x = \sum_{y\in\Lambda} \lambda_y f(y), \quad f \in \Pi_n^d. \tag{11.6.8}$$

Proof. Under the projection $\mathbb{S}_+^d := \{y \in \mathbb{S}^d : y_{d+1} \ge 0\} \mapsto \mathbb{B}^d$, the spherical cap $\mathsf{c}((x, x_{d+1}), \theta) \subset \mathbb{S}_+^d$ becomes $\mathsf{c}_{\mathbb{B}}(x,\varepsilon)$, $x \in \mathbb{B}^d$. For a given $\Lambda \subset \mathbb{B}^d$, we define

$$\Lambda_* := \Lambda_*^+ \cup \Lambda_*^- \quad \text{with} \quad \Lambda_*^\pm := \left\{ \left(y, \pm\sqrt{1-\|y\|^2}\right) : y \in \Lambda \right\}.$$

It follows readily that Λ_* is an extended maximal $\frac{\delta}{n}$-separated subset of $\mathbb{S}^d$. Since $h_\kappa^2(x) = \prod_{i=1}^{d+1} |x_i|^{2\kappa_i}$ is a doubling weight on $\mathbb{S}^d$, by Eq. (6.3.3), there is a cubature formula

$$\int_{\mathbb{S}^d} f(y) h_\kappa^2(y) \mathrm{d}\sigma = \sum_{\eta^+\in\Lambda_*^+} \lambda_{\eta^+} f(\eta^+) + \sum_{\eta^-\in\Lambda_*^-} \lambda_{\eta^-} f(\eta^-), \quad \forall f \in \Pi_n(\mathbb{S}^d). \tag{11.6.9}$$

Furthermore, since $h_\kappa^2 \mathrm{d}\sigma$ is invariant and $\Lambda_*^\pm$ becomes $\Lambda_*^\mp$ under the mapping $(x, x_{d+1}) \mapsto (x, -x_{d+1})$, we can assume that $\lambda_{\eta^+} = \lambda_{\eta^-}$ when $\eta^+ = (\eta', \eta_{d+1})$ and $\eta^+ = (\eta', -\eta_{d+1})$, since otherwise, we can add the formula Eq. (11.6.9) and the same formula applied to $f(y', -y_{d+1})$. Thus, the cubature formula (11.6.9) is of the form Eq. (11.6.4), which implies by Theorem 11.6.3 that Eq. (11.6.8) exists and that $\lambda_y \sim \mathrm{meas}_\kappa^{\mathbb{B}}\, \mathsf{c}_{\mathbb{B}}(y, \varepsilon)$, since $\mathrm{meas}_\kappa^{\mathbb{B}}\, \mathsf{c}_{\mathbb{B}}(y, \varepsilon) = w(\mathsf{c}((y, y_{d+1}), \varepsilon))$ with $w = h_\kappa^2$. □

In this regard, we can also state the Marcinkiewicz–Zygmund inequality for W_κ on the unit ball. Let

$$W_\kappa\left(\mathsf{c}_{\mathbb{B}}\left(\omega, \frac{\delta}{n}\right)\right) = \int_{\mathsf{c}_{\mathbb{B}}\left(\omega, \frac{\delta}{n}\right)} W_\kappa(x) \mathrm{d}x.$$

Theorem 11.6.6. *Let Λ be a $\frac{\delta}{n}$-separated subset of $\mathbb{B}^d$ and $\delta \in (0,1]$.*

(i) For all $0 < p < \infty$ and $f \in \Pi_m^d$ with $m \ge n$,

$$\sum_{y \in \Lambda} \left(\max_{x \in \mathsf{c}_{\mathbb{B}}(y, \frac{\delta}{n})} |f(x)|^p \right) W_\kappa\left(\mathsf{c}_{\mathbb{B}}\left(y, \frac{\delta}{n}\right)\right) \le c_{\kappa,p} \left(\frac{m}{n}\right)^{s_\kappa} \|f\|_{W_\kappa,p}^p, \quad (11.6.10)$$

where $s_\kappa := d + 2|\kappa| - 2\min\kappa$ and $c_{\kappa,p}$ depends on p when p is close to 0.

(ii) If, in addition, Λ is maximal and $\delta \in (0, \delta_r)$, $\delta_r > 0$ for some $r \in (0,1)$, then for $f \in \Pi_n^d$, $\|f\|_\infty \sim \max_{y \in \Lambda} |f(y)|$, and for $r \le p < \infty$,

$$\|f\|_{W_\kappa,p} \sim \left(\sum_{y \in \Lambda} W_\kappa\left(\mathsf{c}_{\mathbb{B}}\left(y, \frac{\delta}{n}\right)\right) \min_{x \in \mathsf{c}_{\mathbb{B}}\left(y, \frac{\delta}{n}\right)} |f(x)|^p \right)^{1/p} \quad (11.6.11)$$

$$\sim \left(\sum_{y \in \Lambda} W_\kappa\left(\mathsf{c}_{\mathbb{B}}\left(y, \frac{\delta}{n}\right)\right) \max_{x \in \mathsf{c}_{\mathbb{B}}\left(y, \frac{\delta}{n}\right)} |f(x)|^p \right)^{1/p}, \quad (11.6.12)$$

where the constants of equivalence depend on r when r is close to 0.

Proof. For a given $\Lambda \subset \mathbb{B}^d$, we define Λ_* as in the proof of the previous theorem. We then apply Theorem 5.3.6 with d replaced by $d+1$ to the function $F(x, x_{d+1}) = f(x)$ over Λ_* for the weight function $w = h_\kappa^2$, and project the resulting inequalities on $\mathbb{S}^d$ to $\mathbb{B}^d$, which gives the stated result, since $W_\kappa\big(\mathsf{c}_{\mathbb{B}}(y, \frac{\delta}{n})\big) = w\big(\mathsf{c}((y, y_{d+1}), \frac{\delta}{n})\big)$ with $w = h_\kappa^2$ and $\|F\|_{L^p(h_\kappa^2, \mathbb{S}^d)} = \|f\|_{W_\kappa,p}$. □

It is worth mentioning that the two theorems in this subsection can be stated for doubling weight functions on the ball if we define the doubling weight on $\mathbb{B}^d$ as those projected down from $\mathbb{S}^d$.

11.6.3 Product-Type Cubature Formulas

In spherical coordinates, the product-type cubature formulas for h_κ^2 on $\mathbb{S}^d$ can be constructed exactly as in Theorem 6.2.3, from which we can derive product-type cubature formulas for W_κ on $\mathbb{B}^d$. We shall give product type cubature formulas on $\mathbb{B}^d$ only with respect to the classical weight function $W_\mu(x) = (1-\|x\|^2)^{\mu-1/2}$ in Eq. (11.1.1), and we shall not deduce them from those on $\mathbb{S}^d$ for $|x_{d+1}|^{2\mu}\mathrm{d}\sigma(x)$ but give an alternative derivation from polar coordinates instead.

We need the Gaussian quadrature for the Jacobi weight function $u_{\alpha,\beta}(t) = t^\beta(1-t)^\alpha$ on $[0,1]$, which is related to the ordinary Jacobi weight $w_{\alpha,\beta}$ by $u_{\alpha,\beta}(t) = 2^{-\alpha-\beta}w_{\alpha,\beta}(2t-1)$. The nodes of the Gaussian quadrature formula of degree $2n-1$ with respect to $u_{\alpha,\beta}$ are zeros of the polynomial $P_n^{(\alpha,\beta)}(2t-1)$, which we denote by

$$0 < t_{1,n}^{(\alpha,\beta)} < t_{2,n}^{(\alpha,\beta)} < \cdots < t_{n,n}^{(\alpha,\beta)} < 1. \qquad (11.6.13)$$

Proposition 11.6.7. *Let $\alpha,\beta > -1$ and $t_{k,n} = t_{k,n}^{(\alpha,\beta)}$. For each $n \in \mathbb{N}$, the Gaussian quadrature of degree $2n-1$ for $u_{\alpha,\beta}$ is given by*

$$\int_0^1 f(t)u_{\alpha,\beta}(t)\mathrm{d}t = \sum_{k=1}^{n} v_{k,n}^{(\alpha,\beta)} f\left(t_{k,n}^{(\alpha,\beta)}\right), \qquad \forall f \in \Pi_{2n-1}, \qquad (11.6.14)$$

where the quadrature weights $v_{k,n}^{(\alpha,\beta)} > 0$ are given by, with $Q(t) = P_n^{(\alpha,\beta)}(t)$,

$$v_{k,n}^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \frac{1}{(1-t_{k,n}^2)[Q_n'(t_{k,n})]^2}. \qquad (11.6.15)$$

This is again classical and can be found in [162]. We now construct product-type cubature formulas for W_μ on $\mathbb{B}^d$. First we consider $d = 2$.

Theorem 11.6.8. *For $n \in \mathbb{N}$, let $\phi_{k,n} = \pi k/n$, $0 \le k \le 2n-1$. Let $m = \lfloor \frac{n}{2} \rfloor$ and let $t_{j,m} = t_{j,m}^{(\mu-\frac{1}{2},0)}$ be defined as in Eq. (11.6.13) and $v_{j,n} = v_{j,n}^{(\mu-\frac{1}{2},0)}$. Then the cubature formula*

$$\int_{\mathbb{B}^2} f(x)W_\mu(x)\mathrm{d}x = \frac{\pi}{2n}\sum_{k=0}^{2n-1}\sum_{j=1}^{m} v_{j,m} f\left(\sqrt{t_{j,m}}\cos\phi_{k,n}, \sqrt{t_{j,m}}\sin\phi_{k,n}\right) \qquad (11.6.16)$$

is of degree $2n-1$, that is, Eq. (11.6.16) holds for all $f \in \Pi_{2n-1}^2$.

Proof. In polar coordinates $x = (r\cos\theta, r\sin\theta)$, we work with the basis of $\mathcal{V}_k^2(W_\mu)$ in Proposition 11.1.13, which consists of, for $d = 2$,

$$f_{j,k}(x) = P_j^{(\mu-\frac{1}{2},k-2j)}(2r^2-1)r^{k-2j}S_{k-2j}(\theta),$$

where $S_{k-2j}(\theta) = \cos(k-2j)\theta$ or $\sin(k-2j)\theta$, for $0 \le 2j \le k$. Thus, we need to verify Eq. (11.6.16) for $f = f_{j,k}$ for $0 \le j \le k \le 2n-1$. In polar coordinates,

$$\int_{\mathbb{B}^2} f(x) W_\mu(x) \mathrm{d}x = \int_0^1 r(1-r^2)^{\mu-\frac{1}{2}} \int_0^{2\pi} f(r\cos\theta, r\sin\theta) \mathrm{d}\theta \mathrm{d}r,$$

from which follows that if $f = f_{j,k}$ and $k-2j > 0$, then both sides of Eq. (11.6.16) are zero. In the remaining case of $k = 2j$, Eq. (11.6.16) becomes

$$\int_0^1 P^{(\mu-\frac{1}{2},0)}(2r^2-1) r(1-r^2)^{\mu-\frac{1}{2}} \mathrm{d}r = \frac{1}{2} \sum_{i=1}^m v_{i,m} P_j^{(\mu-\frac{1}{2},0)}(2t_{i,m}-1)$$

for $0 \le 2j \le 2n-1$ or $0 \le j \le n-1$, which, however, follows from Eq. (11.6.14) on changing variables $t = 2r^2 - 1$. This completes the proof. □

It is evident how to extend the above cubature formula to $d > 2$. In spherical–polar coordinates (1.5.1), let

$$g(r, \theta_1, \ldots, \theta_{d-1}) \\ := f(r\sin\theta_{d-1} \ldots \sin\theta_2 \sin\theta_1, r\sin\theta_{d-1} \ldots \sin\theta_2 \cos\theta_1, \ldots, r\cos\theta_{d-1}).$$

Theorem 11.6.9. *For $n \in \mathbb{N}$, let $\phi_{k,n}$, $\theta_{j,n}^\lambda$ and $\mu_{i,n}^{(\lambda)}$ be as in the cubature formula on $\mathbb{S}^{d-1}$ in Eq.* (6.2.6)*. Let $m = \lfloor \frac{n}{2} \rfloor$ and let $t_{\ell,m} = t_{\ell,m}^{(\mu-\frac{1}{2},\frac{d-2}{2})}$ be defined as in Eq.* (11.6.13) *and $v_{\ell,n} = v_{\ell,n}^{(\mu-\frac{1}{2},\frac{d-2}{2})}$. Then the cubature formula*

$$\int_{\mathbb{B}^d} f(x) W_\mu(x) \mathrm{d}x = \frac{\pi}{2n} \sum_{\ell=1}^m v_\ell \sum_{k=0}^{2n-1} \sum_{j_2=1}^n \cdots \sum_{j_{d-1}=1}^n \prod_{i=2}^{d-1} \mu_{i,n}^{(\frac{i-1}{2})} \\ \times g\left(\sqrt{\theta_{\ell,m}}, \phi_{k,n}, \theta_{j_2,n}^{(\frac{1}{2})}, \ldots, \theta_{j_{d-1},n}^{(\frac{d-2}{2})}\right) \tag{11.6.17}$$

is of degree $2n-1$, that is, Eq. (11.6.17) *holds for all $f \in \Pi_{2n-1}$.*

Proof. Working with the basis of $\mathscr{V}_m^d(W_\mu)$ in Proposition 11.1.13 and using

$$\int_{\mathbb{B}^d} f(x) W_\mu(x) \mathrm{d}x = \int_0^1 r^{d-1}(1-r^2)^{\mu-1} \int_{\mathbb{S}^{d-1}} f(r\xi) \mathrm{d}\sigma(\xi) \mathrm{d}r,$$

the proof follows along the same lines as in the case $d = 2$ after the cubature formula Eq. (6.2.6) on the sphere is used. □

The number of points of the product cubature formula of degree $2n-1$ is n^d when n is even and $n^d - n^{d-1}$ when n is odd. Despite the problem of high concentration of nodes at the center of the ball, its simplicity makes it valuable.

11.7 Orthogonal Structure on Spheres and on Balls

The correspondence of orthogonal structure on $\mathbb{B}^d$ and on $\mathbb{S}^d$ in the first section can be extended further, which will be needed in the next chapter. The extension is based on, instead of Lemma 11.1.3, the following lemma, proved in Lemma A.5.4.

Lemma 11.7.1. *Let d and m be positive integers. Then for every $f \in L(\mathbb{S}^{d+m-1})$,*

$$\int_{\mathbb{S}^{d+m-1}} f(y)\mathrm{d}\sigma_{d+m} = \int_{\mathbb{B}^d} (1-\|x\|^2)^{\frac{m-2}{2}} \left[\int_{\mathbb{S}^{m-1}} f\left(x, \sqrt{1-\|x\|^2}\xi\right) \mathrm{d}\sigma_m(\xi)\right] \mathrm{d}x.$$

Lemma 11.7.2. *Let W_κ be defined as in Eq. (11.1.2) with $\kappa_{d+1} = \frac{m-1}{2}$ on $\mathbb{B}^d$ and let $P_\alpha \in \mathscr{V}_n^d(W_\kappa)$, $|\alpha| = n$ be mutually orthogonal. Then the functions*

$$Y_\alpha^n(x) = \|x\|^n P_\alpha(x'), \quad x = r(x', x'') \in \mathbb{R}^{d+m}, \quad r = \|x\|, \quad x' \in \mathbb{B}^d,$$

are homogeneous polynomials in x, and the Y_α^n are mutually orthogonal with respect to $H(x)\mathrm{d}\sigma_{d+m} = h_\kappa^2(x')$ on $\mathbb{S}^{d+m-1}$, where $h_\kappa^2(x') = \prod_{i=1}^d |x_i|^{2\kappa_i}$.

Proof. Since P_α^n has the same parity as n, we can write $P_\alpha^n(x')$ as a linear combination of monomials of the form x'^β, $|\beta| = n-2k$, $0 \le k \le n$. However, if $y = (y', y'') \in \mathbb{R}^{d+m}$ with $y' \in \mathbb{R}^d$, then $y' = \|y\|x'$ with $x' \in \mathbb{B}^d$. Hence, it follows that Y_α^n is a sum of the terms $c_\beta \|y\|^{2k} y'^\beta$, $|\beta| = n-2k$, and as a consequence, the polynomial $Y_\alpha^n(y)$ is homogeneous of degree n in $y \in \mathbb{R}^{d+m}$. We first prove that Y_α^n is orthogonal to polynomials of lower degrees with respect to $H(x)\mathrm{d}\sigma_{d+m}$, for which we show that Y_α^n is orthogonal to $g_\beta(x) = y^\beta$ for $\beta \in \mathbb{N}^d$ and $|\beta| \le n-1$. From Lemma 11.7.1,

$$\begin{aligned}&\int_{\mathbb{S}^{d+m-1}} Y_\alpha^n(x) g_\beta(x) H(x)\mathrm{d}\sigma_{d+m} \\ &\quad = \int_{\mathbb{B}^d} P_\alpha(x') \left[\int_{\mathbb{S}^{m-1}} g_\beta\left(x', \sqrt{1-\|x'\|^2}\xi\right) \mathrm{d}\sigma(\xi)\right] W_\kappa(x')\mathrm{d}x'.\end{aligned}$$

If g_β is odd in at least one of its variables $x_{d+1}, \ldots, x_{d+m}$, then the integral inside the square brackets is zero. Hence, Y_α^n is orthogonal to g_β in this case. If g_β is even in every one of these variables, then the function inside the square brackets will be a polynomial in x_1 of degree at most $n-1$, from which we conclude that Y_α^n is orthogonal to g_β by the orthogonality of P_α^n with respect to W_κ on $\mathbb{B}^d$. Moreover, we also have

$$\int_{\mathbb{S}^{d+m-1}} Y_\alpha^n(x) Y_\beta^n(x) H(x)\mathrm{d}\sigma_{d+m} = \omega_m \int_{\mathscr{B}^d} P_\alpha^n(x') P_\beta^n(x') W_\kappa(x')\mathrm{d}x',$$

so that $\{Y_\alpha^n\}$ is a mutually orthogonal set of polynomials. $\square$

Theorem 11.7.3. *Let W_κ be defined as in Eq. (11.1.2) with $\kappa_{d+1} = \frac{m-1}{2}$ on $\mathbb{B}^d$. Let $Z_n^\kappa(\cdot,\cdot)$ be the reproducing kernel of $\mathscr{H}_n^{d+m}(h_\kappa^2)$ for $h_\kappa^2(x',x'') = h_\kappa^2(x') = \prod_{i=1}^d |x_i|^{2\kappa_i}$, $(x',x'') \in \mathbb{S}^{d+m-1}$, with $x' \in \mathbb{B}^d$. Then the reproducing kernel $P_n(W_\kappa;\cdot,\cdot)$ satisfies, for $m > 1$,*

$$P_n(W_\kappa;x',y') = \frac{1}{\omega_m}\int_{\mathbb{S}^{m-1}} Z_n^\kappa\left(x,\left(y',\sqrt{1-\|y'\|^2}\xi\right)\right)\mathrm{d}\sigma_m(\xi), \tag{11.7.1}$$

where $y = (y',y'') \in \mathbb{S}^{d+m-1}$ with $y' \in \mathbb{B}^d$ and $y'' = \|y''\|\xi \in \mathbb{B}^m$ with $\xi \in \mathbb{S}^{m-1}$, and it satisfies, for $m = 1$,

$$P_n(W_\kappa;x',y') = \frac{1}{2}\left[Z_n^\kappa\left(x,\left(y',\sqrt{1-\|y'\|^2}\right)\right) + Z_n^k\left(x,\left(y',-\sqrt{1-\|y'\|^2}\right)\right)\right].$$

Proof. We give the proof for the case $m > 1$; the case $m = 1$ is similar. Let the right-hand side of Eq. (11.7.1) be denoted, temporarily, by $Q_n(x,y)$. Let $P_\alpha \in \mathscr{V}_n^d(W_\kappa)$. Using the integral relation in Lemma 11.7.1, which implies $\omega_{d+m}^\kappa = \omega_m/a_\kappa$ on setting $f(x) = h_\kappa^2(x')$, we obtain

$$\begin{aligned}
&a_\kappa\int_{\mathbb{B}^d} Q_n(x',y')P_\alpha(y')W_\kappa(y')\mathrm{d}y' \\
&\quad = \frac{a_\kappa}{\omega_m}\int_{\mathbb{B}^d}\int_{\mathbb{S}^{m-1}} Z_n^\kappa\left(x,\left(y',\sqrt{1-\|y'\|^2}\xi\right)\right)\mathrm{d}\sigma_m(\xi)P_\alpha(y')W_\kappa(y')\mathrm{d}y' \\
&\quad = \frac{1}{\omega_{d+m}^\kappa}\int_{\mathbb{S}^{d+m-1}} Z_n^\kappa(x,y)P_\alpha(y')h_\kappa^2(y')\mathrm{d}\sigma_{d+m}(y).
\end{aligned}$$

Since Z_n^κ is the reproducing kernel of $\mathscr{H}_n^d(h_\kappa^2)$ on $\mathbb{S}^{d+m}$, the last expression becomes, by Lemma 11.7.2,

$$a_\kappa\int_{\mathbb{B}^d} Q_n(x',y')P_\alpha(y')W_\kappa(y')\mathrm{d}y' = P_\alpha(x'), \quad x' \in \mathbb{B}^d,$$

which shows that $Q_n(x',y')$ is a reproducing kernel of $\mathscr{V}_n^d(W_\kappa)$. □

Using the explicit formula of $Z_n^\kappa(\cdot,\cdot)$ in Corollary 7.2.10, we can derive from Eq. (11.7.1) an explicit formula for $P_n(W_\kappa;x,y)$, which gives, by Eq. (A.5.1), exactly the expression (11.1.15). Note, however, that this alternative proof of Eq. (11.1.15) works only when $\kappa_{d+1} = \frac{m-1}{2}$ is a half-integer.

11.8 Notes and Further Results

The connection between orthogonal structure on the ball and the sphere was studied in [178, 184] and used for studying orthogonal expansions in [188, 190]. The closed formula for the reproducing kernel in Eq. (11.1.16) was first established in [181] by summing over a specific orthonormal basis by repeatedly using the addition formula of the Gegenbauer polynomials, and the formula (11.1.15) was later established in [185]. The Funk–Hecke formula and its implication in the first section were studied in [183]. The basis Eq. (11.1.19) appeared early in [110], and it has applications in computerized tomography.

The weight function W_κ is integrable if all κ_i are greater than $-1/2$. The reason that we assume $\kappa_i \ge 0$ can be seen from the closed formulas for the reproducing kernel Eq. (11.1.15), which is undefined if $\kappa_i < 0$. In the case of W_μ, a closed formula is given by performing an integration by parts in Eq. (11.1.16) in [186].

Convolution on the ball was introduced in [188], as was the maximal function in Eq. (11.3.1). The integral formula (11.2.12) for the translation operator was proved in [189]. It is not clear, however, whether such a formula holds for W_κ on $\mathbb{B}^d$. The main results on the maximal function in Sect. 11.3 were established in [47].

The boundedness of projection operators and the Cesàro means in Sect. 11.4 were proved in [48, 49, 108]. For $d = 1$, the results on the (C,δ) means are classical for W_μ, the Gegenbauer polynomials, but are new for W_{κ_1,κ_2} of the generalized Gegenbauer polynomials. It was proved in [161] that the uniform norm of every projection operator $C(\mathbb{B}^d)$ onto Π_n^d is $\ge cn^{\frac{d-1}{2}}$. The lower bound is attained by the operator $\operatorname{proj}_n(W_\mu; f)$ for $-1/2 < \mu \le 0$, as shown in [186].

The proof of the highly localized kernel was given in [139]. The rate of decay can be improved to subexponential as in Eq. (2.7.1) under an additional assumption on the cutoff function [91]. The kernel also satisfies the following relation: For $0 < p \le \infty$,

$$\|L_n(x,\cdot)\|_{W_\mu,p} \sim \left(\frac{n^d}{\mathscr{W}_\mu(n,x)}\right)^{1-\frac{1}{p}}, \quad x \in \mathbb{B}^d.$$

The upper estimate was established in [139], and the lower estimate was proved in [104]. This estimate plays an essential role in the theory of needlets on the ball.

The lower bound (11.6.2) for the cubature formulas is not sharp in general when n is odd. In fact, this bound is attained if and only if the nodes of the cubature formulas are common zeros of all orthogonal polynomials of degree n, and it cannot be attained if W is centrally symmetric, that is, $x \in \Omega$ implies that $-x \in \Omega$ and $W(x) = W(-x)$ [67, 126]. Furthermore, in the case that $d = 2$ and W is centrally symmetric, it is known [123] that for a cubature rule of degree $2n-1$ to exist, it is necessary that

$$N \ge \dim \Pi_{n-1}^2 + \left\lfloor \frac{n}{2} \right\rfloor = \frac{n(n+1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor. \tag{11.8.1}$$

For W_μ on $\mathbb{B}^d$, however, even the above bound is far from sharp when n is large. Indeed, as shown in [187], a cubature formula of degree s for W_0 requires $N \ge 0.13622s^2 + \mathscr{O}(s)$, which is larger than $N \ge 0.125s^2 + \mathscr{O}(s)$ in Eq. (11.6.2) or Eq. (11.8.1).

The relation between cubature formulas on the sphere and on the ball was established in [178].

The product cubature formulas in Eq. (11.6.16) are classical. Despite their drawback of nodes concentrated at zeros, they remain one of few viable choices on the unit ball for moderate or large n. By parameterizing the ball in Cartesian coordinates, one can derive other product-type cubature formulas on the ball that have essentially the same number of points. These are essentially products of quadrature formulas of one dimension. There are few genuine multidimensional cubature formulas that possess a high degree of precision and have reasonably few nodes.

Chapter 12
Polynomial Approximation on the Unit Ball

We study the problem of characterizing the best approximation by polynomials on the unit ball in terms of the smoothness of the function being approximated, similar to what we did on the unit sphere in Chap. 4. There is, however, an essential difference between approximations on the unit ball and those on the unit sphere, which arises from the simple fact that the ball is a domain with boundary, whereas the sphere has no boundary. In the case of $d = 1$, it is well known that the best approximation by algebraic polynomials on a finite interval, say $[-1,1]$, displays better convergence behavior at points close to the end of the interval than at inside points, which renders the usual modulus of smoothness inadequate for describing the convergence behavior accurately and makes the characterization problem much more challenging for the interval. The same phenomenon appears for the unit ball, and satisfactory solutions for the characterization problem on the ball have emerged only recently.

This chapter contains three pairs of moduli of smoothness and equivalent K-functionals on the unit ball, each of which can be used to establish direct and weak inverse theorems for the best approximation. Each pair has its own advantages and has its root in the setting of one variable. In the first section, we provide a short overview of algebraic polynomial approximation on intervals, in which we discuss the background and those results that are relevant to our work on the unit ball. The first pair of modulus of smoothness and K-functional is introduced and studied in the second section; these are inherited from those on the sphere $\mathbb{S}^{d+m-1}$ and are defined with respect to the weight function $(1-\|x\|^2)^{\frac{m-2}{2}}$ on $\mathbb{B}^d$. Most of the results for this pair can be deduced from the corresponding results on the sphere established in Chap. 4; in particular, the modulus of smoothness is reasonably easy to compute. The second pair, developed in the third section, is in analogy with the Ditzian–Totik modulus of smoothness and K-functional on the interval, which differs from the first pair by a single term that takes into account the boundary behavior. The third pair is defined in the fourth section in terms of the generalized translation operator of the orthogonal expansions on the unit ball, which is a complete analogue of the pair defined for the unit sphere in Chap. 10 and works for $h_\kappa^2(x)(1-\|x\|^2)^{\mu-\frac{1}{2}}$, where h_κ

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is the product weight function invariant under $\mathbb{Z}_2^d$. The three moduli of smoothness on $\mathbb{B}^d$ are comparable, and what is known about the comparison is presented in the fifth section.

12.1 Algebraic Polynomial Approximation on an Interval

We denote by Π_n the space of algebraic polynomials of degree at most n on $\mathbb{R}$. Throughout this section, we assume $f \in C[0,1]$ when $f \in L^p[-1,1]$ with $p=\infty$. We consider the problem of characterizing the quantity

$$E_n(f)_p := \inf_{P \in \Pi_n} \|f - P\|_{L^p[-1,1]},$$

the best approximation by polynomials on $[-1,1]$, in terms of the smoothness of the function f. Taking a cue from approximation on the circle, we define the ordinary rth-order modulus of smoothness of a function $f \in L^p[-1,1]$ on the interval by

$$\omega^r(f,t)_p := \sup_{|h| \le t} \|\overrightarrow{\triangle}_h^r f\|_{L^p[-1,1]}, \tag{12.1.1}$$

where $\overrightarrow{\triangle}_h^r$ is the forward difference as in Eq. (4.1.1), and we assume, in addition, that $\overrightarrow{\triangle}_h^r f(x) = 0$ if $x + rh \notin [-1,1]$. We shall write $\omega(f,t)_p$ in place of $\omega^1(f,t)_p$.

This modulus of smoothness is completely analogous to the modulus of smoothness $\omega_r(f,t)_p$ for 2π-periodic functions on the circle $\mathbb{S}^1$, defined in Definition 4.1.1. Recall that for trigonometric polynomial approximation on the unit circle, $\omega_r(f,t)_p$ can be used to establish both the direct Jackson theorem (4.1.3) and the weak inverse inequality (4.1.4), which, in turn, can be applied to characterize function spaces with prescribed rate of best trigonometric polynomial approximation. It is, moreover, computable.

The circle has no endpoints, whereas the interval $[-1,1]$ has two endpoints. This simple geometric difference is the underlying reason for the difference between trigonometric approximation on the circle and algebraic polynomial approximation on the interval. Indeed, in the latter case, while one still has the direct Jackson estimate (see [111, Chap. 5])

$$E_n(f)_p \le c\,\omega^r(f,n^{-1})_p, \qquad n = r, r+1, \ldots,$$

the weak inverse inequality

$$\omega^r(f,n^{-1})_p \le c n^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_p$$

no longer holds, as shown by Nikolskii [131], who pointed out that the quality of approximation by algebraic polynomials increases toward the endpoints of the interval and that for a given $\alpha \in (0,1)$, there exists a continuous function f on $[-1,1]$ for which $\sup_{n\in\mathbb{N}} n^{\alpha} E_n(f)_\infty < \infty$ and $\sup_{t\in(0,1)} t^{-\alpha}\omega(f,t)_\infty = \infty$. A quantitative result was later given by Timan [164], who obtained the following pointwise estimates (see also [167, p. 262]):

Theorem 12.1.1. *For each function $f \in C[-1,1]$, there is a sequence of polynomials $P_n \in \Pi_n$ such that for every $x \in [-1,1]$,*

$$|f(x) - P_n(x)| \le M\omega\left(f, \frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n}\right)_\infty, \tag{12.1.2}$$

where the constant is independent of f, x, and n.

The inequality (12.1.2) shows that there is a substantial increase of approximation order toward the endpoints of the interval $[-1,1]$. As a result, the ordinary moduli of smoothness in Eq. (12.1.1) do not give a satisfactory characterization of the class of functions on $[-1,1]$ that satisfy the condition $\sup_{n\in\mathbb{N}} n^{\alpha} E_n(f)_p < \infty$ for a given $\alpha > 0$. A different modulus of smoothness is then called for, which should capture the smoothness of a function, be relatively easy to compute, and give both the direct and inverse theorems for best algebraic polynomial approximation on the interval. Many authors contributed to this problem, and several candidates were identified. We describe two of them below.

The most successful and widely accepted modulus of smoothness on the interval is the one introduced by Ditzian and Totik in [61], which is satisfactory in almost all accounts for algebraic polynomial approximation on the interval. For $r \in \mathbb{N}$ and $h > 0$, let $\hat{\triangle}_h^r$ denote the central difference of increment h, defined by

$$\hat{\triangle}_h = S_{h/2} - S_{-h/2} \quad \text{and} \quad \hat{\triangle}_h^r = \hat{\triangle}_h^{r-1}\triangle, \quad r = 2,3,\ldots, \tag{12.1.3}$$

where $S_h f(x) = f(x+h)$, which can be written explicitly as

$$\hat{\triangle}_h^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right)h\right).$$

The Ditzian–Totik modulus, as it is now known in the literature, is defined in terms of the central difference with the insight of replacing the increment h by $h\varphi(x)$, where $\varphi(x) := \sqrt{1-x^2}$.

Definition 12.1.2. Let $r \in \mathbb{N}$ and $1 \le p \le \infty$. The Ditzian–Totik moduli of smoothness are defined by

$$\omega_\varphi^r(f,t)_p := \sup_{0<h\le t} \left\| \hat{\triangle}_{h\varphi}^r f \right\|_{L^p[-1,1]}, \tag{12.1.4}$$

where $\hat{\Delta}_{h\varphi(x)}^r f(x) = 0$ if $x \pm rh\varphi(x)/2 \notin [-1,1]$.

The Ditzian–Totik modulus of smoothness is computable and enjoys most of the usual properties of moduli of smoothness. Despite its simplicity of definition, however, its properties and its application in characterizing the best approximation by polynomials require substantial work, which we cannot incorporate in a short section. We collect some of the most relevant properties and results and give precise references for their proofs.

Proposition 12.1.3. *Let $r \in \mathbb{N}$ and $f \in L^p[-1,1]$ for $1 \le p \le \infty$. There exists a constant $t_r \in (0,1)$ depending only on r such that for $0 < t \le t_r$ and $1 \le p \le \infty$,*

(1) $\omega_\varphi^{r+1}(f,t)_p \le c\, \omega_\varphi^r(f,t)_p \le c\|f\|_p$.
(2) *For $\ell > 0$,* $\omega_\varphi^r(f,\ell t)_p \le c(\ell+1)^r \omega_\varphi^r(f,t)_p$.
(3) *For every integer $m > r$,*

$$\omega_\varphi^r(f,t)_p \le c_m \left(t^r \int_t^1 \frac{\omega_\varphi^m(f,u)_p}{u^{r+1}}\, \mathrm{d}u + t^r \|f\|_p \right). \tag{12.1.5}$$

(4) *For $1 \le p \le \infty$,*

$$\omega_\varphi^r(f,t)_p \sim \left(\frac{1}{t} \int_0^t \|\hat{\triangle}_{h\varphi}^r f\|_p^p \, \mathrm{d}h \right)^{\frac{1}{p}}, \tag{12.1.6}$$

where when $p = \infty$, the right-hand side is replaced by $\frac{1}{t}\int_0^t \|\hat{\triangle}_{h\varphi}^r f\|_\infty \, \mathrm{d}h$.

The first two properties in the proposition are given in [61, p. 38], and the third one, the Marchaud inequality (12.1.5), is given in [61, p. 43]. The equivalence (12.1.6) can be deduced from [61, (2.1.4), (2.2.5)].

The Dizian–Totik moduli of smoothness can also be defined in the weighted case. The definition, however, is more complicated (see [61, (8.2.10)]) because of the endpoints. An equivalent K-functional, on the other hand, is defined in a unified way with or without weights. Recall that $w_\lambda(x) := (1-x^2)^{\lambda-1/2}$ for $\lambda > -1/2$ and that $\|\cdot\|_{p,\lambda}$ denotes the L^p norm of $L^p(w_\lambda;[-1,1])$.

Definition 12.1.4. Let $1 \le p \le \infty$, $r \in \mathbb{N}$, and $\mu \ge 0$. The weighted Ditzian–Totik K-functional of $f \in L^p(w_\mu;[-1,1])$ is defined by

$$K_{r,\varphi}(f,t)_{p,\mu} := \inf_{g \in C^r[-1,1]} \left\{ \|f-g\|_{p,\mu} + t^r \|\varphi^r g^{(r)}\|_{p,\mu} \right\}, \quad t \in (0,1). \tag{12.1.7}$$

Since $w_{1/2}(t) = 1$, the case $\mu = 1/2$ corresponds to the unweighted case, for which we write $K_{r,\varphi}(f,t)_p$ instead of $K_{r,\varphi}(f,t)_{p,1/2}$. In this case, the modulus of smoothness $\omega_\varphi^r(f,t)_p$ is equivalent to the K-functional.

Theorem 12.1.5. *Let $1 \le p \le \infty$ and $r \in \mathbb{N}$. Then the following statements hold:*

(i) *There exists a constant $\tau_r \in (0,1)$ depending only on r such that*

$$K_{r,\varphi}(f,t)_p \sim \omega_\varphi^r(f,t)_p, \quad 1 \le p \le \infty, \quad 0 < t < \tau_r. \tag{12.1.8}$$

(ii) *Both the direct inequality*

$$E_n(f)_p \le cK_{r,\varphi}(f,n^{-1})_p, \quad n = r, r+1, \dots, \tag{12.1.9}$$

and the inverse inequality

$$K_{r,\varphi}(f,n^{-1})_p \le cn^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_p, \tag{12.1.10}$$

hold for $f \in L^p[-1,1]$ *if* $1 \le p < \infty$, *and* $f \in C[-1,1]$ *if* $p = \infty$.

These results are stated and proved in [61, Theorem 2.1.1],[61, Chap. 7], and [61, Theorem 7.2.4], respectively.

We now turn to another modulus of smoothness that works for algebraic approximation on the interval, which is defined in terms of the generalized translation operators of the Gegenbauer polynomial expansions. It is closely related to, and in fact a forerunner of, the modulus of smoothness on the sphere defined in Definition 10.1.1. To emphasize the similarity, let us recall that the orthogonal Fourier coefficients of $f \in L^1(w_\lambda; [-1,1])$ are defined by

$$\hat{f}_n^\lambda := c_\lambda \int_{-1}^{1} f(t) R_n^\lambda(t) (1-t^2)^{\lambda - 1/2} \mathrm{d}t,$$

where $R_n^\lambda(t)$ denotes the normalized Gegenbauer polynomial $C_n^\lambda / \|C_n^\lambda\|_{2,\lambda}$, and the orthogonal expansion of f in the Gegenbauer polynomials is given by, for $f \in L^2(w_\lambda; [-1,1])$,

$$f = \sum_{n=0}^{\infty} \operatorname{proj}_n^\lambda f \quad \text{with} \quad \operatorname{proj}_n^\lambda f = \hat{f}_n^\lambda R_n^\lambda. \tag{12.1.11}$$

The polynomial C_n^λ is an eigenfunction of the second-order differential operator

$$\mathscr{D}_\lambda := -\frac{1}{w_\lambda(x)} \frac{\mathrm{d}}{\mathrm{d}x} \left[w_\lambda(x)(1-x^2) \right] \frac{\mathrm{d}}{\mathrm{d}x} \tag{12.1.12}$$

with the eigenvalue $n(n+2\lambda)$. Recall that $C_n^\lambda(\langle x, \cdot \rangle)$ is a zonal spherical harmonic when $\lambda = \frac{d-2}{2}$. For $x \in \mathbb{S}^{d-1}$, the spherical harmonic expansion of $f(\langle x, \cdot \rangle)$ on $\mathbb{S}^{d-1}$ reduces to the Gegenbauer expansion (12.1.11), and the operator $\mathscr{D}_\lambda$ for $\lambda = \frac{d-2}{2}$ is the restriction of the Laplace–Beltrami operator on the zonal functions.

In this regard, it is evident how to define the modulus of smoothness and related K-functional for $L^p(w_\lambda; [-1,1])$ for $\lambda \ne 0$. For $r > 0$, we define the fractional power of the differential operator $\mathscr{D}_\lambda^\alpha$ in a distributional sense by

$$\operatorname{proj}_n^\lambda \left[\mathscr{D}_\lambda^\alpha f \right] = (n(n+2\lambda))^\alpha \operatorname{proj}_n^\lambda f, \quad n = 0, 1, 2, \dots,$$

and use it to define the K-functional for $f \in L^p(w_\lambda;[-1,1])$, $1 \le p \le \infty$, by

$$K_r(f,t)_{p,\lambda} := \inf_g \left\{ \|f-g\|_{p,\lambda} + t^r \|\mathscr{D}_\lambda^{r/2} g\|_{p,\lambda} \right\},$$

where the infimum is taken over all algebraic polynomials g on $[-1,1]$. The modulus of smoothness is defined in terms of the generalized translation operator $T_\theta^\lambda f$ of the Gegenbauer expansions, defined as follows.

Definition 12.1.6. For $\lambda \ge 0$, $\theta \in (0,\pi)$, and $f \in L^1(w_\lambda;[-1,1])$, the generalized translation operator T_θ^λ is defined for $\lambda > 0$ by

$$T_\theta^\lambda f(x) := c_\lambda \int_{-1}^{1} f\left(x\cos\theta + y\sqrt{1-x^2}\sin\theta\right)(1-y^2)^{\lambda-1}\,\mathrm{d}y$$

and taking the limit $\lambda \to 0^+$ of $T_\theta^\lambda f$, defined for $\lambda = 0$ by

$$T_\theta^0 f(x) = \frac{1}{2}\left[f\left(x\cos\theta + \sqrt{1-x^2}\sin\theta\right) + f\left(x\cos\theta - \sqrt{1-x^2}\sin\theta\right)\right].$$

This is evidently a positive operator, and the product formula (B.2.9) of the Gegenbauer polynomials implies that

$$\operatorname{proj}_n^\lambda (T_\theta^\lambda f) = R_n^\lambda(\cos\theta)\operatorname{proj}_n^\lambda f, \qquad n = 0,1,\ldots,$$

which agrees with Eq. (10.1.1). As a simple consequence of the definition, the operator T_θ^λ is a contraction on $L^2(w_\lambda;[-1,1])$ for $1 \le p \le \infty$, that is, $\|T_\theta^\lambda f\|_{p,\lambda} \le \|f\|_{p,\lambda}$.

For $r > 0$, we can then define a modulus of smoothness $\omega_r^*(f,t)_{p,\lambda}$ for $f \in L^p(w_\lambda;[-1,1])$, $1 \le p \le \infty$, in terms of the generalized translation operator in exactly the same way as in Definition 10.1.1, that is,

$$\omega_r^*(f,t)_{p,\lambda} := \sup_{|\theta| \le t} \|\triangle_{\theta,\lambda}^r f\|_{p,\lambda} \quad \text{with} \quad \triangle_{\theta,\lambda}^r f := (I - T_\theta^\lambda)^{r/2} f,$$

for $f \in L^p(w_\lambda;[-1,1])$ if $1 \le p < \infty$, and $f \in C[-1,1]$ if $p = \infty$. Furthermore, following the proof of Theorem 10.4.1 and Corollary 10.3.3, but using the result for the Cesàro means of the Gegenbauer polynomial expansions, we can deduce the following characterization of the best approximation.

Theorem 12.1.7. *Let $f \in L^p(w_\lambda;[-1,1])$ if $1 \le p < \infty$, and $f \in C[-1,1]$ if $p = \infty$.*

(i) *If $t \in (0,1)$ and $r > 0$, then*

$$\omega_r^*(f,t)_{p,\lambda} \sim K_r(f,t)_{p,\lambda}.$$

If, in addition, $\lambda > 0$, then

$$\omega_r^*(f,t)_{p,\lambda} \sim \|\triangle_{t,\lambda}^r f\|_{p,\lambda}.$$

(ii) *We have both the direct Jackson inequality*

$$E_n(f)_{p,\lambda} := \inf_{g\in\Pi_n} \|f-g\|_{p,\lambda} \le c\omega_r^*(f,n^{-1})_{p,\lambda}, \quad n=1,2,\ldots,$$

and the inverse inequality

$$\omega_r^*(f,n^{-1})_{p,\lambda} \le cn^{-r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{p,\lambda}.$$

We conclude this section with a comparison of the two moduli of smoothness, $\omega_\varphi^r(f,t)_p$ and $\omega_r^*(f,t)_{p,\lambda}$. For $\lambda = 1/2$, we write $\omega_r^*(f,t)_{p,\lambda}$ as $\omega_r^*(f,t)_p$. The Ditzian–Totik moduli of smoothness $\omega_\varphi^r(f,t)_p$ are relatively easier to compute, whereas the computation of $\omega_r^*(f,t)_p$ is difficult, if possible at all. On the other hand, the definition of $\omega_\varphi^r(f,t)_p$ in the weighted case is more complicated, while the weighted modulus $\omega_r^*(f,t)_{p,\lambda}$ is easily defined for all $r>0$ and $\mu \ge 0$ and is intimately connected to the multiplier operators of the Gegenbauer polynomial expansions, which allows access to powerful tools in harmonic analysis. Despite their differences, we have the following theorem of equivalence:

Theorem 12.1.8. *If $r\in\mathbb{N}$, $1<p<\infty$, $\mu\ge 0$, and $t\in(0,1)$, then*

$$K_r(f,t)_{p,\mu} \sim K_{r,\varphi}(f,t)_{p,\mu} + t^r E_{r-1}(f)_{p,\mu}. \tag{12.1.13}$$

This theorem was proved in [41, Corollary 7.2], where examples were also given [41, Remark 7.9] to show that the equivalence (12.1.13) fails when $p=1$ and $p=\infty$.

12.2 The First Modulus of Smoothness and K-Functional

For our first pair of moduli of smoothness and K-functionals, we use the connection between the ball $\mathbb{B}^d$ and the sphere $\mathbb{S}^{d+m-1}$ discussed in Sect. 11.7, which allows us to project the unweighted modulus and K-functional defined in Eqs. (4.2.4) and (4.5.2) on the sphere $\mathbb{S}^{d+m-1}$ to those on the ball $\mathbb{B}^d$ with weight $W_\mu(x)$ for $\mu = \frac{m-1}{2}$.

Throughout this section, we denote by $\|\cdot\|_{p,\mu}$ the norm of $L^p(W_\mu;\mathbb{B}^d)$ for $1\le p\le\infty$, while for $p=\infty$, we write $\|f\|_{\infty,\mu} := \|f\|_\infty$ for $f\in C(\mathbb{B}^d)$. When we need to emphasize that the norm is taken over $\mathbb{B}^d$, we write $\|f\|_{L^p(W_\mu;\mathbb{B}^d)}$ instead of $\|f\|_{p,\mu}$.

12.2.1 Projection from Sphere to Ball

Given a function f on $\mathbb{B}^d$, we will frequently need to regard it as a projection onto $\mathbb{B}^d$ of a function F defined on $\mathbb{S}^{d+m-1}$ by

$$F(x,x') := f(x), \qquad (x,x')\in\mathbb{S}^{d+m-1}, \quad x\in\mathbb{B}^d, \quad x'\in\mathbb{B}^m. \tag{12.2.1}$$

Under such an extension of f, the equation in Lemma 11.7.1 becomes, for example,

$$\int_{\mathbb{S}^{d+m-1}} F(y)\mathrm{d}\sigma(y) = \sigma_m \int_{\mathbb{B}^d} f(x)(1-\|x\|^2)^{\frac{m-2}{2}}\mathrm{d}x, \tag{12.2.2}$$

where σ_m denotes the surface area of $\mathbb{S}^{m-1}$ for $m \ge 2$ and $\sigma_1 = 2$. Thus, the space $L^p(W_\mu;\mathbb{B}^d)$ can be identified with a subspace of $L^p(\mathbb{S}^{d+m-1})$ under Eq. (12.2.1). This is the starting point of our study in this section.

The mapping $f \mapsto F$ from $L^p(W_\mu;\mathbb{B}^d)$ to $L^p(\mathbb{S}^{d+m-1})$ in Eq. (12.2.1) preserves the convolution structures of the two spaces. Let $g \in L^2(w_\lambda;[-1,1])$ and $\lambda = \frac{d+m-2}{2}$. For $F \in L^1(\mathbb{S}^{d+m-1})$, the convolution $F * g$ is defined, in Definition 2.1.1, by

$$(F*g)(x) = \frac{1}{\omega_{d+m}} \int_{\mathbb{S}^{d+m-1}} F(y)g(\langle x,y\rangle)\mathrm{d}\sigma_{d+m}(y).$$

On the other hand, for $f \in L^1(W_\mu;\mathbb{B}^d)$ with $\mu = \frac{m-1}{2}$, the convolution $f *_{\mu,\mathbb{B}} g$ is defined, as in Definition 11.2.1, by

$$(f *_{\mu,\mathbb{B}} g)(x) = a_\mu \int_{\mathbb{B}^d} f(y) V_\mu^{\mathbb{B}}\left[g(\langle \cdot,(x,x_{d+1})\rangle)\right](y,y_{d+1}) W_\mu(y)\mathrm{d}y,$$

where $x_{d+1} = \sqrt{1-\|x\|^2}$ and $y_{d+1} = \sqrt{1-\|y\|^2}$. Using the explicit formula of $V_\mu^{\mathbb{B}}$ in Eq. (11.1.13), together with Eq. (7.2.2), it follows that

$$V_\mu^{\mathbb{B}}\left[g(\langle \cdot,(x,x_{d+1})\rangle)\right](y,y_{d+1}) = c_\mu \int_{-1}^1 g(\langle x,y\rangle + x_{d+1}y_{d+1}t)(1-t^2)^{\mu-1}\mathrm{d}t$$

for $\mu > 0$, and the formula holds under the limit $\mu \to 0^+$.

Lemma 12.2.1. *If $f \in L^1(W_\mu;\mathbb{B}^d)$ and $F \in L^1(\mathbb{S}^{d-1+m})$ is defined by Eq. (12.2.1), then for $x \in \mathbb{B}^d$, and $(x,x') \in \mathbb{S}^{d+m-1}$,*

$$(F*g)(x,x') = (f *_{\mu,\mathbb{B}} g)(x), \qquad \mu = \frac{m-1}{2}. \tag{12.2.3}$$

Proof. If $F(y,y') = f(y)$ as in Eq. (12.2.1), then by Lemma 11.7.1 and Eq. (A.5.1),

$$\begin{aligned}(F*g)(x,x') &= \frac{1}{\omega_{d+m}} \int_{\mathbb{B}^d} f(y)\left[\int_{\mathbb{S}^{m-1}} g\big(\langle x,y\rangle + y_{d+1}\langle x',\xi\rangle\big)\mathrm{d}\sigma(\xi)\right] W_\mu(y)\mathrm{d}y \\ &= c\int_{\mathbb{B}^d} f(y)\left[\int_{-1}^1 g\big(\langle x,y\rangle + y_{d+1}x_{d+1}t\big)(1-t^2)^{\frac{m-2}{2}}\mathrm{d}t\right] W_\mu(y)\mathrm{d}y,\end{aligned}$$

where $c = \omega_{m-1}/\omega_{d+m}$ and $x_{d+1} = \sqrt{1-\|x'\|^2}$, which gives the stated identity (12.2.3), since the constant can be verified either directly or using the fact that both convolutions become 1 if $f(x) = 1$ and $g(t) = 1$. ◻

In particular, for the operator L_n on $\mathbb{S}^{d+m-1}$ defined via the cutoff function as in Eq. (2.6.2) and the operator L_n^μ on $\mathbb{B}^d$ defined via the same cutoff function in Eq. (11.5.1) with respect to the weight function W_μ, we have

$$(L_n F)(x,x') = (L_n^\mu f)(x), \quad n = 1,2,\ldots, \tag{12.2.4}$$

for all $x \in \mathbb{B}^d$ and $(x,x') \in \mathbb{S}^{d+m-1}$, which will be useful below.

12.2.2 Modulus of Smoothness

Recall that for $1 \le i,j \le d$ and $\theta \in \mathbb{R}$, we use $Q_{i,j,\theta}$ to denote the rotation through the angle θ in the (x_i,x_j)-plane as in Eq. (4.2.1); for example,

$$Q_{1,2,\theta}x = (x_1\cos\theta + x_2\sin\theta, -x_1\sin\theta + x_2\cos\theta, x_3,\ldots,x_d), \quad x \in \mathbb{R}^d.$$

Furthermore, we define the difference operators $\triangle_{i,j,\theta}^r$ in Eq. (4.2.2) by

$$\triangle_{i,j,\theta}^r = (I - T(Q_{i,j,\theta}))^r, \quad 1 \le i \ne j \le d,$$

where $T(Q_{i,j,\theta})f(x) = f(Q_{i,j,\theta}x)$. These difference operators are used to define a modulus of smoothness $\omega_r(f,t)_p$ on the sphere $\mathbb{S}^{d-1}$ in Definition 4.2.1, which, when defined on the sphere $\mathbb{S}^{d+m-1}$, can be used to define a modulus of smoothness on the unit ball.

With a slight abuse of notation, we write $W_\mu(x) := (1-\|x\|^2)^{\mu-\frac{1}{2}}$ for either the weight function on $\mathbb{B}^d$ or that on $\mathbb{B}^{d+1}$, and write $\triangle_{i,j,\theta}^r$ for either the difference operator on $\mathbb{R}^d$ or that on $\mathbb{R}^{d+1}$. This should not cause any confusion, since which is meant should be clear from context. We denote by $\tilde{f}$ the extension of f in Eq. (12.2.1) in the case of $m = 1$; that is,

$$\tilde{f}(x,x_{d+1}) = f(x), \qquad (x,x_{d+1}) \in \mathbb{B}^{d+1}, \quad x \in \mathbb{B}^d. \tag{12.2.5}$$

Definition 12.2.2. Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Let $f \in L^p(W_\mu;\mathbb{B}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r \in \mathbb{N}$ and $t > 0$,

$$\omega_r(f,t)_{p,\mu} := \sup_{|\theta|\le t}\left\{\max_{1\le i<j\le d}\|\triangle_{i,j,\theta}^r f\|_{L^p(\mathbb{B}^d,W_\mu)}, \max_{1\le i\le d}\|\triangle_{i,d+1,\theta}^r \tilde{f}\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})}\right\}, \tag{12.2.6}$$

where for $m = 1$, $\|\triangle_{i,d+1,\theta}^r\tilde{f}\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})}$ is replaced by $\|\triangle_{i,d+1,\theta}^r\tilde{f}\|_{L^p(\mathbb{S}^d)}$.

The second term in Eq. (12.2.6) can be written more explicitly as, recalling Eq. (4.2.3),

$$\triangle_{i,d+1,\theta}^r\tilde{f}(x,x_{d+1}) = \triangle_\theta^r f(x_1,\ldots,x_{i-1},x_i\cos(\cdot) - x_{d+1}\sin(\cdot),x_{i+1},\ldots,x_d),$$

with the difference on the right-hand side being evaluated at 0. We can also write $\omega_r(f,t)_p$ in an equivalent but more compact form, as the next lemma shows.

Lemma 12.2.3. *With the above notation, we have*

$$\omega_r(f,t)_{p,\mu} := \sup_{|\theta|\le t} \max_{1\le i<j\le d+1} \|\triangle^r_{i,j,\theta}\tilde{f}\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})}. \tag{12.2.7}$$

Proof. If $1 \le i < j \le d$, then by definition,

$$\triangle^r_{i,j,\theta}\tilde{f}(x,x_{d+1}) = \triangle^r_{i,j,\theta}f(x), \quad x \in \mathbb{B}^d. \tag{12.2.8}$$

On the other hand, we observe that for a generic function $g : \mathbb{B}^d \to \mathbb{R}$ and $\lambda > -1$,

$$\begin{aligned}\int_{\mathbb{B}^{d+1}} \tilde{g}(y)(1-\|y\|^2)^\lambda \mathrm{d}y &= \int_{\mathbb{B}^d} g(x) \int_{-\sqrt{1-\|x\|^2}}^{\sqrt{1-\|x\|^2}} (1-\|x\|^2-u^2)^\lambda \mathrm{d}u\mathrm{d}x \\ &= c\int_{\mathbb{B}^d} g(x)(1-\|x\|^2)^{\lambda+1/2}\mathrm{d}x, \end{aligned} \tag{12.2.9}$$

where $c = \int_{-1}^1 (1-t^2)^\lambda \,\mathrm{d}t$, which, together with Eq. (12.2.8), implies that, for $1 \le i < j \le d$,

$$\|\triangle^r_{i,j,\theta}\tilde{f}\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})} = c\|\triangle^r_{i,j,\theta}f\|_{L^p(\mathbb{B}^d,W_\mu)}.$$

The equivalence of Eq. (12.2.6) with Eq. (12.2.7) then follows. □

Our next lemma reveals the idea behind the definition of this modulus of smoothness on $\mathbb{B}^d$. In fact, each function $f \in L^p(W_\mu;\mathbb{B}^d)$ can be identified with a function $F \in L^p(\mathbb{S}^{m+d-1})$ under the mapping Eq. (12.2.1), so that the modulus of smoothness $\omega_r(F,t)_{L^p(\mathbb{S}^{m+d-1})}$ defined as in Eq. (4.2.4) induces a modulus of smoothness for f, which is, as our lemma shows, exactly the modulus of smoothness in Eq. (12.2.6).

Lemma 12.2.4. *Let* $\mu = \frac{m-1}{2}$ *and* $m \in \mathbb{N}$. *Assume that* $f \in L^p(W_\mu;\mathbb{B}^d)$ *if* $1 \le p < \infty$ *and* $f \in C(\mathbb{B}^d)$ *if* $p = \infty$. *Let* F *be defined as in Eq.* (12.2.1)*. Then*

$$\omega_r(f,t)_{L^p(W_\mu;\mathbb{B}^d)} \sim \omega_r(F,t)_{L^p(\mathbb{S}^{d+m-1})},$$

where $\omega_r(F,t)_{L^p(\mathbb{S}^{d+m-1})}$ *is the modulus of smoothness defined in Eq.* (4.2.4)*.*

Proof. If $1 \le i < j \le d$, then $\triangle^r_{i,j,\theta}F(x,x') = \triangle^r_{i,j,\theta}f(x)$ by Eq. (4.2.1), and hence using Eqs. (12.2.2) and (12.2.9),

$$\int_{\mathbb{S}^{d+m-1}} |\triangle^r_{i,j,\theta}F(y)|^p \mathrm{d}\sigma(y) = \sigma_m \int_{\mathbb{B}^d} |\triangle^r_{i,j,\theta}f(x)|^p (1-\|x\|^2)^{\frac{m-2}{2}}\mathrm{d}x.$$

On the other hand, if $1 \le i \le d$ and $d+1 \le j \le d+m$, then it follows from Eq. (4.2.3) that $\triangle^r_{i,j,\theta} F(x,x') = \triangle^r_{i,d+1,\theta} \tilde{f}(x,x_j)$, where $x \in \mathbb{B}^d$, so that for $m \ge 2$,

$$\int_{\mathbb{S}^{d+m-1}} |\triangle^r_{i,j,\theta} F(y)|^p \mathrm{d}\sigma(y) = \sigma_{m-1} \int_{\mathbb{B}^{d+1}} |\triangle^r_{i,d+1,\theta} \tilde{f}(x)|^p (1-\|x\|^2)^{\frac{m-3}{2}} \mathrm{d}x$$

on account of Eq. (12.2.2), whereas there is nothing to prove for $m = 1$ by the modification in the definition of $\omega_r(f,t)_{L^p(W_\mu;\mathbb{B}^d)}$ in that case. Putting these together, we deduce the desired equation. □

12.2.3 Weighted K-Functional and Equivalence

Our first K-functional on the ball is defined, like the one on the sphere, in terms of the differentials $D_{i,j}$ defined in Eq. (1.8.5).

Definition 12.2.5. Let $\mu \ge 0$ and let $f \in L^p(W_\mu;\mathbb{B}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r \in \mathbb{N}$ and $t > 0$, define

$$K_r(f,t)_{p,\mu} := \inf_{g \in C^r(\mathbb{B}^d)} \Big\{ \|f-g\|_{L^p(W_\mu;\mathbb{B}^d)} + t^r \max_{1 \le i<j \le d} \|D^r_{i,j} g\|_{L^p(W_\mu;\mathbb{B}^d)} + t^r \max_{1 \le i \le d} \|D^r_{i,d+1} \tilde{g}\|_{L^p(W_{\mu-1/2};\mathbb{B}^{d+1})} \Big\}, \qquad (12.2.10)$$

where if $\mu = 0$, then $\|D^r_{i,d+1}\tilde{g}\|_{L^p(W_{\mu-1/2};\mathbb{B}^{d+1})}$ is replaced by $\|D^r_{i,d+1}\tilde{g}\|_{L^p(\mathbb{S}^d)}$.

In the case of $\mu > 0$, we can also define the K-functional in the equivalent but more compact form

$$K_r(f,t)_{p,\mu} = \inf_{g \in C^r(\mathbb{B}^d)} \Big\{ \|f-g\|_{L^p(W_\mu;\mathbb{B}^d)} + t^r \max_{1 \le i<j \le d+1} \|D^r_{i,j}\tilde{g}\|_{L^p(W_{\mu-1/2};\mathbb{B}^{d+1})} \Big\}.$$

The equivalence of the two definitions follows from Eq. (12.2.9). Although $\tilde{g}(x,x_{d+1}) = g(x)$ is a constant in the variable x_{d+1}, so that $\partial_{d+1}\tilde{g}(x,x_{d+1}) = 0$, we cannot replace $D^r_{i,d+1}$ by $(x_i\partial_i)^r$, since the operator $D_{i,d+1} = x_i\partial_{d+1} - x_{d+1}\partial_i$ involves x_{d+1}, and $D^r_{i,d+1}\tilde{g}$ is indeed a function of (x,x_{d+1}) in $\mathbb{B}^{d+1}$. The following lemma gives an explicit formula for this term.

Lemma 12.2.6. *Let $f \in C^r(\mathbb{B}^d)$. Assume that $(y,y_{d+1}) = s(x,x_{d+1}) \in \mathbb{B}^{d+1}$ with $s = \|(y,y_{d+1})\|$, $x \in \mathbb{B}^d$ and $(x,x_{d+1}) \in \mathbb{S}$.*

(1) *The function $D^r_{i,d+1}\tilde{f}(x,x_{d+1})$ is even in x_{d+1} if r is even, and odd in x_{d+1} if r is odd.*
(2) *If $x_{d+1} = \varphi(x) := \sqrt{1-\|x\|^2}$, then*

$$(D^r_{i,d+1}\tilde f)(y,y_{d+1}) = \left(-\varphi(x)\frac{\partial}{\partial x_i}\right)^r \Big[f(sx)\Big], \quad 1 \le i \le d.$$

Proof. The proof uses induction. For $r=1$, we have

$$D_{i,d+1}\tilde f(y,y_{d+1}) = (y_i\partial_{d+1} - y_{d+1}\partial_i) f(y) = -y_{d+1}\partial_i f(y).$$

Using the fact that $\frac{\partial}{\partial x_i}[f(sx)] = s(\partial_i f)(sx)$, we have

$$(D_{i,d+1}\tilde f)(sx, sx_{d+1}) = -sx_{d+1}(\partial_i f)(sx),$$

which is clearly odd in x_{d+1}, and it is equal to $-\varphi(x)\frac{\partial}{\partial x_i}[f(sx)]$ when $s=\varphi(x)$.

For $r>1$, let $F_r(x,x_{d+1}) = D^r_{i,d+1}\tilde f(x,x_{d+1})$. Assume that the result has been established for r. Then $F_r(sx, s\varphi(x)) = (-\varphi\partial_i)^r[f(sx)]$. By definition,

$$\begin{aligned} F_{r+1}(sx,sx_{d+1}) &= (D_{i,d+1}F_r)(sx,sx_{d+1}) \\ &= sx_i(\partial_{d+1}F_r)(sx,sx_{d+1}) - sx_{d+1}(\partial_i F_r)(sx,sx_{d+1}). \end{aligned} \tag{12.2.11}$$

The parity of F_r in x_{d+1} follows from induction by Eq. (12.2.11). On the other hand, taking the derivative by the chain rule shows that

$$\begin{aligned} \left(-\varphi(x)\frac{\partial}{\partial x_i}\right)^{r+1}[f(sx)] &= \left(-\varphi(x)\frac{\partial}{\partial x_i}\right)[F_r(sx,s\varphi(x))] \\ &= -s\varphi(x)(\partial_i F_r)(sx,s\varphi(x)) + sx_i(\partial_{d+1}F_r)(sx,s\varphi(x)), \end{aligned}$$

which is the same as the right-hand side of Eq. (12.2.11) with $x_{d+1}=\varphi(x)$. □

Proposition 12.2.7. *Let $g \in C^r(\mathbb{B}^d)$, $\mu \ge 0$, and $1 \le p < \infty$.*

(i) If $\mu = 0$ and $1 \le p < \infty$, then

$$\|D^r_{i,d+1}\tilde g\|_{L^p(\mathbb{S}^d)} = \|(\varphi\partial_i)^r g\|_{L^p(\mathbb{B}^d,W_0)}. \tag{12.2.12}$$

(ii) If $\mu > 0$ and $1 \le p < \infty$, then

$$\begin{aligned} &\|D^r_{i,d+1}\tilde g\|^p_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})} \\ &\quad = a_\mu \int_0^1 s^d(1-s^2)^{\mu-1}\int_{\mathbb{B}^d}\left|(\varphi(x)\partial_i)^r[g(sx)]\right|^p \frac{\mathrm{d}x}{\sqrt{1-\|x\|^2}}\mathrm{d}s. \end{aligned} \tag{12.2.13}$$

(iii) If $p=\infty$ and $\mu \ge 0$, then

$$\max_{y\in\mathbb{B}^{d+1}} |D^r_{i,d+1}\tilde g(y)| = \max_{x\in\mathbb{B}^d, 0\le s\le 1}\left|\left(\varphi(x)\frac{\partial}{\partial x_i}\right)^r[g(sx)]\right|.$$

Proof. For the proof of (i), we need only consider $\mathbb{S}^d_+ = \{x \in \mathbb{S}^d : x_{d+1} \ge 0\}$, by (1) of Lemma 12.2.6, when dealing with $D^r_{i,d+1}\tilde{g}$. By (2) of Lemma 12.2.6 with $s = 1$, we then obtain

$$\int_{\mathbb{S}^d} \left|D^r_{i,d+1}\tilde{g}(x,x_{d+1})\right|^p \mathrm{d}\sigma(x,x_{d+1}) = 2\int_{\mathbb{S}^d_+} |(\varphi(x)\partial_i)^r g(x)|^p \,\mathrm{d}\sigma(x,x_{d+1})$$
$$= \int_{\mathbb{B}^d} |(\varphi(x)\partial_i)^r g(x)|^p \frac{\mathrm{d}x}{\sqrt{1-\|x\|^2}},$$

which is what we needed to prove. The proof of (ii) and (iii) are simple consequences of Lemma 12.2.6. □

Just as in the case of the modulus of smoothness, the K-functional $K_r(f,t)_{p,\mu} \equiv K_r(f,t)_{L^p(W_\mu;\mathbb{B}^d)}$ is related to the K-functional $K_r(F,t)_p \equiv K_r(F,t)_{L^p(\mathbb{S}^{d+m-1})}$ defined by Eq. (4.5.2) on the sphere.

Lemma 12.2.8. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Assume that $f \in L^p(W_\mu;\mathbb{B}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Let F be defined as in Eq. (12.2.1). Then*

$$K_r(f,t)_{L^p(W_\mu;\mathbb{B}^d)} \sim K_r(F,t)_{L^p(\mathbb{S}^{d+m-1})}.$$

Proof. The estimate $K_r(F,t)_{L^p(\mathbb{S}^{d+m-1})} \le cK_r(f,t)_{L^p(W_\mu;\mathbb{B}^d)}$ follows directly from the definition and the fact that for every $g \in C^r(\mathbb{B}^d)$,

$$\|D^r_{i,j}\tilde{g}\|_{L^p(W_{\mu-1/2};\mathbb{B}^{d+1})} = c\|D^r_{i,j}G\|_{L^p(\mathbb{S}^{d+m-1})}, \quad 1 \le i < j \le d+1,$$

where $G(x,x') = g(x)$ for $x \in B^d$ and $(x,x') \in \mathbb{S}^{d+m-1}$.

To prove the inverse inequality, we observe that on account of Lemma 12.2.1 and Eq. (12.2.9), $\|f - L^\mu_n f\|_{L^p(W_\mu;\mathbb{B}^d)} = c\|F - L_n F\|_{L^p(\mathbb{S}^{d+m-1})}$, and

$$\|D^r_{i,j}\widetilde{L^\mu_n f}\|_{L^p(W_{\mu-1/2};\mathbb{B}^{d+1})} = c\|D^r_{i,j}L_n F\|_{L^p(\mathbb{S}^{d+m-1})}, \quad 1 \le i < j \le d+1.$$

The inverse inequality $K_r(f,t)_{L^p(W_\mu;\mathbb{B}^d)} \le cK_r(F,t)_{L^p(\mathbb{S}^{d+m-1})}$ then follows by choosing $g = L^\mu_n f$ with $n \sim \frac{1}{t}$ in Definition 12.2.5 and using Corollary 4.5.4. □

12.2.4 Main Theorems

The modulus of smoothness that we just defined can be used to characterize the best approximation by polynomials. Recall that Π^d_n denotes the space of real algebraic polynomials on $\mathbb{B}^d$ of total degree at most n.

Definition 12.2.9. Let $f \in L^p(W_\mu;\mathbb{B}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Define

$$E_n(f)_{p,\mu} := \inf_{g \in \Pi_n^d} \|f - g\|_{L^p(W_\mu;\mathbb{B}^d)}, \quad n = 0, 1, \ldots.$$

Theorem 12.2.10. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Assume that $f \in L^p(W_\mu;\mathbb{B}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Then for all $1 \le p \le \infty$,*

$$E_n(f)_{p,\mu} \le c\, \omega_r(f, n^{-1})_{p,\mu}, \;\; n = 1, 2, \ldots, \tag{12.2.14}$$

and

$$\omega_r(f, n^{-1})_{p,\mu} \le c n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_{p,\mu}. \tag{12.2.15}$$

Proof. Let F be defined as in Eq. (12.2.1). By Lemma 12.2.1, $(L_n^\mu f)(x) = (L_n F)(x, x')$, so that by Eq. (12.2.2) and the Jackson estimate for F in Eq. (4.4.1),

$$\begin{aligned} \|L_n^\mu f - f\|_{p,\mu}^p &= \int_{\mathbb{B}^d} |L_n^\mu f(x) - f(x)|^p W_\mu(x) \mathrm{d}x \\ &= c \int_{\mathbb{S}^{d+m-1}} |L_n F(y) - F(y)|^p \mathrm{d}\sigma(y) \\ &\le c\, \omega_r(F, n^{-1})_{L^p(\mathbb{S}^{d+m-1})} \le c\, \omega_r(f, n^{-1})_{p,\mu}, \end{aligned}$$

which proves Eq. (12.2.14). The inverse theorem follows likewise from

$$\begin{aligned} E_n(F)_{L^p(\mathbb{S}^{d+m-1})} &\le c\, \|L_{\lfloor \frac{n}{2} \rfloor} F - F\|_{L^p(\mathbb{S}^{d+m-1})} \\ &= c \|L^\mu_{\lfloor \frac{n}{2} \rfloor} f - f\|_{p,\mu} \le c E_{\lfloor \frac{n}{2} \rfloor}(f)_{p,\mu} \end{aligned}$$

and the inverse theorem for F in Eq. (4.4.2). □

As a direct consequence of Theorem 12.2.10, we have the following corollary.

Corollary 12.2.11. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Assume that $f \in L^p(W_\mu;\mathbb{B}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. If $r \in \mathbb{N}$ and $r > \alpha > 0$, then*

$$\sup_{n \in \mathbb{N}} n^\alpha E_n(f)_{p,\alpha} < \infty \;\text{ if and only if }\; \sup_{t \in (0,1)} t^{-\alpha} \omega_r(f, t)_{p,\mu} < \infty.$$

By Lemmas 12.2.4 and 12.2.8 and the equivalence in Theorem 4.5.3, we further arrive at the following.

Theorem 12.2.12. *Let $r \in \mathbb{N}$ and let $f \in L^p(W_\mu;\mathbb{B}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Then for $0 < t < 1$,*

$$\omega_r(f, t)_{p,\mu} \sim K_r(f, t)_{p,\mu}, \qquad 1 \le p \le \infty.$$

Corollary 12.2.13. *If $r \in \mathbb{N}$ and $f \in C^r(\mathbb{B}^d)$, then*

$$E_n(f)_\infty \le cn^{-r} \max_{1 \le i,j \le d} \|D^r_{i,j} f\|_\infty.$$

For $f \in C^r(\mathbb{B}^d)$, we could choose g in the definition of the K-functional as f, so that the corollary follows from the direct theorem. There are also L^p versions of such a result.

12.2.5 The Moduli of Smoothness on $[-1,1]$

When $d = 1$, the ball becomes the interval $B^1 = [-1,1]$, and our modulus of smoothness in Eq. (12.2.6) becomes, written out explicitly,

$$\omega_r(f,t)_{p,\mu} := \sup_{|\theta| \le t} \left(c_\mu \int_{\mathbb{B}^2} |\triangle^r_\theta f(x_1 \cos(\cdot) + x_2 \sin(\cdot))|^p W_{\mu - \frac{1}{2}}(x) \mathrm{d}x \right)^{1/p} \tag{12.2.16}$$

for $1 \le p < \infty$ with the usual modification for $p = \infty$, where $c_\mu^{-1} = \int_{\mathbb{B}^2} W_{\mu-\frac{1}{2}}(x)\mathrm{d}x$. Using polar coordinates, the integral over $\mathbb{B}^2$ can be written as a double integral against θ and r, and it follows that the difference $\triangle^r_\theta$ in Eq. (12.2.16) can be evaluated at any fixed point $t_0 \in [0, 2\pi]$. More precisely, for any fixed $t_0 \in [0, 2\pi]$, $\triangle^r_\theta f(x_1 \cos(\cdot) + x_2 \sin(\cdot)) = \triangle^r_\theta g_{x_1,x_2}(t_0)$, where $g_{x_1,x_2}(\theta) = f(x_1 \cos\theta + x_2 \sin\theta)$. This definition makes sense for all real μ such that $\mu > 0$, whereas for $\mu = 0$, the integral is taken over $\mathbb{S}^1$.

This modulus of smoothness on $[-1,1]$ is new in the sense that it was not known before the result in $\mathbb{B}^d$ was developed. Let us compare it with the Ditzian–Totik modulus of smoothness.

Theorem 12.2.14. *Let $\mu = \frac{m-1}{2}$, $m \in \mathbb{N}$, and $r \in \mathbb{N}$. Let $f \in L^p(w_\mu; [-1,1])$ if $1 \le p < \infty$, and $f \in C[-1,1]$ if $p = \infty$. Assume further that r is odd if $p = \infty$. Then*

$$\omega_r(f,t)_{p,\mu} \le c K_{r,\varphi}(f,t)_{p,\mu} + ct^r \|f\|_{p,\mu}, \quad 0 < t \le t_r, \tag{12.2.17}$$

where the term $t^r \|f\|_{p,\mu}$ can be dropped when $r = 1$.

Proof. By Theorem 12.2.12 and the equivalence (12.1.8), it suffices to prove the inequality for the corresponding K-functionals:

$$K_r(f,t)_{p,\mu} \le cK_{r,\varphi}(f,t)_{p,\mu} + ct^r \|f\|_{p,\mu}, \quad 1 \le p \le \infty,$$

with the additional assumption that r is odd when $p = \infty$. This inequality, together with the equivalence $K_1(f,t)_{p,\mu} \sim K_{1,\varphi}(f,t)_{p,\mu}$, is given in Theorem 12.3.2 of the next section.

In the case of $\mu = 1/2$, that is, $m = 2$, $K_{r,\varphi}(f,t)_p$ is equivalent to the Ditzian–Totik modulus of smoothness $\omega_\varphi^r(f,t)_p$, so that Eq. (12.2.18) implies that

$$\omega_r(f,t)_{p,1/2} \le c\,\omega_\varphi^r(f,t)_p + ct^r\|f\|_{p,\mu}, \quad 0 < t \le t_r, \tag{12.2.18}$$

which shows that the new modulus of smoothness in Eq. (12.2.16) is at least no worse than that of the Ditzian–Totik modulus of smoothness.

It is worthwhile to point out that Eqs. (12.1.6) and (12.2.18) imply that in the unweighted case of $m = 2$,

$$\begin{aligned}\int_{\mathbb{B}^2} |\triangle_\theta^r f(x_1\cos(\cdot) + x_2\sin(\cdot))|^p \frac{\mathrm{d}x_1\mathrm{d}x_2}{\sqrt{1-x_1^2-x_2^2}} \\ \le c\frac{1}{t}\int_0^t \|\hat{\triangle}_{h\varphi}^r f\|_p^p \mathrm{d}h + ct^{rp}\|f\|_p^p,\end{aligned} \tag{12.2.19}$$

with the usual modification when $p = \infty$. This inequality is in fact highly nontrivial, and it will play a crucial role in Sect. 12.3.3.

12.2.6 Computational Examples

One main advantage of the modulus of smoothness defined in this section over two others defined in later sections is that it is reasonably easy to compute. We give several examples below to demonstrate this computability. For simplicity, we will present only the results in the unweighted case.

Example 12.2.15. For $\alpha \ne 0$, define $f_\alpha : \mathbb{B}^d \to \mathbb{R}$ by $f_\alpha(x) = (1 - \|x\|^2 + \|x - y_0\|^2)^\alpha$, where y_0 is a fixed point on $\mathbb{B}^d$. If $\alpha \ne 1 - \frac{d+1}{2p}$, then

$$\omega_2(f_\alpha,t)_{L^p(\mathbb{B}^d)} \sim t^2\|y_0\|(t+1-\|y_0\|)^{2(\alpha-1)+\frac{d+1}{p}} + t^2\|y_0\|,$$

where the constants of equivalence are independent of y_0 and t. Moreover, if $\alpha = 1 - \frac{d+1}{2p}$, then

$$c_\alpha^{-1}t^2\|y_0\| \le \omega_2(f_\alpha,t)_{L^p(\mathbb{B}^d)} \le c_\alpha t^2\|y_0\||\log(t+1-\|y_0\|)|^{\frac{1}{p}},$$

where c_α is independent of t and y_0.

Example 12.2.16. For $\alpha \ne 0$, let $f_\alpha(x) = (1 - \|x\|^2)^\alpha$ for $x \in \mathbb{B}^d$. Then

$$\omega_2(f_\alpha,t)_{L^p(\mathbb{B}^d)} \sim \begin{cases} t^{2\alpha+\frac{2}{p}}, & \text{if } -\frac{1}{p} < \alpha < 1 - \frac{1}{p}, \\ t^2|\log t|^{\frac{1}{p}}, & \text{if } \alpha = 1 - \frac{1}{p}, \\ t^2, & \text{if } \alpha > 1 - \frac{1}{p}. \end{cases}$$

Example 12.2.17. Let $f_\alpha(x) = x^\alpha$ for $x \in \mathbb{B}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \neq 0$. If $0 \le \alpha_i < 1$ for all $1 \le i \le d$, then for $1 \le p \le \infty$,

$$\omega_2(f_\alpha, t)_{L^p(\mathbb{B}^d)} \sim t^{\delta + \frac{1}{p}}, \quad \delta := \min_{\alpha_i \neq 0} \{\alpha_1, \ldots, \alpha_d\}.$$

Example 12.2.18. Let $\alpha \neq 0$, $d \ge 2$ and let $f_\alpha : \mathbb{B}^d \to \mathbb{R}$ be given by $f_\alpha(x) = \|x - e_0\|^{2\alpha}$, where $e_0 = (1, 0, \ldots, 0) \in \mathbb{B}^d$. Then

$$\omega_2(f_\alpha, t)_{L^p(\mathbb{B}^d)} \sim \begin{cases} t^{2\alpha + \frac{d}{p}}, & -\frac{d}{2p} < \alpha < 1 - \frac{d}{2p}, \\ t^2 |\log t|^{\frac{1}{p}}, & \alpha = 1 - \frac{d}{2p}, \quad p \neq \infty, \\ t^2, & \alpha > 1 - \frac{d}{2p}. \end{cases} \tag{12.2.20}$$

The verification of the asymptotic results stated in the above examples uses essentially straightforward, though tedious, computations. The interested reader can find the details in [50].

The above computational examples and Theorem 12.2.10 immediately lead to examples for the asymptotic order of $E_n(f)_{L^p(\mathbb{B}^d)}$. We give two examples. One corresponds to Example 12.2.15, and the other corresponds to Example 12.2.18.

Example 12.2.19. For $\alpha \neq 0$, let $f_\alpha(x) = (1 - \|x\|^2)^\alpha$. Then for $-\frac{1}{p} < \alpha < 1 - \frac{1}{p}$,

$$E_n(f_\alpha)_{L^p(\mathbb{B}^d)} \sim n^{-2\alpha - \frac{2}{p}}.$$

Example 12.2.20. For $\alpha \neq 0$, $d \ge 2$, let $f_\alpha(x) = \|x - e_0\|^{2\alpha}$, where $e_0 = (1, 0, \ldots, 0)$. For $-\frac{d}{2p} < \alpha < 1 - \frac{d}{2p}$,

$$E_n(f_\alpha)_{L^p(\mathbb{B}^d)} \sim n^{-2\alpha - \frac{d}{p}}. \tag{12.2.21}$$

Although our moduli of smoothness on the ball are not rotationally invariant, the best approximation $E_n(f)_{L^p(\mathbb{B}^d)}$ is; that is, $E_n(f)_{L^p(\mathbb{B}^d)} = E_n(f(\rho\{\cdot\}))_{L^p(\mathbb{B}^d)}$ for $\rho \in O(d)$. This implies, since every point x_0 on $\mathbb{S}^{d-1}$ can be rotated to e_0, that Eq. (12.2.21) holds for $f_{\alpha, x_0}(x) := \|x - x_0\|^{2\alpha}$. In particular, Theorem 12.2.10 shows then that

$$\omega_2(f_{\alpha, x_0}, t)_{L^p(\mathbb{B}^d)} \sim t^{2\alpha + \frac{d}{p}}, \qquad -\frac{d}{2p} < \alpha < 1 - \frac{d}{2p}. \tag{12.2.22}$$

12.3 The Second Modulus of Smoothness and *K*-Functional

In this section, we introduce our second pair of modulus of smoothness and *K*-functional on the unit ball, which are analogues of those defined by Ditzian and Totik on $[-1, 1]$, and we use them to characterize best approximation on the unit ball.

Throughout this section, we let $\varphi(x) := \sqrt{1-\|x\|^2}$ for $x \in \mathbb{B}^d$. With a slight abuse of notation, we also use φ to denote $\sqrt{1-x^2}$ on the interval $[-1,1]$.

12.3.1 Analogue of the Ditzian–Totik K-Functional

Recall that the Ditzian–Totik K-functional $K_{r,\varphi}(f,t)_{p,\mu}$ on $[-1,1]$ is defined in Eq. (12.1.7). We now define its higher-dimensional analogue on the ball $\mathbb{B}^d$.

Definition 12.3.1. Let $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r \in \mathbb{N}$ and $t > 0$, define

$$K_{r,\varphi}(f,t)_{p,\mu} := \inf_{g \in C^r(\mathbb{B}^d)} \left\{ \|f-g\|_{p,\mu} + t^r \max_{1\le i<j\le d} \|D_{i,j}^r g\|_{p,\mu} + t^r \max_{1\le i\le d} \|\varphi^r \partial_i^r g\|_{p,\mu} \right\}.$$

Part of our development in this section will rely on a connection between $K_{r,\varphi}(f,t)_{p,\mu}$ and the K-functional $K_r(f,t)_{p,\mu}$ defined in Eq. (12.2.5) of the previous section, which we prove first. In fact, the following theorem has already been called for in the proof of Theorem 12.2.14 of the previous section.

Theorem 12.3.2. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Let $f \in L^p(W_\mu; \mathbb{B}^d)$ if $1 \le p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. We further assume that r is odd when $p = \infty$. Then*

$$K_{1,\varphi}(f,t)_{p,\mu} \sim K_1(f,t)_{p,\mu}, \tag{12.3.1}$$

and for $r > 1$, there is $t_r > 0$ such that

$$K_r(f,t)_{p,\mu} \le c\, K_{r,\varphi}(f,t)_{p,\mu} + c t^r \|f\|_{p,\mu}, \quad 0 < t < t_r. \tag{12.3.2}$$

The proof of Theorem 12.3.2 relies on several lemmas. Our main tool is the following Hardy inequality.

Lemma 12.3.3. *For $1 \le p < \infty$ and $\beta > 0$,*

$$\left(\int_0^\infty \left(\int_x^\infty |f(y)| \mathrm{d}y \right)^p x^{\beta-1} \mathrm{d}x \right)^{1/p} \le \frac{p}{\beta} \left(\int_0^\infty |y f(y)|^p y^{\beta-1} \mathrm{d}y \right)^{1/p}. \tag{12.3.3}$$

Proof. The inequality (12.3.3) for $p = 1$ follows directly from Fubini's theorem. Assume now that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\alpha := \frac{1}{p'} + \frac{\beta-\delta}{p}$, with $\delta \in (0,\beta)$ to be chosen later. Using Hölder's inequality, for every $x > 0$,

$$\begin{aligned}\left(\int_x^\infty |f(y)| \mathrm{d}y \right)^p &\le \left(\int_x^\infty |f(y)|^p |y|^{\alpha p} \mathrm{d}y \right) \left(\int_x^\infty y^{-\alpha p'} \mathrm{d}y \right)^{p/p'} \\ &= \left(\frac{p}{p'(\beta-\delta)} \right)^{p/p'} x^{-\alpha p + p - 1} \int_x^\infty |f(y)|^p y^{\alpha p} \mathrm{d}y.\end{aligned}$$

Integrating both sides of this last inequality with respect to $x^{\beta-1}\,\mathrm{d}x$, and applying Fubini's theorem, we obtain

$$\begin{aligned}
&\int_0^\infty \left(\int_x^\infty |f(y)|\mathrm{d}y\right)^p x^{\beta-1}\mathrm{d}x \\
&\leq \left(\frac{p}{p'(\beta-\delta)}\right)^{p/p'} \int_0^\infty |f(y)|^p y^{\alpha p}\left(\int_0^y x^{-\alpha p+p-1}x^{\beta-1}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \left(\frac{p}{(\beta-\delta)p'}\right)^{p-1} \frac{1}{\delta}\int_0^\infty |yf(y)|^p y^{\beta-1}\,\mathrm{d}y.
\end{aligned}$$

The desired inequality (12.3.3) then follows by choosing $\delta = \beta/p$. □

The second lemma contains two Landau-type inequalities.

Lemma 12.3.4. *Let* $\mu > -\frac{1}{2}$ *and* $r \in \mathbb{N}$. *Assume that* $f \in C^r[-1,1]$ *and* $1 \leq p \leq \infty$.

(i) *If* $1 \leq p < \infty$ *and* $1 \leq i \leq \frac{r}{2}$ *or* $p = \infty$ *and* $1 \leq i < \frac{r}{2}$, *then*

$$\|\varphi^{r-2i}f^{(r-i)}\|_{p,\mu} \leq c_1\|\varphi^r f^{(r)}\|_{p,\mu} + c_2\|\varphi^r f\|_{p,\mu}. \tag{12.3.4}$$

(ii) *If* r *is even, set* $\delta_r := 0$ *and assume* $1 \leq i \leq \frac{r}{2}$, $1 \leq p < \infty$. *If* r *is odd, set* $\delta_r := 1$ *and assume* $1 \leq i \leq \frac{r+1}{2}$, $1 \leq p \leq \infty$. *Then*

$$\|\varphi^{\delta_r} f^{(i)}\|_{p,\mu} \leq c_1\|\varphi^r f^{(r)}\|_{p,\mu} + c_2\|\varphi^r f\|_{p,\mu}. \tag{12.3.5}$$

Proof. We claim that for $j \in \mathbb{N}$, $\ell \in \mathbb{N}$, and $f \in C^{j+\ell}[-1,1]$,

$$\|\varphi^a f^{(j)}\|_{p,\mu} \leq c_{j,\ell}\|\varphi^{a+2\ell} f^{(j+\ell)}\|_{p,\mu} + c_{j,\ell}\int_{-2^{-j}}^{2^{-j}} |f(x)|\mathrm{d}x \tag{12.3.6}$$

whenever $\mu + \frac{1}{2} + \frac{ap}{2} > 0$ and $1 \leq p < \infty$ or $a > 0$ and $p = \infty$. Since $\phi(x) \sim 1$ for $x \in [-2^{-j}, 2^{-j}]$, Eqs. (12.3.4) and (12.3.5) will follow from Eq. (12.3.6) by setting $j = r-i$, $a = r-2i$, and $\ell = i$, and by setting $j = i$, $a = \delta_r$, and $\ell = r-i$, respectively.

Iterating if necessary, we see that it suffices to prove the claim (12.3.6) for $\ell = 1$ and an arbitrary $j \in \mathbb{N}$. For $1 \leq p < \infty$, by the fundamental theorem of calculus, we obtain that for $x \in [0,1]$,

$$|f^{(j)}(x)| \leq \int_{-2^{-j}}^x |f^{(j+1)}(t)|\,\mathrm{d}t + \min_{y\in[-2^{-j},0]} |f^{(j)}(y)|.$$

Recall that the jth-order difference operator $\triangle_h^j$ with step $h < 0$ satisfies the property that $\triangle_h^j f(x) = (-1)^j h^j f^{(j)}(\xi)$ for some $\xi \in [x+jh, x]$. Thus, setting $h_j = -2^{-j-1}/j$, we have

$$\min_{y\in[-2^{-j},0]} |f^{(j)}(y)| \le c_j \min_{x\in[-2^{-j-1},0]} |\triangle_h^j f(x)|$$
$$\le c_j \max_{0\le k\le j} \int_{-2^{-j-1}}^{0} |f(x+kh)|\,\mathrm{d}x \le c_j \int_{-2^{-j}}^{0} |f(x)|\,\mathrm{d}x.$$

Together, the above two displayed equations imply that for $x\in[0,1]$,

$$|f^{(j)}(x)| \le \int_{-2^{-j}}^{x} |f^{(j+1)}(t)|\,\mathrm{d}t + c_j \int_{-2^{-j}}^{0} |f(x)|\,\mathrm{d}x. \tag{12.3.7}$$

On the other hand, using Hardy's inequality (12.3.3), we observe that

$$\int_0^1 \left[\int_{-2^{-j}}^{x} |f^{(j+1)}(t)|\,\mathrm{d}t\right]^p \varphi(x)^{ap} w_\mu(x)\,\mathrm{d}x$$
$$\le c\int_0^1 \left[\int_x^{\frac{3}{2}} |f^{(j+1)}(1-t)|\,\mathrm{d}t\right]^p x^{\frac{ap}{2}+\mu-\frac{1}{2}}\,\mathrm{d}x$$
$$\le c\int_0^{3/2} |f^{(j+1)}(1-x)|^p x^{\frac{ap}{2}+p+\mu-\frac{1}{2}}\,\mathrm{d}x \le c\|\varphi^{a+2} f^{(j+1)}\|_{p,\mu}^p.$$

Thus, combining this inequality and Eq. (12.3.7), we obtain

$$\int_0^1 |\varphi(x)^a f^{(j)}(x)|^p w_\mu(x)\,\mathrm{d}x \le c\|\varphi^{a+2} f^{(j+1)}\|_{p,\mu}^p + c\int_{-2^{-j}}^{0} |f(y)|\,\mathrm{d}y.$$

Similarly, using symmetry, we also have

$$\int_{-1}^0 |\varphi(x)^a f^{(j)}(x)|^p w_\mu(x)\,\mathrm{d}x \le c\|\varphi^{a+2} f^{(j+1)}\|_{p,\mu}^p + c\int_0^{2^{-j}} |f(y)|\,\mathrm{d}y.$$

Combining these two inequalities proves Eq. (12.3.6) for $\ell = 1$ and $1\le p<\infty$. To prove Eq. (12.3.6) for the case in which $\ell=1$ and $p=\infty$, we set $I_j(x) := [-2^{-j}, x]$ for $x\in[0,1]$, and $I_j(x) := [x, 2^{-j}]$ for $x\in[-1,0]$. Then, using Eq. (12.3.7) and symmetry, we obtain

$$|f^{(j)}(x)|\varphi(x)^a \le \varphi(x)^a \int_{I_j(x)} |f^{(j+1)}(t)|\,\mathrm{d}t + c\int_{-2^{-j}}^{2^{-j}} |f(y)|\mathrm{d}y$$
$$\le \|\varphi^{a+2} f^{(j+1)}\|_\infty \varphi(x)^a \int_{I_j(x)} (1-x^2)^{-\frac{a}{2}-1}\,\mathrm{d}x + c\int_{-2^{-j}}^{2^{-j}} |f(y)|\mathrm{d}y$$
$$\le c\|\varphi^{a+2} f^{(j+1)}\|_\infty + c\int_{-2^{-j}}^{2^{-j}} |f(y)|\,\mathrm{d}y,$$

provided that $a>0$. This completes the proof of Eq. (12.3.6). □

Lemma 12.3.5. *Let $\tilde f$ be defined as in Eq. (12.2.5). Then*

$$D^r_{1,d+1}\tilde f(x,x_{d+1}) = \sum_{j=1}^r p_{j,r}(x_1,x_{d+1})\partial_1^j f(x), \quad x\in\mathbb{B}^d,\ (x,x_{d+1})\in\mathbb{B}^{d+1},$$

where $p_{r,r}(x_1,x_{d+1}) = x_{d+1}^r$ and

$$p_{j,2r}(x_1,x_{d+1}) = \sum_{\max\{0,j-r\}\le\nu\le j/2} a^{(2r)}_{\nu,j} x_1^{j-2\nu}x_{d+1}^{2\nu}, \tag{12.3.8}$$

$$p_{j,2r-1}(x_1,x_{d+1}) = \sum_{\max\{0,j-r\}\le\nu\le(j-1)/2} a^{(2r-1)}_{\nu,j} x_1^{j-1-2\nu}x_{d+1}^{2\nu+1} \tag{12.3.9}$$

for $1\le j\le 2r-1$ and $1\le j\le 2r-2$, respectively, and $a^{(r)}_{\nu,j}$ are absolute constants.

Proof. Recall that $\tilde f(x,x_{d+1}) = f(x)$, so that $\partial_{d+1}\tilde f(x,x_{d+1}) = 0$. We use induction. Starting from

$$D^{r+1}_{1,d+1}\tilde f(x_1,x_{d+1}) = (x_{d+1}\partial_1 - x_1\partial_{d+1})\sum_{j=1}^r p_{j,r}(x_1,x_{d+1})\partial_1^j f(x),$$

a simple computation shows that $p_{j,r}$ satisfies the recurrence relation

$$p_{j,r+1} = x_{d+1}p_{j-1,r} + (x_{d+1}\partial_1 - x_1\partial_{d+1})p_{j,r}, \quad 1\le j\le r, \tag{12.3.10}$$

where we define $p_{0,r} := 0$, and $p_{r+1,r+1} = x_{d+1}p_{r,r}$. Since $p_{1,1} = x_{d+1}$, we see that $p_{r,r} = x^r_{d+1}$ by induction. The general case also follows by induction: assuming that $p_{j,r}$ takes the stated form, we apply Eq. (12.3.10) twice to get $p_{j,r+2}$ and verify that $p_{j,2r}$ and $p_{j,2r-1}$ are of the forms (12.3.8) and (12.3.9). □

We will also need the following integral formula, which is a simple consequence of a change of variables.

Lemma 12.3.6. *For $1\le m\le d-1$,*

$$\int_{\mathbb{B}^d} f(x)\mathrm{d}x = \int_{\mathbb{B}^{d-m}}\left[\int_{\mathbb{B}^m} f\left(\sqrt{1-\|v\|^2}u, v\right)\mathrm{d}u\right]\left(1-\|v\|^2\right)^{\frac{m}{2}}\mathrm{d}v. \tag{12.3.11}$$

We are now in a position to prove Theorem 12.3.2.

Proof of Theorem 12.3.2. We give the proof for the case $m\ge 2$ only. The proof for the case $m=1$ follows along the same lines. The only difference in this case is that we need to replace the integral over $\mathbb{B}^{d+1}$ by one over $\mathbb{S}^d$ according to Definition 12.2.5 and use Eq. (A.5.4) instead of Eq. (12.3.11).

By definition, we need to compare $\|D^r_{i,d+1}\tilde g\|_{L^p(W_{\mu-1/2};\mathbb{B}^{d+1})}$ with $\|\varphi^r\partial_i^r g\|_{p,\mu}$, where $\|\cdot\|_{p,\mu}\equiv\|\cdot\|_{L^p(W_\mu;\mathbb{B}^d)}$. More precisely, we need to show that

$$\|D_{i,d+1}\tilde g\|_{L^p(W_{\mu-1/2};\mathbb{B}^{d+1})}\sim\|\varphi\partial_i g\|_{p,\mu},\ \ 1\le i\le d, \tag{12.3.12}$$

and for $r\ge 2$,

$$\|D^r_{i,d+1}\tilde g\|_{L^p(W_{\mu-1/2};\mathbb{B}^{d+1})}\le c\|\varphi^r\partial_i^r g\|_{p,\mu}+c\|g\|_{p,\mu},\ \ 1\le i\le d \tag{12.3.13}$$

If $r=1$, then by Eq. (4.2.3), $D_{1,d+1}\tilde g(x,x_{d+1})=x_{d+1}\partial_1 g(x)$. Hence by Eq. (12.3.11),

$$\begin{aligned}\|D_{i,d+1}\tilde g\|^p_{L^p(W_{\mu-1/2};\mathbb{B}^{d+1})}&=\int_{\mathbb{B}^{d+1}}|x_{d+1}\partial_1 g(x)|^p(1-\|x\|^2-x^2_{d+1})^{\mu-1}\mathrm{d}x\mathrm{d}x_{d+1}\\&=c\int_{\mathbb{B}^d}|\varphi(x)\partial_1 g(x)|^p(1-\|x\|^2)^{\mu-1/2}\mathrm{d}x=c\|\varphi\partial_1 g\|^p_{p,\mu},\end{aligned}$$

where $c=\int_{-1}^1|s|^p(1-s^2)^{\mu-1}\mathrm{d}s$. The above argument with slight modification works equally well for $p=\infty$. This proves Eq. (12.3.12).

Next, we prove Eq. (12.3.13) for $r\ge 2$. By symmetry, we need to consider only the case $i=1$. We start with the case of even $r=2\ell$ with $\ell\in\mathbb{N}$. In this case, $1\le p<\infty$, and by Eq. (12.3.8), we have

$$|D^{2\ell}_{1,d+1}\tilde g(x,x_{d+1})|\le c\sum_{j=1}^{2\ell}\max_{\max\{0,j-\ell\}\le\nu\le j/2}\left|x_1^{j-2\nu}x_{d+1}^{2\nu}\partial_1^j g(x)\right|.$$

This implies

$$\|D^{2\ell}_{1,d+1}\tilde g\|_{L^p(W_{\mu-1/2};\mathbb{B}^{d+1})}\le c\sum_{j=1}^{\ell}\max_{0\le\nu\le j/2}I_{j,\nu}+c\sum_{j=\ell+1}^{2\ell}\max_{j-\ell\le\nu\le j/2}I_{j,\nu}, \tag{12.3.14}$$

with

$$\begin{aligned}I_{j,\nu}&:=\int_{\mathbb{B}^{d+1}}\left|x_1^{j-2\nu}x_{d+1}^{2\nu}\partial_1^j g(x)\right|^p(1-\|x\|^2-x^2_{d+1})^{\mu-1}d(x,x_{d+1})\\&=c\int_{\mathbb{B}^d}\left|x_1^{j-2\nu}\varphi^{2\nu}(x)\partial_1^j g(x)\right|^p(1-\|x\|^2)^{\mu-1/2}\mathrm{d}x,\end{aligned}$$

where the last equation follows by Eq. (12.3.11). Let $x=(x_1,x')\in\mathbb{B}^d$. Using Eq. (12.3.11) again, and setting $g_{x'}(t)=g(t\varphi(x'),x')$, we see that

$$\begin{aligned}I_{j,\nu}&=c\int_{\mathbb{B}^{d-1}}\int_{-1}^1\left|t^{j-2\nu}\varphi^j(x')\varphi^{2\nu}(t)\partial_1^j g(\varphi(x')t,x')\right|^p(1-t^2)^{\mu-1/2}\mathrm{d}t(1-\|x'\|^2)^\mu\mathrm{d}x'\\&\le c\int_{\mathbb{B}^{d-1}}\left[\int_{-1}^1|\varphi^{2\nu}(t)g_{x'}^{(j)}(t)|^p(1-t^2)^{\mu-1/2}\mathrm{d}t\right](1-\|x'\|^2)^\mu\mathrm{d}x',\end{aligned}$$

where the inequality results from $|t^{j-2\nu}|\le 1$.

If $1 \le j \le \ell = \frac{r}{2}$ and $\nu \ge 0$, then $\varphi^{2\nu}(t) \le 1$, so that we can apply Eq. (12.3.5) in Lemma 12.3.4 to conclude that

$$I_{j,\nu} \le c \int_{\mathbb{B}^{d-1}} \left[\int_{-1}^{1} \left| \varphi^{2\ell}(t) \frac{\mathrm{d}^{2\ell}}{\mathrm{d}t^{2\ell}} \left[g(\varphi(x')t, x') \right] \right|^p (1-t^2)^{\mu-1/2} \mathrm{d}t \right] (1 - \|x'\|^2)^{\mu} \mathrm{d}x'$$
$$+ c \int_{\mathbb{B}^{d-1}} \int_{-1}^{1} \left| g(\varphi(x')t, x') \right|^p (1-t^2)^{\mu-1/2} \mathrm{d}t (1 - \|x'\|^2)^{\mu} \mathrm{d}x'$$
$$= c \int_{\mathbb{B}^d} \left| \varphi^{2\ell}(x) \partial_1^{2\ell} g(x) \right|^p (1 - \|x\|^2)^{\mu-1/2} \mathrm{d}x + c\|g\|_{p,\mu}^p.$$

If $\ell + 1 \le j \le 2\ell$ and $\nu \ge j - \ell$, then $\varphi^{2\nu}(t) \le \varphi^{2j-2\ell}(t)$, so that we can apply Eq. (12.3.4) in Lemma 12.3.4 with $i = 2\ell - j$ and $r = 2\ell$ to the integral over t, which leads, exactly as in the previous case, to $I_{j,\nu} \le c\|\varphi^{2\ell}\partial_1^{2\ell} g\|_{p,\mu}^p + c\|g\|_{p,\mu}^p$.

Putting these together, and using Eq. (12.3.14), we have established the desired result for the case of even $r = 2\ell$. The proof for the case of odd r follows along the same lines. This completes the proof. □

12.3.2 Direct and Inverse Theorems Using the K-Functional

Using the K-functional in Definition 12.3.1, we establish both the direct and the inverse inequalities.

Theorem 12.3.7. *Let $\mu = \frac{m-1}{2}$, $m \in \mathbb{N}$ and $r \in \mathbb{N}$. Let $f \in L^p(W_\mu; \mathbb{B}^d)$ if $1 \le p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Then*

$$E_n(f)_{p,\mu} \le c\, K_{r,\varphi}(f, n^{-1})_{p,\mu} + cn^{-r}\|f\|_{p,\mu} \tag{12.3.15}$$

and

$$K_{r,\varphi}(f, n^{-1})_{p,\mu} \le c\, n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_{p,\mu}. \tag{12.3.16}$$

Furthermore, the additional term $n^{-r}\|f\|_{p,\mu}$ on the right-hand side of Eq. (12.3.15) can be dropped when $r = 1$.

Proof. When $1 \le p < \infty$ and $r \in \mathbb{N}$ or $p = \infty$ and r is odd, the Jackson-type estimate (12.3.15) follows immediately from Eq. (12.2.14) and Theorem 12.3.2. Thus, it remains to prove Eq. (12.3.15) for even $r = 2\ell$ and $p = \infty$. Since we have already proved Eq. (12.3.15) for $K_{2\ell+1,\varphi}(f,t)_\infty$, it suffices to prove the inequality

$$K_{2\ell+1,\varphi}(f,t)_\infty \le cK_{2\ell,\varphi}(f,t)_\infty.$$

For $d = 1$, this inequality has already been proved in [61, p. 38], whereas in the case of $d \ge 2$, it is a consequence of the following inequalities:

$$\|D_{i,j}^{r+1} g\|_\infty \le c\|D_{i,j}^r g\|_\infty \quad \text{and} \quad \|\varphi^{r+1}\partial_i^{r+1} g\|_\infty \le c\|\varphi^r \partial_i^r g\|_\infty,$$

which can be deduced directly from the corresponding results for functions of one variable; see, for example, Eq. (12.3.6).

The inverse estimate Eq. (12.3.16) follows as usual from the Bernstein inequalities: For $1 \le p \le \infty$ and $P \in \Pi_n^d$,

$$\max_{1\le i<j\le d} \|D_{i,j}^r P\|_{p,\mu} \le cn^r \|P\|_{p,\mu} \quad \text{and} \quad \max_{1\le i\le d} \|\varphi^r \partial_i^r P\|_{p,\mu} \le cn^r \|P\|_{p,\mu}. \tag{12.3.17}$$

We shall prove only the first inequality in Eq. (12.3.17), since the second inequality follows along the same lines. Without loss of generality, we may assume $(i,j) = (1,2)$. We then have, for $1 \le p < \infty$,

$$\begin{aligned}
\|D_{1,2}^r f\|_{p,\mu}^p &= \int_{\mathbb{B}^{d-2}} \left[\int_{\mathbb{B}^2} |D_{1,2}^r f(\varphi(u)x_1, \varphi(u)x_2, u)|^p (1-x_1^2-x_2^2)^{\mu-\frac{1}{2}} \,\mathrm{d}x_1 \mathrm{d}x_2 \right] \\
&\quad \times (1-\|u\|^2)^{\mu+\frac{1}{2}} \,\mathrm{d}u \\
&= \int_{\mathbb{B}^{d-2}} \int_0^1 \left[\int_0^{2\pi} |D_{1,2}^r f(\varphi(u)\rho\cos\theta, \varphi(u)\rho\sin\theta, u)|^p \,\mathrm{d}\theta \right] \\
&\quad \times (1-\rho^2)^{\mu-\frac{1}{2}} \rho \,\mathrm{d}\rho \,(1-\|u\|^2)^{\mu+\frac{1}{2}} \,\mathrm{d}u \\
&= \int_{\mathbb{B}^{d-2}} \int_0^1 \left[\int_0^{2\pi} |f_{u,\rho}^{(r)}(\theta)|^p \,\mathrm{d}\theta \right] (1-\rho^2)^{\mu-\frac{1}{2}} \rho \,\mathrm{d}\rho \,(1-\|u\|^2)^{\mu+\frac{1}{2}} \,\mathrm{d}u \\
&\le cn^{rp} \int_{\mathbb{B}^{d-2}} \int_0^1 \left[\int_0^{2\pi} |f_{u,\rho}(\theta)|^p \,\mathrm{d}\theta \right] (1-\rho^2)^{\mu-\frac{1}{2}} \rho \,\mathrm{d}\rho \,(1-\|u\|^2)^{\mu+\frac{1}{2}} \,\mathrm{d}u \\
&= cn^{rp} \|f\|_{p,\mu}^p,
\end{aligned}$$

where $f_{u,\rho}(\theta) = f(\varphi(u)\rho\cos\theta, \varphi(u)\rho\sin\theta, u)$, and the inequality step uses the usual Bernstein inequality for trigonometric polynomials. Using Eq. (4.5.1), the same argument works for $p = \infty$. This completes the proof of the inverse estimate.

□

Remark 12.3.8. Since the Bernstein inequality (12.3.17) is proved for all $\mu > -\frac{1}{2}$, the inverse estimate (12.3.16) holds for all $\mu > -\frac{1}{2}$ as well.

12.3.3 Analogue of the Ditzian–Totik Modulus of Smoothness on $\mathbb{B}^d$

We define an analogue of the Ditzian–Totik modulus of smoothness, defined in Eq. (12.1.4), on the unit ball $\mathbb{B}^d$. Since the definition for the weighted version has an additional complication, we consider only the unweighted case, that is, the case $W_{1/2}(x)\mathrm{d}x = \mathrm{d}x$, in this section. Let e_i be the ith coordinate vector of $\mathbb{R}^d$ and let $\hat{\Delta}_{he_i}^r$ be the rth central difference in the direction of e_i, more precisely,

$$\hat{\Delta}^r_{he_i} f(x) := \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right) he_i\right).$$

As in the case of $[-1,1]$, we assume that $\hat{\Delta}^r_{he_i}$ is zero if either of the points $x \pm r\frac{h}{2}e_i$ does not belong to $\mathbb{B}^d$. We write $L^p(\mathbb{B}^d)$, $\|f\|_p$, and $K_{r,\varphi}(f,t)_p$ for $L^p(W_{1/2};\mathbb{B}^d)$, $\|f\|_{W_{1/2};L^p(\mathbb{B}^d)}$, and $K_{r,\varphi}(f,t)_{p,1/2}$ respectively. The modulus of smoothness $\omega^r_\varphi(f,t)_p$ in Eq. (12.1.4) for the case $d=1$ suggests the following definition.

Definition 12.3.9. Let $f \in L^p(\mathbb{B}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r \in \mathbb{N}$ and $t > 0$,

$$\omega^r_\varphi(f,t)_p = \sup_{0<|h|\le t} \left\{ \max_{1\le i<j\le d} \|\triangle^r_{i,j,h} f\|_p, \max_{1\le i\le d} \|\hat{\triangle}^r_{h\varphi e_i} f\|_p \right\}. \qquad (12.3.18)$$

Many of the properties of the modulus of smoothness $\omega^r_\varphi(f,t)_p$ follow from the corresponding properties of the moduli of smoothness on the sphere and on $[-1,1]$. For example, we have the following lemma.

Lemma 12.3.10. *Let $f \in L^p(\mathbb{B}^d)$ for $1 \le p < \infty$ and $f \in C(\mathbb{B}^d)$ for $p = \infty$.*

(1) *For $0 < t < t_0$, $\omega^{r+1}_\varphi(f,t)_p \le c\, \omega^r_\varphi(f,t)_p$.*
(2) *For $\lambda > 0$, $\omega^r_\varphi(f,\lambda t)_p \le c(\lambda+1)^r \omega^r_\varphi(f,t)_p$.*
(3) *For $0 < t < \frac{1}{2}$ and every $m > r$,*

$$\omega^r_\varphi(f,t)_p \le c_m \left(t^r \int_t^1 \frac{\omega^m_\varphi(f,u)_p}{u^{r+1}} \mathrm{d}u + t^r \|f\|_p \right).$$

(4) *For $0 < t < t_0$, $\omega^r_\varphi(f,t)_p \le c\|f\|_p$.*

Proof. For $1 \le p < \infty$ and the $\triangle^r_{i,j,\theta} f$ part, we use the integral formula

$$\|\triangle^r_{i,j,\theta} f\|_p^p = \int_0^1 s^{d-1} \int_{\mathbb{S}^{d-1}} \left|\triangle^r_{i,j,\theta} f(sx')\right|^p \mathrm{d}\sigma(x')\mathrm{d}s$$

and apply Lemma A.5.4. For the $\hat{\triangle}^r_{\theta\varphi e_i} f$ part, we use Eq. (12.3.11) with $m = 1$ and the fact that if $x = (\varphi(u)s, u)$, then $\varphi(x) = \varphi(s)\varphi(u)$, to conclude that

$$\begin{aligned}
\|\varphi^r \hat{\triangle}^r_{\theta\varphi e_i} f\|_p^p &= \int_{\mathbb{B}^d} \left|\varphi^r(x) \hat{\triangle}^r_{\theta\varphi(x)e_i} f(x)\right|^p \mathrm{d}x \\
&= \int_{\mathbb{B}^{d-1}} \int_{-1}^1 \left|\varphi^r(u)\varphi^r(s) \hat{\triangle}^r_{\theta\varphi(s)\varphi(u)e_i} f(\varphi(u)s,u)\right|^p \mathrm{d}s\varphi(u)\mathrm{d}u \\
&= \int_{\mathbb{B}^{d-1}} (\varphi(u))^{rp+1} \left[\int_{-1}^1 \left|\varphi^r(s) \hat{\triangle}^r_{\theta\varphi(s)e_i} f_u(s)\right|^p \mathrm{d}s\right] \mathrm{d}u,
\end{aligned}$$

where $f_u(s) = f(\varphi(u)s, u)$, and then apply the result for one variable in [61, pp. 38, 43] to the inner integral and use the equivalence Eq. (12.1.6). □

Next we establish the direct and the inverse theorems in $\omega_\varphi^r(f,t)_p$, two of the central results in this section.

Theorem 12.3.11. *Let $f \in L^p(\mathbb{B}^d)$ if $1 \le p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. Then for $r \in \mathbb{N}$,*

$$E_n(f)_p \le c\,\omega_\varphi^r(f, n^{-1})_p + n^{-r}\|f\|_p \tag{12.3.19}$$

and

$$\omega_\varphi^r(f, n^{-1})_p \le c n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_p. \tag{12.3.20}$$

Furthermore, the additional term $n^{-r}\|f\|_p$ on the right-hand side of Eq. (12.3.19) can be dropped when $r = 1$.

Proof. We start with the proof of the Jackson-type inequality (12.3.19). By Eq. (12.2.14), it suffices to show that for the modulus $\omega_r(f,t)_p$ given in Definition 12.2.2,

$$\omega_r(f, n^{-1})_p \le c\,\omega_\varphi^r(f, n^{-1})_p + cn^{-r}\|f\|_p. \tag{12.3.21}$$

However, using Definitions 12.2.2 and 12.3.9, this amounts to showing that for $1 \le i \le d$,

$$\sup_{|\theta| \le t} \|\triangle_{i,d+1,\theta}^r \tilde{f}\|_{L^p(W_0;\mathbb{B}^{d+1})} \le c\,\omega_\varphi^r(f,t)_p + ct^r\|f\|_p, \tag{12.3.22}$$

where $\tilde{f}(x, x_{d+1}) = f(x)$ for $x \in \mathbb{B}^d$ and $(x, x_{d+1}) \in \mathbb{B}^{d+1}$. By symmetry, we need to consider only $i = 1$. Set

$$f_v(s) = f(\varphi(v)s, v), \quad v \in \mathbb{B}^{d-1},\ s \in [-1,1],$$

where $\varphi(v) = \sqrt{1 - \|v\|^2}$. We can then write, by Eq. (12.3.11),

$$\begin{aligned}
&\|\triangle_{1,d+1,\theta}^r \tilde{f}\|_{L^p(W_0;\mathbb{B}^{d+1})}^p \\
&= \int_{\mathbb{B}^{d+1}} \left|\overrightarrow{\triangle}_\theta^r f(x_1\cos(\cdot) + x_{d+1}\sin(\cdot), x_2, \ldots, x_d)\right|^p \frac{dx}{\sqrt{1-\|x\|^2}} \\
&= \int_{\mathbb{B}^{d-1}} \left[\int_{\mathbb{B}^2} \left|\overrightarrow{\triangle}_\theta^r f_v(x_1\cos(\cdot) + x_{d+1}\sin(\cdot))\right|^p \frac{dx_1\,dx_{d+1}}{\sqrt{1 - x_1^2 - x_{d+1}^2}}\right] \varphi(v)\,dv.
\end{aligned}$$

Applying Eq. (12.2.19) to the inner integral, we see that the last expression is bounded by, for $|\theta| \le t$,

$$c\frac{1}{t}\int_0^t\int_{\mathbb{B}^{d-1}}\left[\int_{-1}^1\left|\hat{\triangle}^r_{h\varphi(s)}f_v(s)\right|^p\mathrm{d}s\right]\varphi(v)\,\mathrm{d}v\,\mathrm{d}h+t^{rp}\int_{\mathbb{B}^{d-1}}\int_{-1}^1|f_v(s)|^p\,\mathrm{d}s\,\varphi(v)\,\mathrm{d}v$$
$$=c\frac{1}{t}\int_0^t\int_{\mathbb{B}^{d-1}}\int_{-1}^1\left|\hat{\triangle}^r_{h\varphi(\varphi(v)s,v)e_1}f(\varphi(v)s,v)\right|^p\mathrm{d}s\varphi(v)\,\mathrm{d}v\,\mathrm{d}h+ct^{rp}\|f\|_p^p$$
$$=c\frac{1}{t}\int_0^t\int_{\mathbb{B}^d}\left|\hat{\triangle}^r_{h\varphi(x)e_1}f(x)\right|^p\mathrm{d}x\,\mathrm{d}h+ct^{rp}\|f\|_p^p\le c\,\omega^r_\varphi(f,t)_p^p+ct^{rp}\|f\|_p^p.$$

For $r=1$, the additional term $t^{rp}\|f\|_p^p$ can be dropped, because of Theorem 12.2.14. Obviously, the above argument with slight modification works equally well for the case $p=\infty$. This proves the Jackson inequality (12.3.22).

Finally, the inverse estimate (12.3.20) follows by Eq. (12.3.16) and the inequality $\omega^r_\varphi(f,t)_{p,\mu}\le cK_{r,\varphi}(f,t)_{p,\mu}$, which will be given in Theorem 12.3.12 in the next subsection. □

12.3.4 Equivalence of $\omega^r_\varphi(f,t)_p$ and $K_{r,\varphi}(f,t)_p$

As a consequence of Theorem 12.3.11, we can now establish the equivalence of the modulus of smoothness $\omega^r_\varphi(f,t)_p$ and the K-functional $K_{r,\varphi}(f,t)_p$.

Theorem 12.3.12. *Let $f\in L^p(\mathbb{B}^d)$ if $1\le p<\infty$, and $f\in C(\mathbb{B}^d)$ if $p=\infty$. Then for $r\in\mathbb{N}$ and $0<t<t_r$,*

$$c^{-1}\omega^r_\varphi(f,t)_p\le K_{r,\varphi}(f,t)_p\le c\,\omega^r_\varphi(f,t)_p+ct^r\|f\|_p.$$

Furthermore, the term $t^r\|f\|_p$ on the right-hand side can be dropped when $r=1$.

For the proof of Theorem 12.3.12, we need the following lemma.

Lemma 12.3.13. *For $1\le p\le\infty$ and $f\in\Pi^d_n$, we have*

$$n^{-r}\|D^r_{i,j}f\|_{p,\mu}\sim\sup_{|\theta|\le n^{-1}}\|\triangle^r_{i,j,\theta}f\|_{p,\mu},\quad 1\le i<j\le d,\tag{12.3.23}$$

and

$$n^{-rp}\|\varphi^r\partial_i^r f\|_p^p\sim n\int_0^{n^{-1}}\|\hat{\triangle}^r_{h\varphi e_i}f\|_p^p\,\mathrm{d}h,\quad 1\le i\le d,\tag{12.3.24}$$

with the usual change when $p=\infty$.

Proof. The relation (12.3.23) follows directly from the inequality at the end of the proof of Theorem 12.3.7 and the corresponding trigonometric inequality in Lemma 4.1.4. The relation (12.3.24) can be proved similarly. In fact, setting $i=1$ and $f_u(s)=f(\varphi(u)s,u)$, we have

$$n^{-rp}\|\varphi^r\partial_1^r f\|_p^p = n^{-rp}\int_{\mathbb{B}^{d-1}}\left[\int_{-1}^1\left|\varphi^r(s)f_u^{(r)}(s)\right|^p\mathrm{d}s\right]\varphi(u)\,\mathrm{d}u$$
$$\sim n\int_{\mathbb{B}^{d-1}}\left[\int_0^{n^{-1}}\int_{-1}^1\left|\hat{\triangle}^r_{h\varphi(s)}f_u(s)\right|^p\mathrm{d}s\,\mathrm{d}h\right]\varphi(u)\,\mathrm{d}u$$
$$= n\int_0^{n^{-1}}\|\hat{\triangle}^r_{h\varphi e_1}f\|_p^p\,\mathrm{d}h,$$

where we have used the equivalence relation of one variable in [89, p. 191] and Eq. (12.1.6). □

Proof of Theorem 12.3.12. We start with the proof of the inequality

$$\omega_\varphi^r(f,t)_p \le cK_{r,\varphi}(f,t)_p, \quad 0<t<t_r. \tag{12.3.25}$$

Let $g_t \in C^r(\mathbb{B}^d)$ be chosen such that

$$\|f-g_t\|_p \le 2K_{r,\varphi}(f,t)_p, \qquad t^r\max_{1\le i<j\le d}\|D^r_{i,j}g_t\|_p \le 2K_{r,\varphi}(f,t)_p,$$

and

$$t^r\max_{1\le i\le d}\|\varphi^r\partial_i^r g_t\|_p \le 2K_{r,\varphi}(f,t)_p.$$

From the definition of $\omega_\varphi^r(f,t)_p$ and (4) of Lemma 12.3.10, it follows that

$$\omega_\varphi^r(f,t)_p \le \omega_\varphi^r(f-g_t,t)_p + \omega_\varphi^r(g_t,t)_p \le cK_{r,\varphi}(f,t)_p + \hat{\omega}_\varphi^r(g_t,t)_p.$$

Consequently, for the proof of the inequality of Eq. (12.3.25), it suffices to show that for $g\in C^r(\mathbb{B}^d)$,

$$\|\triangle^r_{i,j,\theta}g\|_p \le c\,\theta^r\|D^r_{i,j}g\|_p \quad\text{and}\quad \|\hat{\triangle}^r_{\theta\varphi e_i}g\|_p \le c\,\theta^r\|\varphi^r\partial_i^r g\|_p. \tag{12.3.26}$$

First we consider $\hat{\Delta}^r_{\theta\varphi e_i}f$, for which we will need the corresponding result for $[-1,1]$. By Eqs. (12.1.7) and (12.1.8), there exists $t_r\in(0,1)$ such that for $0<h<t_r$,

$$\|\hat{\triangle}^r_{h\varphi}g_t\|_{L^p[-1,1]} \le c\,h^r\|\varphi^r g_t^{(r)}\|_{L^p[-1,1]}. \tag{12.3.27}$$

For $p=\infty$, the proof of Eq. (12.3.26) follows from the usual relation between forward differences and derivatives. For $1\le p<\infty$, we need to consider only the case $i=1$. Using Eq. (12.3.11) with d replaced by $d-1$, we obtain by Eq. (12.3.27) that

$$\left\|\hat{\triangle}^r_{\theta\varphi e_1}g\right\|_p^p = \int_{\mathbb{B}^{d-1}}\int_{-1}^1\left|\hat{\triangle}^r_{\theta\varphi(y)\varphi(s)e_1}g\left(\varphi(y)s,y\right)\right|^p\mathrm{d}s\,\varphi(y)\mathrm{d}y$$

$$= \int_{\mathbb{B}^{d-1}} \int_{-1}^{1} \left| \hat{\triangle}^r_{\theta\varphi(s)} g_y(s) \right|^p \mathrm{d}s\, \varphi(y) \mathrm{d}y$$

$$\le c \int_{\mathbb{B}^{d-1}} \theta^{rp} \int_{-1}^{1} \left| \varphi^r(s) \frac{\mathrm{d}^r}{\mathrm{d}s^r} \left[g\left(\varphi(y)s, y\right)\right] \right|^p \mathrm{d}s\, \varphi(y) \mathrm{d}y$$

$$= c\, \theta^{rp} \int_{\mathbb{B}^d} |\varphi^r(x) \partial_1^r g(x)|^p \mathrm{d}x = c\, \theta^{rp} \|\varphi^r \partial_1^r g\|_p^p,$$

where $g_y(s) = g(\varphi(y)s, y)$. This proves the second inequality of Eq. (12.3.26).

Next, we consider $\Delta^r_{i,j,\theta} g$, for which we will need the corresponding result for trigonometric functions. Let h be a 2π-periodic function in $L^p[0,2\pi]$ and let $\|h\|_p := \left(\int_0^{2\pi} |h(\theta)|^p \mathrm{d}\theta \right)^{1/p}$ in the rest of this proof. Then using Lemma 4.1.4,

$$\|\overrightarrow{\triangle}^r_h h\|_p \le c h^r \|h^{(r)}\|_p. \tag{12.3.28}$$

We consider only the case $(i, j) = (1, 2)$. By Eq. (4.2.3),

$$\|\triangle^r_{1,2,\theta} g\|_p^p = \int_{\mathbb{B}^{d-2}} \int_{\mathbb{B}^2} \left| \triangle^r_{1,2,\theta} g\left(v, \varphi(v)u\right) \right|^p \varphi(v)^{d-2} \mathrm{d}v\, \mathrm{d}u$$

$$= \int_{\mathbb{B}^{d-2}} \int_0^1 \rho \int_0^{2\pi} \left| \overrightarrow{\triangle}^r_\theta g\left(\rho\cos t, \rho\sin t, \varphi(\rho)u\right) \right|^p \mathrm{d}t\, \varphi(\rho)^{d-2} \mathrm{d}\rho\, \mathrm{d}u.$$

Setting $g_{\rho,u}(t) = g\left(\rho\cos t, \rho\sin t, \varphi(\rho)u\right)$, we deduce from Eq. (12.3.28) that

$$\|\triangle^r_{1,2,\theta} f\|_p^p \le c\, \theta^{rp} \int_{\mathbb{B}^{d-2}} \int_0^1 \rho \int_0^{2\pi} \left| g^{(r)}_{\rho,u}(t) \right|^p \mathrm{d}t\, \varphi(\rho)^{d-2} \mathrm{d}\rho\, \mathrm{d}u$$

$$= c\, \theta^{rp} \int_{\mathbb{B}^d} \left| D^r_{1,2} g(x) \right|^p \mathrm{d}x = c\, \theta^{rp} \|D^r_{1,2} f\|_p^p,$$

which proves the first inequality of Eq. (12.3.26). Consequently, we have proved the inequality (12.3.25).

We now prove the reverse inequality

$$K_{r,\varphi}(f,t)_p \le c\, \omega^r_\varphi(f,t)_p + c t^r \|f\|_p. \tag{12.3.29}$$

Setting $n = \lfloor \frac{1}{t} \rfloor$, we have

$$K_{r,\varphi}(f,t)_p \le \|f - L^\mu_n f\|_p + t^r \max_{1\le i<j\le d} \|D^r_{i,j} L^\mu_n f\|_p + t^r \max_{1\le i\le d} \|\varphi^r \partial_i^r L^\mu_n f\|_p.$$

The first term is bounded by $c\, \omega^r_\varphi(f,t)_p + c n^{-r} \|f\|_p$ by Eq. (12.3.19). For the second term, we use Eq. (12.3.23) to obtain

$$t^r \max_{1\le i<j\le d} \|D^r_{i,j} L^\mu_n f\|_p \le c\, \omega^r_\varphi(L^\mu_n f, n^{-1})_p \le c\, \omega^r_\varphi(L^\mu_n f - f, n^{-1})_p + c\, \omega^r_\varphi(f, n^{-1})_p$$

$$\le c \|f - L^\mu_n f\|_p + c\, \omega^r_\varphi(f, n^{-1})_p \le c\, \omega^r_\varphi(f,t)_p + c t^r \|f\|_p.$$

The third term can be treated similarly, using Lemma 12.3.13. This completes the proof of Eq. (12.3.29). ◻

12.4 The Third Modulus of Smoothness and *K*-Functional

The results we obtained in the previous two sections have been established for W_μ, and in most cases, under the restriction $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Our third pair of modulus of smoothness and K-functional can be defined in the more general setting of $L^p(\mathbb{B}^d, W_\kappa)$, where W_κ is defined in Eq. (11.1.2), which is

$$W_\kappa(x) = \prod_{i=1}^{d} |x_i|^{\kappa_i}(1-\|x\|^2)^{\mu-1/2}, \quad \kappa_i \ge 0,\ \mu \ge 0.$$

The analysis in $L^p(W_\kappa;\mathbb{B}^d)$ is the subject of the previous chapter. The third modulus of smoothness is defined in terms of the generalized translation operator, and our development is parallel to that of Chap. 10.

For $\theta \in \mathbb{R}$, let $T_\theta(W_\kappa)$ be the generalized translation operator in Definition 11.2.4 defined with respect to the weight W_κ. Properties of this operator are stated in Proposition 11.2.5. In the special case of W_μ, an explicit integral representation of $T_\theta(W_\mu)$ is given in Eq. (11.2.12). Our third modulus of smoothness on the ball is defined, following the approach in Sect. 10.1, in terms of the operator $T_\theta(W_\kappa)$. For $r > 0$, we define the difference operator $\triangle^r_{\theta,\kappa}$ by

$$\triangle^r_{\theta,\kappa} f := \big(I - T_\theta(W_\kappa)\big)^{r/2} f = \sum_{k=0}^{\infty}(-1)^k \binom{r/2}{k} \big(T_\theta(W_\kappa)\big)^k f.$$

Definition 12.4.1. Let $f \in L^p(W_\kappa;\mathbb{B}^d)$ if $1 \le p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. The rth-order modulus of smoothness of f is defined by

$$\omega_r^*(f,t)_{p,\kappa} := \sup_{|\theta|\le t} \|\triangle^r_{\theta,\kappa} f\|_{p,\kappa}, \quad t \in (0,1). \tag{12.4.1}$$

This modulus of smoothness is closely related to the one defined in Definition 10.1.1 for $L^p(h_\kappa^2;\mathbb{S}^d)$, where $h_\kappa^2(x) = \prod_{j=1}^{d+1}|x_j|^{\kappa_j}$, which is denoted by $\omega_r^*(f,t)_{L^p(h_\kappa^2;\mathbb{S}^d)}$. Using the relation (11.2.6), it is easy to see that

$$\omega_r^*(f,t)_{p,\kappa} = \omega_r(F,t)_{L^p(h_\kappa^2;\mathbb{S}^d)}, \quad F(x,x_{d+1}) = f(x). \tag{12.4.2}$$

The K-functional that is equivalent to $\omega_r^*(f,t)_{p,\kappa}$ is defined in terms of the fractional powers of the differential–difference operator

$$\mathscr{D}_{\kappa,\mathbb{B}} := \Delta_h - \langle x,\nabla\rangle^2 - (2|\kappa|+d-1)\langle x,\nabla\rangle, \tag{12.4.3}$$

defined in Eq. (11.1.11), where Δ_h is the Dunkl Laplacian, defined in Eq. (7.1.2), for $h_\kappa^2(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$. As shown in Theorem 11.1.5, the orthogonal polynomials with respect to W_κ on $\mathbb{B}^d$ are eigenfunctions of $\mathscr{D}_{\kappa,\mathbb{B}}$. More precisely, by Eq. (11.1.10),

$$\mathscr{D}_{\kappa,\mathbb{B}} P = -n(n+2|\kappa|+d-1)P, \quad \forall P \in \mathscr{V}_n^d(W_\kappa),$$

where $\mathscr{V}_n^d(W_\mu)$ is the space of orthogonal polynomials of degree n with respect to the weight function W_μ on $\mathbb{B}^d$. Recall that $\operatorname{proj}_n(W_\kappa; f)$ denotes the projection operator from $L^2(W_\kappa, \mathbb{B}^d)$ to $\mathscr{V}_n^d(W_\kappa)$ as in Eq. (11.1.12). In analogy to Eq. (10.2.1), for $r > 0$, we define the fractional power of $(-\mathscr{D}_{\kappa,\mathbb{B}})^r$ in the sense of distributions by

$$\operatorname{proj}_n\left(W_\kappa; (-\mathscr{D}_{\kappa,\mathbb{B}})^r f\right) = (n(n+2|\kappa|+d-1))^r \operatorname{proj}_n(W_\kappa; f), \quad n = 0, 1, 2, \ldots,$$

which is used to define our third K-functional as follows.

Definition 12.4.2. Let $f \in L^p(W_\kappa; \mathbb{B}^d)$ if $1 \le p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r > 0$, define a K-functional by

$$K_r^*(f,t)_{p,\kappa} := \inf_{g \in C^\infty(\mathbb{B}^d)} \left\{ \|f-g\|_{p,\kappa} + t^r \left\| \left(-\mathscr{D}_{\kappa,\mathbb{B}}\right)^{r/2} g \right\|_{p,\kappa} \right\}. \tag{12.4.4}$$

For $f \in L^p(W_\kappa; \mathbb{B}^d)$ if $1 \le p < \infty$, or $f \in C(\mathbb{B}^d)$ if $p = \infty$, define

$$E_n(f)_{p,\kappa} := \inf_{g \in \Pi_n} \|f-g\|_{p,\kappa}.$$

Theorem 12.4.3. *Let $f \in L^p(W_\kappa; \mathbb{B}^d)$ if $1 \le p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$.*

(i) *If $t \in (0,1)$ and $r > 0$, then*

$$\omega_r^*(f,t)_{p,\kappa} \sim K_r^*(f,t)_{p,\kappa}.$$

(ii) *We have the direct inequality*

$$E_n(f)_{p,\kappa} := \inf_{g \in \Pi_n} \|f-g\|_{p,\kappa} \le c\,\omega_r^*(f, n^{-1})_{p,\kappa}, \quad n = 1, 2, \ldots,$$

and the weak inverse inequality

$$\omega_r^*(f, n^{-1})_{p,\kappa} \le c n^{-r} \sum_{k=0}^n (k+1)^{r-1} E_k(f)_{p,\kappa}.$$

Proof. Let $K_r(F,t)_{L^p(h_\kappa^2;\mathbb{S}^d)}$ denote the K-functional defined, in Eq. (10.3.6), in terms of the spherical h-Laplacian $\Delta_{h,0}$ associated with $h_\kappa(x) = \prod_{j=1}^{d+1} |x_j|^{\kappa_j}$ on $\mathbb{S}^d$. For the proof of (i), we need only show, by Eq. (12.4.2) and the equivalent relation in Theorem 10.4.1, that

$$K_r^*(f,t)_{p,\kappa} = K_r(F,t)_{L^p(h_\kappa^2;\mathbb{S}^d)}, \quad F(x, x_{d+1}) = f(x). \tag{12.4.5}$$

By the proof of Theorem 11.1.5, the operator $\mathscr{D}_{\kappa,\mathbb{B}}$ is deduced from $\Delta_{h,0}$, so that we obtain from Eq. (11.1.6) that $\|(-\mathscr{D}_{\kappa,\mathbb{B}})^{r/2}\|_{\kappa,p} = \|(-\Delta_{h,0})^{r/2}F\|_{L^p(h_\kappa^2;\mathbb{S}^d)}$, which implies, by Eq. (11.1.6) and the definition of the K-functional,

$$K_r^*(f,t)_{p,\kappa} = \inf_G \left\{ \|F-G\|_{L^p(h_\kappa^2;\mathbb{S}^d)} + t^r \|(-\Delta_{h,0})^{r/2}G\|_{L^p(h_\kappa^2;\mathbb{S}^d)} \right\},$$

where the infimum is over all $G \in C^\infty(\mathbb{S}^d)$ such that G is even in x_{d+1}. It follows, in particular, that $K_r^*(f,t)_{p,\kappa} \ge K_r(F,t)_{L^p(h_\kappa^2;\mathbb{S}^d)}$. On the other hand, for $G \in C^\infty(\mathbb{S}^d)$, we define $G_e(x,x_{d+1}) = [G(x,x_{d+1}) + G(x,-x_{x+1})]/2$. Then G_e is even in x_{d+1}, and it is easy to see that $\|(-\Delta_{h,0})^{r/2}G_e\|_{L^p(h_\kappa^2;\mathbb{S}^d)} \le \|(-\Delta_{h,0})^{r/2}G\|_{L^p(h_\kappa^2;\mathbb{S}^d)}$ and $\|F - G_e\|_{L^p(h_\kappa^2;\mathbb{S}^d)} \le \|F-G\|_{L^p(h_\kappa^2;\mathbb{S}^d)}$, which allows us to conclude that $K_r^*(f,t)_{p,\kappa} \le K_r(F,t)_{L^p(h_\kappa^2;\mathbb{S}^d)}$. Thus, we have proved Eq. (12.4.5) and hence (i). The above idea of taking G_e of G can also be used to show that

$$E_n(f)_{\kappa,p} = E_n(F)_{L^p(h_\kappa^2;\mathbb{S}^d)}, \quad F(x,x_{d+1}) = f(x),$$

where $E_n(F)_{L^p(h_\kappa^2;\mathbb{S}^d)}$ denotes the quantity of best approximation by polynomials with respect to h_κ^2 on $\mathbb{S}^d$, as defined in Definition 10.1.5. Consequently, both (ii) and (iii) follow from (i) and the corresponding result in Corollary 10.3.3. □

12.5 Comparisons of Three Moduli of Smoothness

Among the three pairs of moduli of smoothness and K-functionals that we introduced, the first pair is defined for $W_\mu(x)$ with $\mu = \frac{d-2}{2}$, the second pair for dx, although the K-functional is defined for W_μ, and the third part, most generally, for $W_\kappa(x)$. In each case, we have the equivalence of modulus of smoothness and the K-functional in the same pair, and the direct and the inverse theorems for the algebraic polynomial approximation on $\mathbb{B}^d$. The first pair is essentially the projections of those on the sphere $\mathbb{S}^{d+m-1}$ studied in Chap. 4, and as such, the modulus can be relatively easily computed, as demonstrated by the examples. The second pair gives complete analogues of the Ditzian–Totik pair on the interval $[-1,1]$, which captures the boundary behavior most visibly. The third pair is the most general, defined not only for all W_κ but also for all $r > 0$. It is, however, also the most complicated pair in structure, which allows access to tools in harmonic analysis, such as multiplier theorems, as shown in Chap. 10, but the modulus is essentially incomputable.

Beyond these apparent differences, we can also look for other points of comparison between them. The fact that each pair can be used to characterize the best approximation shows that they must have similar behavior. However, since the inverse theorem is of weak type, there may be subtle differences among the three. Below are several comparison theorems between the three pairs, which reflect what is known at the time of writing. In these comparisons, we restrict W_κ to W_μ and write the third modulus as $\omega_r^*(f,t)_{p,\mu}$, and we further write $\omega_r(f,t)_{p,\mu}$ as $\omega_r(f,t)_p$ when $\mu = 0$.

Theorem 12.5.1. *Let $f \in L^p(\mathbb{B}^d)$ if $1 \le p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. We further assume that r is odd when $p = \infty$. Then*

$$\omega_\varphi^1(f,t)_p \sim \omega_1(f,t)_p,$$

and for $r > 1$, there is a $t_r > 0$ such that

$$\omega_r(f,t)_p \le c\,\omega_\varphi^r(f,t)_p + ct^r\|f\|_p, \qquad 0 < t < t_r.$$

Furthermore, the above estimates hold if $\omega_r(f,t)_p$ is replaced by $K_r(f,t)_{p,\mu}$, and $\omega_\varphi^r(f,t)_p$ is replaced by $K_{r,\varphi}(f,t)_p$.

Theorem 12.5.1 is a direct consequence of Theorems 12.2.12, 12.3.2, and 12.3.12. Note that we state the estimates only for unweighted moduli of smoothness, since we did not define their weighted counterparts.

Theorem 12.5.2. *Let $\mu \ge 0$. Then for $f \in L^p(W_\mu;\mathbb{B}^d)$, $1 < p < \infty$, and $0 < t \le 1$,*

$$c_1 K_{2,\varphi}(f;t)_{p,\mu} \le K_2(f,t)_{p,\mu} \le c_2 K_{2,\varphi}(f;t)_{p,\mu} + c_2 t^2 \|f\|_{p,\mu}.$$

Theorem 12.5.3. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Then for $f \in L^p(W_\mu;\mathbb{B}^d)$, $1 < p < \infty$, $r \in \mathbb{N}$, and $0 < t < 1$,*

$$\omega_r(f,t)_{p,\mu} \le c\omega_r^*(f,t)_{p,\mu}.$$

Theorems 12.5.2 and 12.5.3 are partial results, and their proofs are quite involved. We omit the proofs and give references in the next section.

12.6 Notes and Further Results

The main reference for the Ditzian–Totik modulus of smoothness is [61]. The idea of using generalized translation operators to introduce moduli of smoothness on intervals can be traced back to Butzer and his collaborators. In the case that the weight $w(x)$ is given by $w_\mu(x) = (1-x^2)^{\mu-\frac{1}{2}}$, their moduli are defined by

$$\omega_r(f,t)_{p,\mu} := \sup_{\substack{|\theta_j| \le t \\ 1 \le j \le r}} \|\triangle_{\theta_1,\mu} \cdots \triangle_{\theta_r,\mu} f\|_{p,\mu},$$

where $\triangle_{\theta,\mu} f = f - T_\theta^\mu f$. Properties of $\omega_r(f,t)_{p,\mu}$ and their relation to best weighted algebraic polynomial approximation were described in [27], which is a survey of the authors' work that also includes further results. Potapov further explored relations between the rate of approximation by algebraic polynomials and generalized translations (see [140] and the references therein).

Besides the two moduli on the interval discussed in the first section, there are several other alternative moduli of smoothness on the interval. The most notable one is due to Ivanov (see [90] and the references therein), who introduced the following averaged moduli for $1 \le p, q \le \infty$ and used them to study polynomial approximation on $[-1,1]$: Let $\psi(t,x) := t^2 + t\sqrt{1-x^2}$ for $x \in [-1,1]$. Define

$$\tau^r(f,t)_{q,p} := \left\| \left(\frac{1}{2\psi(t,x)} \int_{-\psi(t,x)}^{\psi(t,x)} |\triangle_u^r f(x)|^q \, \mathrm{d}u \right)^{\frac{1}{q}} \right\|_p, \quad t \in (0,1),$$

where the L^p norm is taken with respect to $\mathrm{d}x$ on $[-1,1]$ and the L^q-average on the right is replaced by $\sup_{|u| \le \psi(t,x)} |\triangle_u^r f(x)|$ in the case of $q = \infty$. Among other things, he proved

$$\tau^r(f,t)_{p,p} \sim K_{r,\varphi}(f,t)_p, \quad 1 \le p \le \infty, \quad r \in \mathbb{N}.$$

For further results on moduli of smoothness and algebraic polynomial approximation, we refer to the comprehensive survey [58] by Ditzian.

For the unit ball $\mathbb{B}^d$, an early result in [141] gave a direct theorem in terms of $\sup_{\|h\| \le t} |f(x+h) - f(x)|$, which, however, does not take into account the boundary of $\mathbb{B}^d$. In fact, using this modulus of smoothness and its higher-order analogue, it is possible to establish a direct estimate for continuous functions on any compact set, and one can do so simultaneously for the derivatives of the functions as well [10]. However, the direct estimate given in terms of such moduli of smoothness is weaker for many functions. For example, for those functions in Examples 12.2.15, 12.2.16, and 12.2.18, such moduli of smoothness give an estimate of order $n^{-\alpha}$ for $E_n(f)_\infty$, in contrast to the accurate order $n^{-2\alpha}$. In particular, no matching inverse theorems can be given for these moduli of smoothness.

Most of the results in Sects. 12.2 and 12.3 were proved in [50]; see also [51]. The weighted moduli of smoothness $\omega_r^*(f,t)_{p,\kappa}$ and the weighted K-functionals $K_r(f,t)_{p,\kappa}$ in Sect. 12.4 were introduced and discussed in [190]. Although we believe that the results for the first pair of modulus of smoothness and K-functional, $\omega_r(f,t)_{p,\mu}$ and $K_r(f,t)_{p,\mu}$, hold for all $\mu \ge 0$ instead of $\mu = \frac{m-1}{2}$ of half-integers, our approach does not seem to allow such extensions, and this appears to be a difficult problem.

Like the results in Chap. 5, many weighted polynomial inequalities, including the second inequality in Eq. (12.3.17), the Nikolskii inequality, and the Remeze-type inequality, can be established under doubling or a slightly stronger A_∞-condition on the weights on $\mathbb{B}^d$; see [38, Sect. 8].

Theorem 12.5.2 was stated in [50, Theorem 7.5]. Its proof relies on the decomposition of the differential operator $\mathscr{D}_\mu$ in [50, Proposition 7.1],

$$\mathscr{D}_\mu = \sum_{i=1}^{d} U_{i,\mu} + \sum_{1 \le i < j \le d} D_{i,j}^2,$$

where $D_{i,j} = x_j \partial_i - x_i \partial_j$ for $i \neq j$, and

$$U_{i,\mu} := [W_\mu(x)]^{-1} \partial_i \left[(1 - \|x\|^2) W_\mu(x)\right] \partial_i, \quad 1 \le i \le d,$$

and the following result of [50, Theorem 7.3]: for every $g \in C^2(\mathbb{B}^d)$,

$$\|\mathscr{D}_\mu g\|_{p,\mu} \sim \sum_{1 \le i < j \le d} \|D_{i,j}^2 g\|_{p,\mu} + \sum_{i=1}^{d} \|U_{i,\mu} g\|_{p,\mu}, \quad 1 < p < \infty,$$

which holds for all $\mu \ge 0$ and follows from a corresponding result of [45, Remark 1.2] for the simplex. Theorem 12.5.3 is a consequence of Theorem 8.2 of [44].

Using the first modulus of smoothness, $\omega_r(f,t)_{\mu,p}$ in Eq. (12.2.6), a Lipschitz space can be defined in complete analogy with the one that we defined on the sphere in Sect. 4.8, in which $E_n(f)_{p,\mu}$ has order $n^{-r-\alpha}$ for $r \in \mathbb{N}_0$ and $\alpha \in (0,1)$. For precise statements and proofs, see [51].

Chapter 13
Harmonic Analysis on the Simplex

The simplex in $\mathbb{R}^d$ is another compact domain with boundary. We consider the setting of the standard simplex equipped with Jacobi-type weight functions that have singularities at the boundary. Analysis on this domain, as it turns out, is closely related to analysis on the unit ball. In fact, the simpleminded map from the simplex to the positive quadrant of the unit ball is an isomorphism between orthogonal polynomials on the simplex and those orthogonal polynomials on the unit ball that are even in every variable. A large portion of the harmonic analysis on the simplex can be deduced from the corresponding part on the unit ball.

The orthogonal structures on the simplex and its connection on the unit ball are studied in the first section, which allows us to define a convolution structure and use it to study orthogonal expansions on the simplex in the second section. The connection to the unit ball allows us to deduce several essential results on the simplex, including maximal functions and a multiplier theorem in the third section, and boundedness of projection operators and Cesàro means in the fourth section. The list, however, does not include the near-best-approximation operator and highly localized kernel, which can nevertheless be deduced using a similar approach, as shown in the fifth section. Using again the connection to the unit ball, weighted best approximation is deduced in the sixth section, and cubature formulas in the seventh section.

13.1 Orthogonal Structure on the Simplex

The simplex $\mathbb{T}^d$ of $\mathbb{R}^d$ is defined by, with $|x| = x_1 + \cdots + x_d$,

$$\mathbb{T}^d := \{x \in \mathbb{R}^d : x_1 \ge 0, \ldots, x_d \ge 0, |x| \le 1\}.$$

We will be working with orthogonal structures with respect to the weight function on the simplex

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$$U_\kappa(x) := \prod_{i=1}^{d} |x_i|^{\kappa_i - \frac{1}{2}} (1-|x|)^{\kappa_{d+1}-\frac{1}{2}}, \quad \kappa_i \ge 0, \quad x \in \mathbb{T}^d, \tag{13.1.1}$$

which is the analogue of the Jacobi weight function on $[0,1]$. The reason that the parameters are chosen to be of the form $\kappa_i - 1/2$ instead of κ_i lies in the connection of U_κ with the weight function W_κ in (11.1.2) on the unit ball. As in the case of W_κ on $\mathbb{B}^d$, we set $\lambda_k := |\kappa| + \frac{d-1}{2}$.

Definition 13.1.1. For $n \in \mathbb{N}_0$, let $\mathscr{V}_n^d(U_\kappa)$ denote the space of orthogonal polynomials of degree exactly n with respect to the inner product

$$\langle f, g \rangle_{U_\kappa} := a_\kappa \int_{\mathbb{T}^d} f(x) g(x) U_\kappa(x) \mathrm{d}x,$$

where a_κ is the normalization constant of U_κ, $a_\kappa := 1/\int_{\mathbb{T}^d} U_\kappa(x)\mathrm{d}x$.

From the Gram–Schmidt process applied to the monomials, it follows that

$$\dim \mathscr{V}_n^d(U_\kappa) = \binom{n+d-1}{n}, \quad n = 0, 1, 2, \ldots.$$

The orthogonal structure on the simplex is closely related to the corresponding structure on the unit ball. We start with a simple relation on polynomials over these two domains.

Let $\mathbb{B}_+^d$ denote the positive quadrant of the ball $\mathbb{B}^d$, defined more precisely as $\mathbb{B}_+^d := \{x \in \mathbb{B}^d : x_1 \ge 0, \ldots, x_d \ge 0\}$. Then

$$(x_1, \ldots, x_d) \in \mathbb{B}_+^d \iff (x_1^2, \ldots, x_d^2) \in \mathbb{T}^d. \tag{13.1.2}$$

A polynomial P of the form $P(x) = p(x_1^2, \ldots, x_d^2)$ is invariant under sign changes of its coordinates; that is, it is invariant under the group $G = \mathbb{Z}_2^d$. Let ψ denote the map

$$\psi : (x_1, \ldots, x_d) \in \mathbb{B}^d \mapsto (x_1^2, \ldots, x_d^2) \in \mathbb{T}^d. \tag{13.1.3}$$

The domain B_+^d can be considered a fundamental domain for the polynomials invariant under $\mathbb{Z}_2^d$. Let us define

$$G\Pi_{2n}^d := \{P \in \Pi_{2n}^d : P \text{ invariant under } \mathbb{Z}_2^d\}.$$

The relation (13.1.2) leads to a correspondence between polynomial spaces.

Lemma 13.1.2. *The map ψ introduces a one-to-one correspondence between Π_n^d and $G\Pi_{2n}^d$; more precisely, $p \in \Pi_n^d$ corresponds to $p \circ \psi \in G\Pi_{2n}^d$.*

Proof. If $P \in G\Pi_{2n}^d$, then P is even in each of its variables. Hence, it is easy to see that $P(x) = p(x_1^2, \ldots, x_d^2) = (p \circ \psi)(x)$ for some $p \in \Pi_n^d$. The correspondence between P and p is evidently one-to-one. □

Using (13.1.2) as a change of variables leads immediately to the relation

$$\int_{\mathbb{B}^d} f(x_1^2,\ldots,x_d^2)\mathrm{d}x = \int_{\mathbb{T}^d} f(x_1,\ldots,x_d)\frac{\mathrm{d}x}{\sqrt{x_1\cdots x_d}}. \tag{13.1.4}$$

Under the mapping (13.1.2), U_κ on $\mathbb{T}^d$ is related to W_κ on $\mathbb{B}^d$ via

$$W_k(x) = (U_\kappa \circ \psi(x))|x_1\cdots x_d|, \quad x \in \mathbb{B}^d,$$

which shows, by (13.1.4), that the normalization constants for U_κ on $\mathbb{T}^d$ and W_κ on $\mathbb{B}^d$ are identical. Moreover, the inner product $\langle\cdot,\cdot\rangle_{U_\kappa}$ on the simplex is related to the inner product $\langle\cdot,\cdot\rangle_{W_\kappa}$ on the unit ball by

$$\langle f,g\rangle_{U_\kappa} = \langle f\circ\psi, g\circ\psi\rangle_{W_\kappa}, \tag{13.1.5}$$

from which a relation between the spaces of orthogonal polynomials $\mathscr{V}_n(U_\kappa,\mathbb{T}^d)$ and $\mathscr{V}_n(W_\kappa,\mathbb{B}^d)$ follows immediately, where here and in the following, we include the $\mathbb{T}^d$ and $\mathbb{B}^d$ in the notation of $\mathscr{V}_n^d$ to emphasize the domain whenever necessary.

Let us define $G\mathscr{V}_{2n}(W_\kappa,\mathbb{B}^d) := \mathscr{V}_{2n}(W_\kappa,\mathbb{B}^d)\cap G\Pi_{2n}^d$ on $\mathbb{B}^d$, which contains polynomials in $\mathscr{V}_{2n}(W_\kappa,\mathbb{B}^d)$ that are invariant under $\mathbb{Z}_2^d$.

Proposition 13.1.3. *The mapping (13.1.3) induces a one-to-one correspondence between $R\in\mathscr{V}_n(U_\kappa,\mathbb{T}^d)$ and $R\circ\psi\in G\mathscr{V}_{2n}(W_\kappa,\mathbb{B}^d)$.*

Using the above proposition and the mapping ψ, we can deduce from (11.1.10) that the elements in $\mathscr{V}_n^d(U_\kappa)$ are the eigenfunctions of a differential operator.

Theorem 13.1.4. *Let $\mathscr{D}_{\kappa,\mathbb{T}}$ be the second-order differential operator*

$$\mathscr{D}_{\kappa,\mathbb{T}} := \sum_{i=1}^d x_i(1-x_i)\partial_i^2 - 2\sum_{1\le i<j\le d} x_ix_j\partial_i\partial_j + \sum_{i=1}^d\left(\left(\kappa_i+\frac{1}{2}\right)-\lambda_\kappa x_i\right)\partial_i. \tag{13.1.6}$$

The orthogonal polynomials in $\mathscr{V}_n^d(U_\kappa)$ are eigenfunctions of $\mathscr{D}_{\kappa,\mathbb{T}}$,

$$\mathscr{D}_{\kappa,\mathbb{T}}u = -n(n+2\lambda_k)u, \qquad u\in\mathscr{V}_n^d(U_\kappa), \quad \lambda_\kappa = |\kappa|+\frac{d-1}{2}. \tag{13.1.7}$$

Proof. Using (13.1.2) as a change of variables, it is easy to verify that

$$4(\mathscr{D}_{\kappa,\mathbb{T}}f)\circ\psi = \mathscr{D}_{\kappa,\mathbb{B}}(f\circ\psi), \tag{13.1.8}$$

and Eq. (11.1.10) becomes (13.1.7) for polynomials in $G\mathscr{V}_{2n}(W_\kappa,\mathbb{B}^d)$, where we use the fact that for functions that are even in each of its variables, Δ_h for h_κ invariant under $\mathbb{Z}_2^d$ in (11.1.10) becomes the differential operator Δ. We leave the details to the interested reader. □

From Proposition 13.1.3, orthogonal bases of $\mathcal{V}_n^d(U_\kappa)$ can be deduced from those for $G\mathcal{V}_{2n}^d(W_\kappa,\mathbb{B}^d)$. We shall not need an explicit basis but will need a formula for the reproducing kernel. Let $P(U_\kappa;\cdot,\cdot)$ denote the reproducing kernel of $\mathcal{V}_n^d(U_\kappa)$. It is uniquely determined by the fact that $P_n(U_\kappa;x,\cdot)\in\mathcal{V}_n^d(U_\kappa)$ for every $x\in\mathbb{T}^d$ and the reproducing property

$$\langle P_n(U_\kappa;x,\cdot),q\rangle_{U_\kappa}=q(x),\quad \forall q\in\mathcal{V}_n^d(U_\kappa),\quad x\in\mathbb{T}^d.$$

Let $\mathrm{proj}_n(U_\kappa;f)$ denote the projection operator from $L^2(U_\kappa,\mathbb{T}^d)$ to $\mathcal{V}_n^d(U_\kappa)$. Then

$$\mathrm{proj}_n(U_\kappa;f,x)=a_\kappa\int_{\mathbb{T}^d}f(y)P_n(U_\kappa;x,y)U_\kappa(y)\mathrm{d}y. \tag{13.1.9}$$

Proposition 13.1.5. *For $n=0,1,2,\dots$ and $x,y\in\mathbb{T}^d$,*

$$P_n(U_\kappa;x,y)=2^{-d}\sum_{\varepsilon\in\mathbb{Z}_2^d}P_{2n}(W_\kappa;\sqrt{x},\varepsilon\sqrt{y}), \tag{13.1.10}$$

where $\sqrt{x}:=(\sqrt{x_1},\dots,\sqrt{x_d})$ and $\varepsilon u=(\varepsilon_1u_1,\dots,\varepsilon_du_d)$. Furthermore,

$$\mathrm{proj}_n(U_\kappa;f,x)\circ\psi=\mathrm{proj}_{2n}(W_\kappa;f\circ\psi,\sqrt{x}),\quad x\in\mathbb{T}^d. \tag{13.1.11}$$

Proof. This follows from (13.1.4) and Proposition 13.1.3. Indeed, denote temporarily the right-hand side of (13.1.10) by $Q_n(x,y)$ and let $R\in\mathcal{V}_n^d(U_\kappa)$; then by (13.1.4) and the invariance of W_κ under $\mathbb{Z}_2^d$,

$$\begin{aligned}\langle Q_n(x,\cdot),R\rangle_{U_\kappa}&=2^{-d}\sum_{\varepsilon\in\mathbb{Z}_2^d}a_\kappa\int_{\mathbb{B}^d}P_{2n}(W_\kappa;\sqrt{x},\varepsilon y)R(y)W_\kappa(y)\mathrm{d}y\\&=a_\kappa\int_{\mathbb{B}^d}P_{2n}(W_\kappa;\sqrt{x},y)(R\circ\psi)(y)W_\kappa(y)\mathrm{d}y\\&=R\circ\psi(\sqrt{x})=R(x),\end{aligned}$$

which shows that Q_n is the reproducing kernel of $\mathcal{V}_n^d(U_k)$. The proof of (13.1.11) follows easily by the same reasoning. □

Corollary 13.1.6. *Let $p_n^{(\alpha,\beta)}$ denote the orthonormal Jacobi polynomial of degree n with respect to the normalized weight function in* (B.1.1)*. Then*

$$\begin{aligned}P_n(U_\kappa;x,y)&=p_n^{(\lambda_\kappa-1/2,-1/2)}(1)\\&\quad\times c_\kappa\int_{[-1,1]^{d+1}}p_n^{(\lambda_\kappa-1/2,-1/2)}\left(2z(x,y,t)^2-1\right)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t,\end{aligned} \tag{13.1.12}$$

where $z(x,y,t):=\sqrt{x_1}\sqrt{y_1}t_1+\cdots+\sqrt{x_{d+1}}\sqrt{y_{d+1}}t_{d+1}$ with $x_{d+1}=1-|x|$ and $y_{d+1}=1-|y|$.

Proof. Using the quadratic transform (B.2.4), it is easy to verify that

$$\frac{2n+\lambda}{\lambda} C_{2n}^{\lambda}(t) = p_n^{(\lambda-1/2,-1/2)}(1) p_n^{(\lambda-1/2,-1/2)}(2t^2-1).$$

The stated result then follows from (13.1.10) and (11.1.15), in which the factor $\prod_{i=1}^d (1+t_i)$ drops out because of the summation over $\varepsilon \in \mathbb{Z}_2^d$. □

The relation between the orthogonal structures allows us to work with more general weight functions on the simplex. For example, what we have done in this section, except Corollary 13.1.6, works for weight functions of the form $h_\kappa^2(\sqrt{x})(1-|x|)^{\mu-1/2}$ whenever h_κ is even in each of its variables.

13.2 Convolution and Orthogonal Expansions

We denote by $\|\cdot\|_{U_\kappa,p}$ the norm of the space $L^p(U_\kappa;\mathbb{T}^d)$,

$$\|f\|_{U_\kappa,p} := \left(a_\kappa \int_{\mathbb{T}^d} |f(x)|^p U_\kappa(x)\mathrm{d}x\right)^{1/p}, \qquad 1 \le p < \infty,$$

and as usual, consider $C(\mathbb{T}^d)$ with $\|f\|_{U_\kappa,\infty} = \|f\|_\infty$ for $p = \infty$.

The relations (13.1.10) and (13.1.12) between the reproducing kernels suggest the following definition.

Definition 13.2.1. Let $V_\kappa^{\mathbb{B}}$ denote the operator defined in (11.1.13). Define an operator $V_\kappa^{\mathbb{T}}$ acting on functions on $\mathbb{R}^{d+1}$ by

$$V_\kappa^{\mathbb{T}} F(x, x_{d+1}) = 2^{-d} \sum_{\varepsilon \in \mathbb{Z}_2^d} V_\kappa^{\mathbb{B}} F(\varepsilon x, x_{d+1}). \tag{13.2.1}$$

In terms of this operator, we can write, for example,

$$P_n(U_\kappa; x, y) = p_n^{(\lambda_\kappa - \frac{1}{2}, -\frac{1}{2})}(1) V_\kappa^{\mathbb{T}} \left[p_n^{(\lambda_\kappa - \frac{1}{2}, -\frac{1}{2})}\left(2\langle \cdot, \sqrt{Y}\rangle^2 - 1\right)\right](\sqrt{X}),$$

where $\sqrt{X} =: (\sqrt{x}, \sqrt{1-|x|})$ and $\sqrt{Y} := (\sqrt{y}, \sqrt{1-|y|})$. The operator $V_\kappa^{\mathbb{T}}$ can be used to define a convolution structure $*_{\kappa,\mathbb{T}}$.

Definition 13.2.2. For $f \in L^1(U_\kappa; \mathbb{T}^d)$ and $g \in L^1(w_{\lambda_\kappa - \frac{1}{2}, -\frac{1}{2}}; [-1,1])$,

$$(f *_{\kappa,\mathbb{T}} g)(x) := a_\kappa \int_{\mathbb{T}^d} f(y) V_\kappa^{\mathbb{T}} \left[g\left(2\langle \sqrt{X}, \cdot \rangle^2 - 1\right)\right](\sqrt{Y}) U_\kappa(y)\mathrm{d}y. \tag{13.2.2}$$

This convolution is closely related to the convolution $f *_{\kappa,\mathbb{B}}$ on $\mathbb{B}^d$. In fact, we have the following result.

Proposition 13.2.3. *For* $f \in L^1(U_\kappa;\mathbb{T}^d)$ *and* $g \in L^1(w_{\lambda-\frac{1}{2},-\frac{1}{2}};[-1,1])$,

$$\left((f *_{\kappa,\mathbb{T}} g)\circ\psi\right)(x) = \left((f\circ\psi) *_{\kappa,\mathbb{B}} g(2\{\cdot\}^2-1)\right)(x). \tag{13.2.3}$$

Proof. From the identity (13.1.4), it is easy to see that $f \in L^1(U_\kappa;\mathbb{T}^d)$ implies $f\circ\psi \in L^1(W_\kappa;\mathbb{B}^d)$, and furthermore, it follows that

$$\begin{aligned}(f *_{\kappa,\mathbb{T}} g)(x_1^2,\ldots,x_d^2) &= a_\kappa \int_{\mathbb{B}^d} (f\circ\psi)(y) V_\kappa^{\mathbb{T}}\left[g\left(2\langle X,\cdot\rangle^2-1\right)\right](Y) W_\kappa(y)\mathrm{d}y \\ &= a_\kappa \int_{\mathbb{B}^d} (f\circ\psi)(y)\frac{1}{2^d}\sum_{\varepsilon\in\mathbb{Z}_2^d} V_\kappa^{\mathbb{B}}\left[g\left(2\langle X,\cdot\rangle^2-1\right)\right](\varepsilon Y) W_\kappa(y)\mathrm{d}y,\end{aligned}$$

where $\varepsilon Y = (\varepsilon_1 y_1,\ldots \varepsilon_d y_d, 1-|y|)$. Since $W_\kappa(y)$ is even in each of its variables, changing variables $y_i \mapsto \varepsilon_i y_i$ shows that the summation can be removed from the above formula. □

The relation (13.2.3) establishes the close connection between the convolutions on the simplex and on the ball. To see how it is applied, we prove Young's inequality. Recall that $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$ and $w_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ with respective normalization constants c_λ and $c_{\alpha,\beta}$.

Lemma 13.2.4. *For* $p,q,r \ge 1$ *and* $p^{-1} = r^{-1}+q^{-1}-1$, $f \in L^q(U_\kappa;\mathbb{T}^d)$ *and* $g \in L^r(w_{\lambda_\kappa-\frac{1}{2},-\frac{1}{2}};[-1,1])$,

$$\left\|f *_{\kappa,\mathbb{T}} g\right\|_{U_\kappa,p} \le \|f\|_{U_\kappa,q}\|g\|_{L^r(w_{\lambda_\kappa-\frac{1}{2},\frac{1}{2}},[-1,1])}. \tag{13.2.4}$$

Proof. By (13.1.4) and Young's inequality (11.2.5) on the unit ball,

$$\begin{aligned}\left\|f *_{\kappa,\mathbb{T}} g\right\|_{U_\kappa,p} &= \left\|(f *_{\kappa,\mathbb{T}} g)\circ\psi\right\|_{W_\kappa,p} = \left\|\left((f\circ\psi) *_{\kappa,\mathbb{B}} g(2\{\cdot\}^2-1)\right)\right\|_{W_\kappa,p} \\ &\le \|f\circ\psi\|_{W_\kappa,p}\|g(2\{\cdot\}^2-1)\|_{\lambda_k,r} = \|f\|_{U_\kappa,q}\|g\|_{L^r(w_{\lambda_\kappa-\frac{1}{2},\frac{1}{2}},[-1,1])},\end{aligned}$$

where in the last step we have used $\|g(2\{\cdot\}^2-1)\|_{\lambda_k,r} = \|g\|_{L^r(w_{\lambda_\kappa-\frac{1}{2},\frac{1}{2}};[-1,1])}$, which can be easily verified. □

We can also define a translation operator $T(U_\kappa;f)$ on the simplex.

Definition 13.2.5. For $0\le\theta\le\pi$, the translation operator $T_\theta(U_\kappa)$ of the orthogonal expansion (13.2.7) is defined by

$$\operatorname{proj}_n\left(U_\kappa;T_\theta(U_\kappa;f)\right) = \frac{P_n^{(\lambda_\kappa-\frac{1}{2},-\frac{1}{2})}(\cos 2\theta)}{P_n^{(\lambda_\kappa-\frac{1}{2},-\frac{1}{2})}(1)}\operatorname{proj}_n(U_\kappa;f), \quad n=0,1,\ldots. \tag{13.2.5}$$

This operator is closely related to the translation operator $T_\theta(W_\kappa;f)$ on the ball and to the convolution operator $*_{\kappa,\mathbb{T}}$.

Proposition 13.2.6. *The translation operator $T_\theta(U_\kappa)$ is well defined for all $f \in L^1(U_\kappa, \mathbb{T}^d)$, and it satisfies the following properties:*

(i) $T_\theta(U_\kappa; f) \circ \psi = T_\theta(W_\kappa; f \circ \psi)$.
(ii) For $f \in L^2(U_\kappa; \mathbb{T}^d)$ *and* $g \in L^1(w_{\lambda_\kappa - \frac{1}{2}, -\frac{1}{2}}, [-1,1])$,

$$(f *_{\kappa,\mathbb{T}} g)(x) = c_{\lambda_\kappa} \int_0^\pi T_\theta(U_\kappa; f, x) g(\cos 2\theta)(\sin\theta)^{2\lambda_\kappa} \mathrm{d}\theta. \tag{13.2.6}$$

(iii) $T_\theta(U_\kappa; f)$ *preserves positivity, i.e.,* $T_\theta(U_\kappa; f) \ge 0$ *if* $f \ge 0$.
(iv) For $f \in L^p(U_\kappa; \mathbb{T}^d)$, $1 \le p < \infty$, *or* $f \in C(\mathbb{T}^d)$,

$$\|T_\theta(U_\kappa; f)\|_{U_\kappa,p} \le \|f\|_{U_\kappa,p} \quad \textit{and} \quad \lim_{\theta\to 0} \|T_\theta(U_\kappa; f) - f\|_{U_\kappa,p} = 0.$$

Proof. Statement (i) follows form the definitions of (13.2.5) and (11.2.9) and the quadratic transform (B.2.4) between the Gegenbauer polynomials and the Jacobi polynomials. Assertion (ii) follows from (13.2.3) and the elementary relation $g(\cos 2\theta) = g(2x^2 - 1)$ if $x = \cos\theta$. The other two properties follow from (i), (13.1.4), and the corresponding properties in Proposition 11.2.5. □

The Fourier orthogonal series with respect to U_κ on the simplex $\mathbb{T}^d$ are defined in terms of $\mathscr{V}_n^d(U_\kappa)$. For $f \in L^2(U_\kappa; \mathbb{T}^d)$,

$$f(x) = \sum_{n=0}^\infty \operatorname{proj}_n(U_\kappa; f, x). \tag{13.2.7}$$

For convergence of the series (13.2.7) beyond the L^2 setting, we consider the summability method. Because of (13.1.10), however, the summability on the simplex does not follow directly from that on the sphere. In fact, as shown by (13.1.12), the kernel for the summability on the simplex is expanded into the Jacobi series, whereas the kernel for the summability on the ball is in the Gegenbauer series. The quadratic transformation between $P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2t^2-1)$ and $C_{2n}^\lambda(t)$ does not preserve the summability. On the other hand, many of the tools that we developed on the ball, such as convolution operators and maximal functions, can be extended to the simplex, and these allow us to study the summability on the simplex analogously to that on the ball.

Let us consider the Cesàro (C,δ) means of the series (13.2.7), defined by

$$S_n^\delta(U_\kappa; f) := \frac{1}{A_n^\delta} \sum_{j=0}^n A_{n-j}^\delta \operatorname{proj}_j(U_\kappa; f). \tag{13.2.8}$$

By (13.1.12) and the definition of $*_{\kappa,\mathbb{T}}$, we can write

$$S_n^\delta(U_\kappa; f) = f *_{\kappa,\mathbb{T}} K_n^\delta(U_\kappa) \quad \text{with} \quad K_n^\delta(U; t) := k_n^\delta\left(w_{\lambda_\kappa-\frac{1}{2},-\frac{1}{2}}; 1, t\right), \tag{13.2.9}$$

where $k_n^\delta(w_{\alpha,\beta}; \cdot, \cdot)$ denotes the (C,δ) kernel of the Jacobi series. The following is an analogue, although not a consequence, of Theorem 11.2.3.

Theorem 13.2.7. *The Cesàro means of the orthogonal expansions with respect to U_κ on $\mathbb{T}^d$ satisfy the following:*

1. *If $\delta \ge 2\lambda_\kappa + 1$, then $S_n^\delta(U_\kappa)$ is a nonnegative operator.*
2. *If $\delta > \lambda_\kappa$, then $S_n^\delta(U_\kappa; f)$ converges to f in $L^p(U_\kappa; \mathbb{T}^d)$ for $1 \le p \le \infty$.*

Proof. Statement (i) is a consequence of the positivity of the kernel $k_n^\delta(w_{\alpha,\beta}; \cdot, \cdot)$ for the Jacobi series; see [75]. To prove (ii), we use Young's inequality (13.2.4) with $q = p$ and $r = 1$ to reduce the proof to the boundedness of $k_n^\delta\big(w_{\lambda_\kappa - \frac{1}{2}, -\frac{1}{2}}; 1, t\big)$ in $L^1(w_{\lambda - \frac{1}{2}, -\frac{1}{2}}; [-1, 1])$, which is classical [162]. □

We can also consider the Poisson summation defined by

$$P_r(U_\kappa; f, x) := \sum_{n=0}^{\infty} r^n \operatorname{proj}_n(U_\kappa; f, x) = \big(f *_{\kappa,\mathbb{T}} P_r(U_\kappa; \{\cdot\})\big)(x), \quad x \in \mathbb{T}^d, \tag{13.2.10}$$

where the kernel $P_r(U_\kappa; t)$ is the Poisson kernel of the Jacobi series

$$P_r(U_\kappa; t) = \sum_{n=0}^{\infty} r^n p_n^{(\lambda_\kappa + \mu - \frac{1}{2}, -\frac{1}{2})}(1) p_n^{(\lambda_\kappa + \mu - \frac{1}{2}, -\frac{1}{2})}(t).$$

The kernel has an explicit representation in terms of the hypergeometric function ([8, p. 102, Ex. 19] and use [71, Vol. 1, p. 64, 2.1.4(23)])

$$P_r(U_\kappa; t) = \frac{(1-r)(1+r)^{\lambda_\kappa}}{(1 - 2rt + r^2)^{\lambda_\kappa + 1}}\, {}_2F_1\left(\begin{matrix} \frac{\lambda_\kappa}{2}, \frac{\lambda_\kappa - 1}{2} \\ \frac{1}{2} \end{matrix}; \frac{2r(1+t)}{(1+r)^2}\right). \tag{13.2.11}$$

It is known that this kernel is positive, that is, $P_r(U_\kappa; t) \ge 0$ for $-1 \le t \le 1$ and $0 \le r < 1$ [9]

Theorem 13.2.8. *For $f \in L^p(U_\kappa; \mathbb{T}^d)$, $1 \le p < \infty$, or $f \in C(\mathbb{T}^d)$, $p = \infty$, we have* $\lim_{r \to 1-} \|P_r(U_\kappa; f) - f\|_{U_\kappa, p} = 0$.

Proof. The formula (13.2.11) implies that the kernel is bounded by

$$0 \le P_r(U_\kappa; \cos\theta) \le c \frac{(1 - r^2)}{(1 - 2r\cos\theta + r^2)^{\lambda_\kappa + 1}}.$$

Using this inequality and (13.2.6), the proof follows almost exactly that of Theorem 7.4.10. □

13.3 Maximal Functions and a Multiplier Theorem

In analogy to Definition 11.3.1, we define a maximal function on the simplex.

Definition 13.3.1. For $f \in L^1(U_\kappa;\mathbb{T}^d)$, the maximal function $\mathscr{M}_\kappa^{\mathbb{T}} f$ is defined by

$$\mathscr{M}_\kappa^{\mathbb{T}} f(x) = \sup_{0\le\theta\le\pi} \frac{\int_0^\theta T_\phi(U_\kappa;|f|,x)(\sin\phi)^{2\lambda_\kappa} d\phi}{\int_0^\theta (\sin\phi)^{2\lambda_\kappa} d\phi}.$$

Since $T_\phi(U_\kappa;f)$ is related to $T_\theta(W_\kappa;f)$, the maximal function $\mathscr{M}_\kappa^{\mathbb{T}} f$ is related to $\mathscr{M}_\kappa^{\mathbb{B}} f$ defined in Definition 11.3.1.

Proposition 13.3.2. *For $f \in L^1(U_\kappa;\mathbb{T}^d)$,*

$$\left(\mathscr{M}_\kappa^{\mathbb{T}} f\right)\circ\psi = \mathscr{M}_\kappa^{\mathbb{B}}(f\circ\psi). \tag{13.3.1}$$

Proof. This is a simple consequence of (i) in Proposition 13.2.6. $\square$

With this relation to $\mathscr{M}_\kappa^{\mathbb{B}} f$, we can also give an alternative definition of $\mathscr{M}_k^{\mathbb{T}}$ in terms of $V_\kappa^{\mathbb{T}}$ in analogy to (11.3.3). Despite the connection (13.3.1), the following theorem does not follow as a consequence of its counterpart Theorem 11.3.3 on the ball. This can be seen, at a technical level, from the fact that $g(2x^2-1)$ appears on the right-hand side of (13.2.3) instead of $g(x)$, and it reflects, in fact, a characteristic difference between the structures on the ball and on the simplex.

Theorem 13.3.3. *Assume that $g \in L^1(w_{\lambda_\kappa-\frac12,-\frac12};[-1,1])$ and $|g(\cos\theta)| \le k(\theta)$ for all θ, where $k(\theta)$ is a continuous, nonnegative, and decreasing function on $[0,\pi]$. Then for $f \in L^1(U_\kappa;\mathbb{T}^d)$,*

$$|(f *_{\kappa,\mathbb{T}} g)(x)| \le c\,\mathscr{M}_\kappa^{\mathbb{T}}(|f|)(x), \quad x \in \mathbb{T}^d,$$

where $c = \int_0^\pi k(\theta)(\sin\frac{\theta}{2})^{2\lambda_\kappa} d\theta$.

Proof. First we note that changing variables $\theta \mapsto \pi-\theta$ in (13.2.5) shows that

$$T_{\pi-\theta}(U_\kappa;f,x) = T_\theta(U_\kappa;f,x). \tag{13.3.2}$$

Consequently, by (13.2.6), we can write

$$\begin{aligned}(f *_{\kappa,\mathbb{T}} g)(x) &= c_\lambda \int_0^\pi T_\phi(U_\kappa;f,x) g(\cos 2\phi)(\sin\phi)^{2\lambda_\kappa} \mathrm{d}\phi \\ &= c_\lambda \frac{1}{2}\int_0^{2\pi} T_{\phi/2}(U_\kappa;f,x) g(\cos\phi)\left(\sin\frac{\phi}{2}\right)^{2\lambda_\kappa} \mathrm{d}\phi.\end{aligned}$$

We split the integral over $[0,2\pi]$ into two integrals, one over $[0,\pi]$ and the other over $[\pi,2\pi]$. Changing variables $\phi \mapsto 2\pi-\phi$ in the second integral and using (13.3.2) shows that the integral over $[\pi,2\pi]$ is equal to the one over $[0,\pi]$. Consequently,

$$(f *_{\kappa,\mathbb{T}} g)(x) = c_\lambda \int_0^\pi T_{\phi/2}(U_\kappa;f,x) g(\cos\phi)\left(\sin\frac{\phi}{2}\right)^{2\lambda_\kappa} \mathrm{d}\phi.$$

Let us define

$$\Lambda(\theta,x) = \int_0^\theta T_{\phi/2}(U_\kappa;|f|,x)\left(\sin\frac{\phi}{2}\right)^{2\lambda_\kappa} \mathrm{d}\phi.$$

On changing variables $\phi \mapsto \phi/2$, it follows from the definition of $\mathscr{M}_\kappa^{\mathbb{T}} f(x)$ that

$$\Lambda(\theta,x) \le \mathscr{M}_\kappa^{\mathbb{T}} f(x) \int_0^\theta \left(\sin\frac{\phi}{2}\right)^{2\lambda_\kappa} \mathrm{d}\phi.$$

We can now follow the proof of Theorem 2.3.6, using an integration by parts to finish the proof. □

To show that the maximal function $\mathscr{M}_\kappa^{\mathbb{T}}$ is of weak type $(1,1)$, we define

$$\mathrm{meas}_\kappa^{\mathbb{T}} E := \int_E U_\kappa(x)\mathrm{d}x, \qquad E \subset \mathbb{T}^d.$$

Theorem 13.3.4. *If $f \in L^1(U_\kappa;\mathbb{T}^d)$, then $\mathscr{M}_k^{\mathbb{T}}$ satisfies*

$$\mathrm{meas}_\kappa^{\mathbb{T}} \left\{x \in \mathbb{T}^d : \mathscr{M}_\kappa^{\mathbb{T}} f(x) \ge \alpha\right\} \le c\frac{\|f\|_{U_\kappa,1}}{\alpha}, \quad \forall \alpha > 0.$$

Furthermore, if $f \in L^p(U_\kappa;T^d)$ for $1 < p \le \infty$, then $\|\mathscr{M}_\kappa^{\mathbb{T}} f\|_{U_\kappa,p} \le c\|f\|_{U_\kappa,p}$.

Proof. Using the relation (13.3.1) and (13.1.4), we obtain

$$\int_{\mathbb{T}^d} \chi_{\{x\in\mathbb{T}^d:\mathscr{M}_\kappa^{\mathbb{T}} f(x)\ge\alpha\}}(x)U_\kappa(x)\mathrm{d}x = \int_{\mathbb{B}^d} \chi_{\{x\in\mathbb{B}^d:\mathscr{M}_\kappa^{\mathbb{B}}(f\circ\psi)(x)\ge\alpha\}}(x)W_\kappa(x)\mathrm{d}x.$$

Hence, by Theorem 11.3.4, we conclude that

$$\mathrm{meas}_\kappa^{\mathbb{B}} \left\{x \in \mathbb{B}^d : \mathscr{M}_\kappa^{\mathbb{B}}(f\circ\psi)(x) \ge \alpha\right\} \le c\frac{\|f\circ\psi\|_{W_\kappa,1}}{\alpha} = c\frac{\|f\|_{U_\kappa,1}}{\alpha},$$

where the last step follows again from (13.1.4). □

Next we relate to the maximal function $\mathscr{M}_\kappa^{\mathbb{T}} f$ to the Hardy–Littlewood maximal function defined with respect to an appropriate distance $\mathtt{d}_{\mathbb{T}}$ on $\mathbb{T}^d$. This distance function needs to take into consideration the boundary,

$$\mathtt{d}_{\mathbb{T}}(x,y) = \arccos\left(\langle\sqrt{x},\sqrt{y}\rangle + \sqrt{1-|x|}\sqrt{1-|y|}\right), \quad x,y \in \mathbb{T}^d,$$

where $\sqrt{x} = (\sqrt{x_1},\ldots,\sqrt{x_d})$ for $x \in \mathbb{T}^d$. Directly from the definition, we have

$$\mathtt{d}_{\mathbb{B}}(x,y) = \mathtt{d}_{\mathbb{T}}(\psi(x),\psi(y)). \tag{13.3.3}$$

Using the distance on the simplex, we define the weighted Hardy–Littlewood maximal function as

$$M_\kappa^{\mathbb{T}} f(x) := \sup_{0<\theta\le\pi} \frac{\int_{\mathrm{d}_{\mathbb{T}}(x,y)\le\theta} |f(y)| U_\kappa(y)\mathrm{d}y}{\int_{\mathrm{d}_{\mathbb{T}}(x,y)\le\theta} U_\kappa(y)\mathrm{d}y}, \quad x\in\mathbb{T}^d.$$

It turns out that the maximal function $\mathscr{M}_\kappa^{\mathbb{T}} f$ is dominated by $M_\kappa^{\mathbb{T}} f$.

Theorem 13.3.5. *Let $f\in L^1(U_\kappa;\mathbb{T}^d)$. Then for every $x\in\mathbb{T}^d$,*

$$\mathscr{M}_\kappa^{\mathbb{T}} f(x) \le c M_\kappa^{\mathbb{T}} f(x). \tag{13.3.4}$$

Proof. Using (13.1.4), it follows readily from the definitions of $M_\kappa^{\mathbb{B}} f$ and $M_\kappa^{\mathbb{T}} f$ that $(M_\kappa^{\mathbb{T}} f)\circ\psi = M_\kappa^{\mathbb{B}}(f\circ\psi)$. Hence, using the fact that if g is invariant under sign changes, then $M_\kappa^{\mathbb{B}} g(x\varepsilon) = M_\kappa^{\mathbb{B}} g(x)$ by a simple change of variables, it follows from (13.3.1) and Theorem 11.3.4 that

$$\begin{aligned}(\mathscr{M}_\kappa^{\mathbb{T}} f)\circ\psi(x) &= \mathscr{M}_\kappa^{\mathbb{B}}(f\circ\psi)(x) \le c\sum_{\varepsilon\in\mathbb{Z}_2^d} M_\kappa^{\mathbb{B}}(f\circ\psi)(x\varepsilon)\\ &= c' M_\kappa^{\mathbb{B}}(f\circ\psi)(x) = c'(M_\kappa^{\mathbb{T}} f)\circ\psi(x)\end{aligned}$$

for $x\in\mathbb{B}^d$, from which the stated result follows immediately. □

As a consequence of Theorem 13.3.5, we have the following analogues of Corollaries 11.3.8 and 11.3.9 on the maximal function $\mathscr{M}_\kappa^{\mathbb{T}} f$.

Corollary 13.3.6. *If $-\frac{\mathbb{1}}{2} < \tau \le \kappa$ and $f\in L^1(U_\tau;\mathbb{T}^d)$, then $\mathscr{M}_\kappa f$ satisfies*

$$\mathrm{meas}_\tau^{\mathbb{T}}\{x : \mathscr{M}_\kappa^{\mathbb{T}} f(x)\ge\alpha\} \le c\frac{\|f\|_{U_\tau,1}}{\alpha}, \quad \forall\alpha>0.$$

Furthermore, if $1<p<\infty$, $-\frac{\mathbb{1}}{2}<\tau<p\kappa+\frac{p-1}{2}\mathbb{1}$ and $f\in L^p(U_\tau;\mathbb{T}^d)$, then

$$\left\|\mathscr{M}_\kappa^{\mathbb{T}} f\right\|_{U_\tau,p} \le c\|f\|_{U_\tau,p}.$$

Corollary 13.3.7. *Let $1<p<\infty$, $-\frac{\mathbb{1}}{2}<\tau<p\kappa+\frac{p-1}{2}\mathbb{1}$, and let $\{f_j\}_{j=1}^\infty$ be a sequence of functions. Then*

$$\left\|\left(\sum_{j=1}^\infty \left|\mathscr{M}_\kappa^{\mathbb{T}} f_j\right|^2\right)^{1/2}\right\|_{U_\tau,p} \le c\left\|\left(\sum_{j=1}^\infty |f_j|^2\right)^{1/2}\right\|_{U_\tau,p}.$$

As a related result, we also have the following analogue of Theorem 7.5.8.

Corollary 13.3.8. *For $\delta > \lambda_\kappa$, $1 < p < \infty$, and any sequence $\{n_j\}$ of positive integers,*

$$\left\| \left(\sum_{j=0}^{\infty} \left| S_{n_j}^{\delta}(U_\kappa; f_j) \right|^2 \right)^{1/2} \right\|_{U_\kappa,p} \le c \left\| \left(\sum_{j=0}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{U_\kappa,p}. \tag{13.3.5}$$

Proof. From Lemma B.1.2, we can derive an estimate for the kernel $k_n^\delta(w_{\alpha,\beta})$ in the form $|k_n^\delta(w_{\alpha,\beta}; 1, \cos\theta)| \le k(\theta)$, which allows us to apply Theorem 13.3.3 to show, by (13.2.9), that

$$\sup_n |S_n^\delta(U_\kappa; f, x)| \le c\mathscr{M}_\kappa^{\mathbb{T}}(x), \qquad \text{if} \quad \delta > \lambda_\kappa.$$

Consequently, the inequality (13.3.5) follows from Corollary 13.3.7. □

Like its analogue (7.5.13) for the weighted sphere, the inequality (13.3.5) is an essential ingredient in the proof of the multiplier theorem. Using the Poisson operators $P_r(U_\kappa; f)$ on the simplex, defined in (13.2.10), we can again define a semigroup by setting $T^t f := P_r(U_\kappa; f)$ with $r = e^{-t}$. Thus, the corresponding Littlewood–Paley function, defined as in (3.2.1), is bounded in $L^p(U_\kappa; \mathbb{T}^d)$ for $1 < p < \infty$. Hence, all the essential ingredients of the proof of the multiplier theorem in Theorem 3.3.1 hold for the orthogonal expansion with respect to U_κ. As a consequence, we have the following multiplier theorem.

Theorem 13.3.9. *Let $\{\mu_j\}_{j=0}^\infty$ be a sequence of complex numbers that satisfies*

1. $\sup_j |\mu_j| \le c < \infty$,
2. $\sup_j 2^{j(k-1)} \sum_{l=2^j}^{2^{j+1}} |\triangle^k u_l| \le c < \infty$,

where k is the smallest integer greater than or equal to $\lambda_\kappa + 1$. Then $\{\mu_j\}$ defines an $L^p(U_\kappa; \mathbb{T}^d)$, $1 < p < \infty$, multiplier; that is,

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j(U_\kappa; f) \right\|_{U_\kappa,p} \le c\|f\|_{U_\kappa,p}, \quad 1 < p < \infty,$$

where c is independent of f and μ_j.

We leave the details of the proof to the interested reader.

13.4 Projection Operator and Cesàro Means

Since the projection operator $\operatorname{proj}_n(U_\kappa; f)$ can be expressed, as in (13.1.11), in terms of the projection operator on the unit ball, its properties can be deduced from those of $\operatorname{proj}_{2n}(W_\kappa; f)$. As an example, recall that

$$\sigma_\kappa = \frac{d-1}{2} + |\kappa| - \kappa_{\min} = \lambda_\kappa - \kappa_{\min} \quad \text{with} \quad \kappa_{\min} = \min_{1\le i\le d+1} \kappa_i.$$

We can then deduce from Theorem 11.4.3 the following theorem.

Theorem 13.4.1. *Let $d \ge 2$ and $n \in \mathbb{N}$. Then*

(i) For $1 \le p \le \frac{2(\sigma_k+1)}{\sigma_k+2}$,

$$\|\mathrm{proj}_n(U_\kappa;f)\|_{U_\kappa,2} \le cn^{\delta_\kappa(p)}\|f\|_{U_\kappa,p};$$

(ii) For $\frac{2(\sigma_k+1)}{\sigma_k+2} \le p \le 2$,

$$\|\mathrm{proj}_n(U_\kappa;f)\|_{U_\kappa,2} \le cn^{\sigma_k(\frac{1}{p}-\frac{1}{2})}\|f\|_{U_\kappa,p}.$$

Furthermore, the estimate in (i) is sharp.

Proof. If $\|\,\mathrm{proj}_n(W_\kappa;f)\|_{W_k,p} \le A_n\|f\|_{W_\kappa;p}$, then by (13.1.11) and (13.1.4),

$$\begin{aligned}\|\,\mathrm{proj}_n(U_\kappa;f)\|_{U_\kappa;p} &= \|\,\mathrm{proj}_n(U_\kappa;f)\circ\psi\|_{W_\kappa;p} = \|\,\mathrm{proj}_{2n}(W_\kappa;f\circ\psi)\|_{W_\kappa;p}\\ &\le A_{2n}\|f\circ\psi\|_{W_\kappa;p} = A_{2n}\|f\|_{U_\kappa;p},\end{aligned}$$

from which the results follow immediately from Theorem 11.4.3. □

We can also state a localized result for the projection operator, for which we need to define an analogue of the spherical cap on $\mathbb{T}^d$. For $0 \le \theta \le \pi$, let

$$\mathtt{c}_\mathbb{T}(x,\theta) := \{y \in \mathbb{T}^d : \mathtt{d}_\mathbb{T}(x,y) \le \theta\}.$$

Theorem 13.4.2. *Suppose $1 \le p \le \frac{2\sigma_\kappa+2}{\sigma_k+2}$ and f is supported on the set $\mathtt{c}_\mathbb{T}(x,\theta)$ with $\theta \in (n^{-1},\pi]$ and $x \in \mathbb{T}^d$. Then*

$$\|\mathrm{proj}_n(U_\kappa;f)\|_{U_\kappa,2} \le cn^{\delta_\kappa(p)}\theta^{\delta_\kappa(p)+\frac{1}{2}}\left[\int_{\mathtt{c}_\mathbb{T}(x,\theta)} U_\kappa(y)\mathrm{d}y\right]^{\frac{1}{2}-\frac{1}{p}}\|f\|_{U_\kappa,p}.$$

Proof. By (11.2.6) and (13.1.11), we can relate $\mathrm{proj}_n(U_\kappa;f)$ to the projection operator $\mathrm{proj}_n^\kappa : L^2(h_\kappa^2;\mathbb{S}^d) \mapsto \mathscr{H}_n^{d+1}(h_\kappa^2)$ with $h_\kappa^2(x) = \prod_{i=1}^{d+1}|x_i|^{2\kappa_i}$. Indeed, let $F(x,x_{d+1}) = (f\circ\psi)(x)$; then

$$\mathrm{proj}_n(U_\kappa;f,x) = \mathrm{proj}_{2n}^\kappa F\left(\sqrt{x},\sqrt{1-|x|}\right), \quad x \in \mathbb{T}^d. \tag{13.4.1}$$

Since the distance $\mathtt{d}_\mathbb{T}(x,y)$ on $\mathbb{T}^d$ is related to the geodesic distance on $\mathbb{S}^d$ by

$$\mathtt{d}_\mathbb{T}(\psi(x),\psi(y)) = \mathtt{d}(X,Y), \quad X = \left(x,\sqrt{1-\|x\|^2}\right), \quad Y = \left(y,\sqrt{1-\|y\|^2}\right),$$

from (11.1.6) and (13.1.4) it follows readily that

$$\int_{\mathsf{c}_{\mathbb{T}}(x,\theta)} U_\kappa(x)\mathrm{d}x = \int_{\mathsf{c}(X,\theta)} h_\kappa^2(y)\mathrm{d}\sigma(y), \quad X = \left(x, \sqrt{1-\|x\|^2}\right).$$

Consequently, by (13.4.1), we can deduce from (11.1.6) and (13.1.4) that the stated result follows from Theorem 9.1.3. □

Let $K_n^\delta(U_\kappa;x,y)$ be the kernel of the Cesàro means $S_n^\delta(U_\kappa;f)$. Despite the relation (13.1.11) between the projection operators on the simplex and on the ball, there is no direct relation between the kernel $K_n^\delta(U_\kappa;\cdot,\cdot)$ on the simplex and the kernel $K_n^\delta(W_\kappa;\cdot,\cdot)$ on the ball. As a result, most of the results for the (C,δ) means $S_n^\delta(U_\kappa;f)$ need to be proved directly. Fortunately, the proofs mostly follow along the same lines as those we have encountered in the cases of the unit sphere and the unit ball, so that we can afford to be brief.

We start with a pointwise estimate of the kernel function, for which we need to introduce the following notation: for $x=(x_1,\ldots,x_d), y=(y_1,\ldots,y_d)\in\mathbb{T}^d$, let

$$\xi := (\sqrt{x_1},\ldots,\sqrt{x_d},\sqrt{x_{d+1}}), \quad \zeta := (\sqrt{y_1},\ldots,\sqrt{y_d},\sqrt{y_{d+1}})$$

with $x_{d+1} := 1-|x|$ and $y_{d+1} := 1-|y|$. Both of these are points in $\mathbb{S}^d$, since $|x| = x_1+\cdots+x_d$ by definition.

Theorem 13.4.3. *Let $\delta > -1$. For $x,y\in\mathbb{T}^d$,*

$$|K_n^\delta(U_\kappa;x,y)| \le c\left[\frac{\prod_{j=1}^{d+1}\left(\sqrt{x_jy_j}+n^{-1}\|\xi-\zeta\|+n^{-2}\right)^{-\kappa_j}}{n^{\delta-(d-1)/2}(\|\xi-\zeta\|+n^{-1})^{\delta+(d+1)/2}} + \frac{\prod_{j=1}^{d+1}\left(\sqrt{x_jy_j}+\|\xi-\zeta\|^2+n^{-2}\right)^{-\kappa_j}}{n(\|\xi-\zeta\|+n^{-1})^{d+1}}\right]. \tag{13.4.2}$$

Furthermore, for the kernel of the projection operator,

$$|P_n(U_\kappa;x,y)| \le c\frac{\prod_{j=1}^{d+1}\left(\sqrt{x_jy_j}+n^{-1}\|\xi-\zeta\|+n^{-2}\right)^{-\kappa_j}}{n^{-(d-1)/2}(\|\xi-\zeta\|+n^{-1})^{(d-1)/2}}. \tag{13.4.3}$$

Proof. The estimate (13.4.3) follows from (8.3.2) using the relations (11.2.6) and (13.1.11). For $\delta>-1$, by (13.1.12) and (13.2.9), we have

$$K_n^\delta(U_\kappa;x,y) = c_\kappa\int_{[-1,1]^{d+1}} k_n^\delta\left(w_{\lambda_\kappa-\frac12,-\frac12};1,2z(x,y,t)^2-1\right)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t,$$

where $z(x,y,t) := \sum_{j=1}^{d+1}\sqrt{x_jy_j}t_j$ with $x_{d+1}=\sqrt{1-|x|}$ and $y_{d+1}=\sqrt{1-|y|}$. Setting $\alpha=\lambda_\kappa-\frac12$ and $J=\lfloor\alpha+3/2\rfloor$, as in the proof of Theorem 8.3.2, we use Lemma 8.3.1 to break the kernel $K_n^\delta(U_\kappa;x,y)$ into the sum

$$K_n^\delta(U_\kappa;x,y) = \sum_{j=0}^{J} b_j(\alpha,-1/2,\delta,n)\Omega_j(x,y) + \Omega_*(x,y),$$

where

$$\Omega_j(x,y) = c_\kappa \int_{[-1,1]^{d+1}} P_n^{(\alpha+\delta+j+1,-\frac{1}{2})}\left(2z(x,y,t)^2-1\right)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t$$

and

$$\Omega_*(x,y) = c_\kappa \int_{[-1,1]^{d+1}} G_n^\delta\left(2z(x,y,t)^2-1\right)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t.$$

Using the quadratic transform $P_n^{(\lambda,-\frac{1}{2})}(2t^2-1) = a_n P_{2n}^{(\lambda,\lambda)}(t)$ with $a_n = O(1)$, we can further write

$$\Omega_j(x,y) = O(1)\int_{[-1,1]^{d+1}} P_{2n}^{(\alpha+\delta+j+1,\alpha+\delta+j+1)}(z(x,y,t))\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t.$$

Since $\xi,\zeta \in \mathbb{S}^d$, we have $1-z(x,y,t) \ge \|\xi-\zeta\|^2/2$. Hence, we can follow the same procedure as in the proof of Theorem 8.3.2 and use the general estimate in Theorem 8.2.5 to complete the proof. □

Theorem 13.4.4. *Let $\delta > -1$. For $p = 1$ or ∞,*

$$\|\operatorname{proj}_n(U_\kappa)\|_{U_k,p} \sim n^{\sigma_\kappa} \quad \text{and} \quad \|S_n^\delta(U_\kappa)\|_{U_k,p} \sim \begin{cases} 1, & \delta > \sigma_\kappa, \\ \log n, & \delta = \sigma_\kappa, \\ n^{-\delta+\sigma_\kappa}, & -1 < \delta < \sigma_\kappa. \end{cases}$$

In particular, $S_n^\delta(U_\kappa;f)$ converges in $L^p(U_\kappa;\mathbb{T}^d)$ for all $1 \le p \le \infty$ if and only if $\delta > \sigma_\kappa$.

Proof. The order of $\|\operatorname{proj}_n(U_\kappa)\|_{U_k,p}$ follows from the relation (13.1.11) and Theorem 11.4.1. Since $\operatorname{proj}_n(U_\kappa)$ is related to $\operatorname{proj}_{2n}(W_\kappa)$, the lower estimate in the equivalence works for all n for $\operatorname{proj}_n(U_\kappa)$. For $\delta > -1$, the upper estimate of $\|S_n^\delta(U_\kappa)\|_{U_k,p}$ is based on the estimate (13.4.2), which can be worked out as in the proof of Theorem 8.1.1 or, in fact, can be converted, using (13.1.4), into the estimate there.

For the lower bound estimate, we need the following identities:

$$K_n^\delta(U_\kappa;x,e_j) = k_n^\delta\left(w_{\lambda_\kappa-\kappa_j-\frac{1}{2},\kappa_j-\frac{1}{2}};1,2x_j-1\right), \quad 1 \le j \le d, \tag{13.4.4}$$

$$K_n^\delta(U_\kappa;x,0) = k_n^\delta\left(w_{\lambda_\kappa-\kappa_{d+1}-\frac{1}{2},\kappa_{d+1}-\frac{1}{2}};1,1-2|x|\right), \tag{13.4.5}$$

where e_j is the jth coordinate vector, whose jth component is 1 and all other components are zero. To prove (13.4.4), we set $x = e_j$ to get

$$K_n^\delta(U_\kappa; e_j, y) = c_{\kappa_j} \int_{-1}^{1} k_n^\delta \left(w_{\lambda_\kappa - \frac{1}{2}, -\frac{1}{2}}; 1, 2y_j t_j^2 - 1 \right) (1 - t_j^2)^{\kappa_j - 1} \mathrm{d}t_j.$$

By definition, $k_n^\delta(w_{\lambda-\frac{1}{2},-\frac{1}{2}}; 1, s)$ is the (C, δ) mean of the Jacobi polynomials $[h_n^{(\alpha,\beta)}]^{-2} P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(1) P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(s)$, so that (13.4.4) follows from (B.1.6). Similarly, setting $x = 0$, we have

$$K_n^\delta(U_\kappa; 0, y) = c_{\kappa_{d+1}} \int_{-1}^{1} k_n^\delta \left(w_{\lambda_\kappa - \frac{1}{2}, -\frac{1}{2}}; 1, 2(1 - |y|)t^2 - 1 \right) (1 - t^2)^{\kappa_{d+1} - 1} \mathrm{d}t,$$

from which (13.4.5) follows again from (B.1.6). Now assume that $\kappa_1 = \kappa_{\min}$, for example. Then by (13.4.4), changing variables $y_i = (1 - y_1)u_i$ for $2 \le i \le d$, we obtain

$$\int_{\mathbb{T}^d} \left| K_n^\delta(U_\kappa; e_1, y) \right| U_\kappa(y) \mathrm{d}y = c \int_{-1}^{1} \left| k_n^\delta \left(w_{\lambda_\kappa - \kappa_1 - \frac{1}{2}, \kappa_1 - \frac{1}{2}}; 1, t \right) \right| w_{\lambda_\kappa - \kappa_1 - \frac{1}{2}, \kappa_1 - \frac{1}{2}}(t) \mathrm{d}t,$$

which is bounded if and only if $\delta > \lambda_\kappa - \kappa_1$ by the classical result on Jacobi series [162, Theorem 9.1.4]. Similarly, if $\kappa_{d+1} = \kappa_{\min}$, we make a change of variables $y = sy'$ with $|y'| = 1$ to obtain

$$\begin{aligned}
\int_{\mathbb{T}^d} \left| K_n^\delta(U_\kappa; 0, y) \right| U_\kappa(y) \mathrm{d}y &= c \int_0^1 s^{\lambda_k - \kappa_{d+1} - \frac{1}{2}} \left| k_n^\delta \left(w_{\lambda_\kappa - \kappa_{d+1} - \frac{1}{2}, \kappa_{d+1} - \frac{1}{2}}; 1, 1 - 2s \right) \right| \\
&\qquad \times (1 - s)^{\kappa_{d+1} - \frac{1}{2}} \mathrm{d}s \\
&= c \int_{-1}^{1} \left| k_n^\delta \left(w_{\sigma_\kappa - \frac{1}{2}, \kappa_{d+1} - \frac{1}{2}}; 1, t \right) \right| w_{\sigma_\kappa - \frac{1}{2}, \kappa_{d+1} - \frac{1}{2}}(t) \mathrm{d}t,
\end{aligned}$$

which is bounded if and only if $\delta > \sigma_\kappa$ by the same result on the Jacobi series. □

We can also state an analogue of Theorem 11.4.2.

Theorem 13.4.5. *Let f be continuous on $\mathbb{T}^d$. If $\delta > (d-1)/2$, then $S_n^\delta(U_\kappa; f)$ converges to f for every x in the interior of $\mathbb{T}^d$, and the convergence is uniform over each compact set contained inside $\mathbb{T}^d$.*

Proof. The proof relies on the estimate (13.4.2) and can be carried out as in the proof of Theorem 8.1.3. In fact, setting $x^2 = (x_1^2, \ldots, x_d^2)$ for $x \in \mathbb{R}^d$, the upper bound for $K_n^\delta(U_\kappa; x^2, y^2)$ derived from (13.4.2) is the same as that of the upper bound of $K_n^\delta(h_\kappa^2; x, y)$ in the proof of Theorem 8.1.3, so that the proof can be carried over using (11.1.6) and (13.1.4). □

Further results on the L^p convergence of the (C,δ) means with respect to U_κ on the simplex can also be deduced, although they again do not follow directly from the corresponding results on the ball.

Theorem 13.4.6. *Suppose that* $f \in L^p(U_\kappa;\mathbb{T}^d)$, $1 \le p \le \infty$, $|\frac{1}{p}-\frac{1}{2}| \ge \frac{1}{2\sigma_\kappa+2}$ *and*

$$\delta > \delta_\kappa(p) := \max\left\{(2\sigma_\kappa+1)\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2},0\right\}.$$

Then $S_n^\delta(U_\kappa;f)$ *converges to* f *in* $L^p(U_\kappa;\mathbb{T}^d)$ *and*

$$\sup_{n\in\mathbb{N}}\left\|S_n^\delta(U_\kappa;f)\right\|_{U_\kappa,p} \le c\|f\|_{U_\kappa,p}.$$

Proof. We follow the decomposition in Sect. 9.2.1.1 to define

$$S_{n,v}^\delta(U_\kappa;f) = \sum_{j=0}^n \hat{S}_{n,v}^\delta(j)\,\mathrm{proj}_j(U_\kappa;f), \qquad v=1,2,\ldots,\lfloor \log_2 n\rfloor+2.$$

The same argument shows that it suffices to prove the analogue of (9.2.4),

$$\left\|S_{n,v}^\delta(U_\kappa;f)\right\|_{U_\kappa,p} \le c2^{-v\varepsilon}\|f\|_{U_\kappa;p}, \qquad v=2,\ldots,\lfloor \log_2 n\rfloor+2. \tag{13.4.6}$$

Denote the kernel of $S_{n,v}^\delta(U_\kappa;f)$ by $K_{n,v}^\delta(U_\kappa;x,y)$. Then we have by (13.1.12) that

$$K_{n,v}^\delta(U_\kappa;x,y) := c_\kappa\int_{[-1,1]^{d+1}} D_{n,v}^\delta(U_\kappa;2z(x,y,t)^2-1)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t,$$

where

$$D_{n,v}^\delta(U_\kappa;t) := \sum_{j=0}^n \hat{S}_{n,v}^\delta(j)\frac{(2j+\lambda_\kappa)\Gamma(\frac{1}{2})\Gamma(j+\lambda_k)}{\Gamma(\lambda_\kappa+1)\Gamma(j+\frac{1}{2})}P_j^{(\lambda_k-\frac{1}{2},\frac{1}{2})}(t).$$

Consequently, defining analogues of $a_{n,v,\ell}$ in Sect. 9.2.1.2 by

$$\begin{aligned} a_{n,v,0}^{\mathbb{T}}(j) &= (2j+\lambda_k)\hat{S}_{n,v}^\delta(j) \\ a_{n,v,\ell+1}^{\mathbb{T}}(j) &= \frac{a_{n,v,\ell}^{\mathbb{T}}(j)}{2j+2\lambda_\kappa+\ell}-\frac{a_{n,v,\ell}^{\mathbb{T}}(j+1)}{2j+2\lambda_\kappa+\ell+2}, \end{aligned}$$

we can then write, again following the proof of Theorem 2.6.7,

$$D_{n,v}^\delta(U_\kappa;t) = c\sum_{j=0}^n a_{n,v,\ell}^{\mathbb{T}}(j)\frac{\Gamma(j+2\lambda_\kappa+\ell)}{\Gamma(j+\lambda_\kappa+\frac{1}{2})}P_j^{(\lambda_k+\ell-\frac{1}{2},-\frac{1}{2})}(t).$$

The analogue of the estimate (9.2.5) holds in exact form for $a^{\mathbb{T}}_{n,\nu,\ell}(j)$. Thus, to follow the proof of Lemma 9.2.3, we need to estimate

$$\int_{[-1,1]^{d+1}} P_j^{(\lambda_k+\ell-\frac{1}{2},-\frac{1}{2})}(2z(x,y,t)^2-1)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t. \tag{13.4.7}$$

Using the quadratic transform to write $P_n^{(\alpha,-1/2)}(2t^2-1)$ in terms of $P_n^{(\alpha,\alpha)}$, we can estimate (13.4.7) again by (8.2.5). The result is

$$|K^\delta_{n,\nu}(U_\kappa;x,y)|\,[U_\kappa(y)]^{-1}\le cn^d 2^{\nu(\ell-1-\delta)}\,(1+nd(\bar{x},\bar{y}))^{-\lambda_\kappa-\ell+d-1},$$

which implies that the analogue of Corollary 9.2.4 holds; that is, for every $\gamma>0$, there is an $\varepsilon_0>0$ such that

$$\sup_{x\in\mathbb{T}^d}\int_{\{y:\mathtt{d}_{\mathbb{T}}(x,y)\ge 2^{(1+\nu)\gamma}/n\}}|K_{n,\nu}(U_\kappa;x,y)|U_\kappa(y)\mathrm{d}y\le c2^{-\nu\varepsilon_0}.$$

In order to prove (13.4.6), we then define Λ to be a maximal separate subset of $\mathbb{T}^d$ exactly like the one we defined in Sect. 9.2.1.3, except with $\mathtt{d}(\varpi,\varpi')$ replaced by $\mathtt{d}_{\mathbb{T}}(x,x')$. Define

$$f_y(x):=f(x)\chi_{\mathtt{c}_{\mathbb{T}}(y,\frac{2^{\nu}1}{n})}(x)[A(x)]^{-1},\qquad A(x)=\sum_{y\in\Lambda}\chi_{\mathtt{c}_{\mathbb{T}}(y,\frac{2^{\nu}1}{n})}(x).$$

Then the same argument shows that it suffices to prove that

$$\left\|S^\delta_{n,\nu}(U_\kappa;f_y)\right\|_{U_\kappa,p}\le c2^{-\nu\varepsilon_0}\|f_y\|_{U_\kappa;p}.$$

This last inequality can be established exactly as in (9.2.7), and there is no need to introduce the additional set $\mathtt{c}^*(\varpi,\theta)$. $\square$

As an accompaniment to Theorem 13.4.6, we have an analogue of Theorem 11.4.5.

Theorem 13.4.7. *Assume $1\le p\le\infty$ and $0<\delta\le\delta_\kappa(p)$. Then there exists a function $f\in L^p(U_\kappa;\mathbb{T}^d)$ such that $S_n^\delta(U_\kappa;f)$ diverges in $L^p(U_\kappa;\mathbb{T}^d)$.*

Proof. We first note that the analogue of the Nikolskii inequality (5.5.1) holds, with a similar proof, and it gives, in particular, for $1\le p<\infty$,

$$\|Q\|_\infty:=\max_{x\in\mathbb{T}^d}|Q(x)|\le cn^{(2\sigma_\kappa+1)/p}\|Q\|_{U_\kappa,p}$$

for every polynomial Q of degree n on $\mathbb{R}^d$. Hence, following the proof of Theorem 9.2.2, it is sufficient to prove that

$$\left\| K_n^\delta(U_\kappa;x,\cdot) \right\|_{\kappa,q} \le c n^{(2\sigma_\kappa+1)/p}, \quad \frac{1}{p}+\frac{1}{q}=1,$$

where $y \in \mathbb{T}^d$ is fixed, does not hold for $p = p_1 := \frac{2\sigma_\kappa+1}{\sigma_\kappa-\delta}$. By considering $x = e_j$ and $x = 0$, we can follow the proof of Theorem 13.4.4 to reduce the problem to that of the Jacobi series, for which the desired result is known [30]. □

13.5 Near-Best-Approximation Operators and Highly Localized Kernels

In analogy to Definition 11.5.1, we define near-best-approximation operators on the simplex.

Definition 13.5.1. Let η be a C^∞-function on $[0,\infty)$ such that $\eta(t) = 1$ for $0 \le t \le 1$ and $\eta(t) = 0$ for $t \ge 2$. Define

$$L_n(U_\kappa; f, x) := f *_{\kappa,\mathbb{T}} L_n(x) = a_\kappa \int_{\mathbb{T}^d} f(y) L_n(U_\kappa; x, y) U_\kappa(y) \mathrm{d}y$$

for $x \in \mathbb{T}^d$ and $n = 0, 1, 2, \ldots$, where

$$L_n(U_\kappa; x, y) := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) P_k(U_\kappa; x, y).$$

For $f \in L^p(U_\kappa; \mathbb{T}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{T}^d)$ if $p = \infty$, the error of best approximation to f by polynomials of degree at most n is defined by

$$E_n(f)_{U_\kappa,p} := \inf_{g \in \Pi_n} \|f - g\|_{U_\kappa,p}, \qquad 1 \le p \le \infty. \tag{13.5.1}$$

The following theorem is an analogue of Theorem 11.5.2 but not its consequence, despite (13.1.11). The proof of Theorem 2.6.3, on the other hand, applies with few changes.

Theorem 13.5.2. *Let $f \in L^p(U_\kappa; \mathbb{T}^d)$ if $1 \le p < \infty$ and $f \in C(\mathbb{T}^d)$ if $p = \infty$. Then*

(1) $L_n(U_\kappa; f) \in \Pi_{2n-1}^d$ *and* $L_n(U_\kappa; f) = f$ *for* $f \in \Pi_n^d$.
(2) *For* $n \in \mathbb{N}$, $\|L_n(U_\kappa; f)\|_{U_\kappa,p} \le c \|f\|_{U_\kappa,p}$.
(3) *For* $n \in \mathbb{N}$,

$$\|f - L_n(U_\kappa; f)\|_{U_\kappa,p} \le (1 + c) E_n(f)_{U_\kappa,p}.$$

As in the case of the classical weight function W_μ on the unit ball, the kernel $K_n(U_\mu, x, y)$ is highly localized around the main diagonal $x = y$ in $\mathbb{T}^d \times \mathbb{T}^d$.

Theorem 13.5.3. *Let $\mu \ge 0$ and let ℓ be a positive integer. There exists a constant c_ℓ depending only on ℓ, d, μ, and η such that*

$$|L_n(U_\kappa;x,y)| \le c_\ell \frac{n^d}{\sqrt{\mathscr{U}_\kappa(n;x)}\sqrt{\mathscr{U}_\kappa(n;y)}(1+n\,\mathtt{d}_{\mathbb{T}}(x,y))^\ell} \tag{13.5.2}$$

for $x,y \in \mathbb{T}^d$, where

$$\mathscr{U}_\kappa(n;x) := \prod_{i=1}^{d+1}(x_i+n^{-2})^{\kappa_i}, \quad x_{d+1} := 1-|x|. \tag{13.5.3}$$

Proof. We only prove the case $\kappa_i > 0$ for $1 \le i \le d+1$; the case in which some κ_i are zeros can be treated similarly and is in fact easier. By (13.1.12), we can express the kernel $L_n(U_\kappa;x,y)$ in terms of a univariate kernel L_n,

$$L_n(U_\kappa;x,y) = c(\kappa,d)\int_{[-1,1]^{d+1}} L_n\left(2z(x,y,t)^2-1\right)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t,$$

where $z(x,y,t) = \sqrt{x_1}\sqrt{y_1}t_1+\cdots+\sqrt{x_{d+1}}\sqrt{y_{d+1}}t_{d+1}$ and L_n is defined by

$$L_n(t) := \sum_{j=0}^{\infty}\eta\left(\frac{j}{n}\right)p_j^{(\lambda_\kappa-1/2,-1/2)}(1)p_j^{(\lambda_\kappa-1/2,-1/2)}(t), \quad t\in[-1,1]. \tag{13.5.4}$$

Let $\theta(x,y,t) := \arccos(2z(x,y,t)^2-1)$. We use the fact that

$$1-z(x,y,t)^2 = \frac{1}{2}(1-\cos\theta(x,y,t)) = \sin^2\frac{\theta(x,y,t)}{2} \sim \theta(x,y,t)^2$$

and apply the estimate (2.6.8) with $j=0$, $\alpha=\lambda_k-1/2$, and $\beta=-1/2$ to obtain

$$|L_n(U_\kappa;x,y)| \le cn^{2\lambda_\kappa+1}\int_{[-1,1]^{d+1}}\frac{1}{(1+n\sqrt{1-z(x,y,t)^2})^\ell}\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t.$$

Since it is evident that

$$1-z(x,y,t)^2 \ge 1-|z(x,y,t)| \ge 1-\sqrt{x_1y_1}|t_1|-\cdots-\sqrt{x_{d+1}y_{d+1}}|t_{d+1}|,$$

using the symmetry of the integrand with respect to $t\in[-1,1]^{d+1}$, we obtain

$$|L_n(U_\kappa;x,y)| \le cn^{2\lambda_\kappa+1}\int_{[0,1]^{d+1}}\frac{1}{(1+n\sqrt{1-z(x,y,t)^2})^\ell}\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t.$$

Let J denote the integral in the above inequality. Then the proof of (13.5.2) will follow once the following claim is established: for $\ell > 3|\kappa| + d + 1$,

$$J \le c \frac{n^{-2|\kappa|}}{\sqrt{\mathscr{U}_\kappa(n;x)}\sqrt{\mathscr{U}_\kappa(n;y)}(1 + n\,\mathtt{d}_{\mathbb{T}}(x,y))^{\ell-3|\kappa|-d-1}}. \tag{13.5.5}$$

First, from the definition of $z(x,y,t)$, we obtain the lower estimate

$$\begin{aligned} 1 - z(x,y,t) &\ge 1 - \sqrt{x_1 y_1} - \cdots - \sqrt{x_{d+1} y_{d+1}} = 1 - \cos \mathtt{d}_{\mathbb{T}}(x,y) \\ &= 2\sin^2 \frac{\mathtt{d}_{\mathbb{T}}(x,y)}{2} \ge \frac{2}{\pi^2} \mathtt{d}_{\mathbb{T}}(x,y)^2, \end{aligned} \tag{13.5.6}$$

which enables us to deduce the estimate

$$J \le \frac{c}{(1 + n\mathtt{d}_{\mathbb{T}}(x,y))^{\ell}}. \tag{13.5.7}$$

Second, from the definition of $z(x,y,t)$, we have

$$1 - z(x,y,t) = 1 - \cos \mathtt{d}_{\mathbb{T}}(x,y) + \sum_{i=1}^{d+1} \sqrt{x_i y_i}(1 - t_i) \ge \sum_{i=1}^{d+1} \sqrt{x_i y_i}(1 - t_i),$$

which, together with (13.5.6), implies that

$$J \le \frac{c}{(1 + n\mathtt{d}_{\mathbb{T}}(x,y))^{\ell-2|\kappa|-(d+1)}} \int_{[0,1]^{d+1}} \frac{\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i - 1}\mathrm{d}t}{\left(1 + n\left[\sum_{i=1}^{d+1} \sqrt{x_i y_i}(1-t_i)\right]^{1/2}\right)^{\gamma}},$$

where $\gamma := 2|k| + d + 1$. Denote the integral on the right-hand side of the above inequality by $I_{d+1}(\gamma)$. In order to estimate this integral, we first establish the following inequality for $A > 0$, $B \ge 0$, $\gamma \ge 2\kappa + 1$, and $\kappa > 0$:

$$\int_0^1 \frac{(1-t^2)^{\kappa-1}\mathrm{d}t}{(1 + n\sqrt{B + A(1-t)})^{\gamma}} \le \frac{cn^{-2\kappa}}{A^{\kappa}\left(1 + n\sqrt{B}\right)^{\gamma - 2\kappa - 1}}. \tag{13.5.8}$$

Indeed, substituting $s = n^2 A(1-t)$, we see that

$$\begin{aligned} \int_0^1 \frac{(1-t^2)^{\kappa-1}\mathrm{d}t}{\left(1 + n\sqrt{B + A(1-t)}\right)^{\gamma}} &\le \frac{2^{\kappa-1}}{(An^2)^{\kappa}} \int_0^{An^2} \frac{s^{\kappa-1}\mathrm{d}s}{(1 + \sqrt{n^2 B + s})^{\gamma}} \\ &\le \frac{2^{\kappa-1}}{(An^2)^{\kappa}(1 + n\sqrt{B})^{\gamma-2\kappa-1}} \int_0^{\infty} \frac{s^{\kappa-1}\mathrm{d}s}{(1 + \sqrt{s})^{2\kappa+1}} \le \frac{cn^{-2\kappa}}{A^{\kappa}\left(1 + n\sqrt{B}\right)^{\gamma-2\kappa-1}}. \end{aligned}$$

We now set $B := 1 + n\sum_{i=1}^{d}\sqrt{x_i y_i}t_i$ and $A := \sqrt{x_{d+1}y_{d+1}}$, and apply inequality (13.5.8) to the integral in $I_{d+1}(\gamma)$ with respect to t_{d+1}, which leads to

$$I_{d+1}(\gamma) \le \frac{cn^{-2\kappa_{d+1}}}{(\sqrt{x_{d+1}y_{d+1}})^{\kappa_{d+1}}}\int_{[0,1]^d}\frac{\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}\mathrm{d}t}{\left(1+n\left[\sum_{i=1}^{d}\sqrt{x_i y_i}(1-t_i)\right]^{1/2}\right)^{\gamma-2\kappa_{d+1}-1}}$$

$$= \frac{cn^{-2\kappa_{d+1}}}{(\sqrt{x_{d+1}y_{d+1}})^{\kappa_{d+1}}}I_d(\gamma-2\kappa_{d+1}-1).$$

Iterating this process with respect to $t_d, t_{d-1}, \ldots, t_1$, we obtain then

$$I_{d+1}(\gamma) \le \frac{cn^{-2|\kappa|}}{\prod_{i=1}^{d+1}(\sqrt{x_i y_i})^{\kappa_i}} \le \frac{cn^{-2|\kappa|}}{\prod_{i=1}^{d+1}(\sqrt{x_i y_i}+n^{-2})^{\kappa_i}},$$

where the second inequality follows because we trivially have $I_{d+1}(\gamma) \le 1$. Consequently, we conclude that

$$J \le \frac{cn^{-2|\kappa|}}{(1+n\mathtt{d}_{\mathbb{T}}(x,y))^{\ell-2|\kappa|-(d+1)}\prod_{i=1}^{d+1}(\sqrt{x_i y_i}+n^{-2})^{\kappa_i}}.$$

Applying the elementary inequalities (11.5.13) and (A.1.5), we obtain

$$\begin{aligned}\sqrt{x_i+n^{-2}}\sqrt{y_i+n^{-2}} &\le (\sqrt{x_i}+n^{-1})(\sqrt{y_i}+n^{-1})\\ &\le 3(\sqrt{x_i y_i}+n^{-2})(1+n|\sqrt{x_i}-\sqrt{y_i}|)\\ &\le 3(\sqrt{x_i y_i}+n^{-2})(1+n\mathtt{d}_{\mathbb{T}}(x,y)),\end{aligned}$$

which implies further that

$$J \le \frac{cn^{-2|\kappa|}}{(1+n\mathtt{d}_{\mathbb{T}}(x,y))^{\ell-3|\kappa|-(d+1)}\sqrt{\mathscr{U}_\kappa(n;x)}\sqrt{\mathscr{U}_\kappa(n;y)}}.$$

This inequality, together with (13.5.7), proves (13.5.5) and completes the proof. □

13.6 Weighted Best Approximation on the Simplex

We can also define a modulus of smoothness on $\mathbb{T}^d$ using the generalized translation operator $T_\theta(U_\kappa; f)$, in analogy with the third modulus of smoothness defined on the unit ball in Definition 12.4.1.

Definition 13.6.1. For $r > 0$, define

$$\omega_r(f;t)_{U_\kappa,p} := \sup_{\theta\le t}\left\|(T_\theta(U_\kappa)-I)^{r/2}f\right\|_{U_\kappa,p}.$$

This modulus of smoothness is closely related to the third modulus of smoothness on $\mathbb{B}^d$, which we denote below by $\omega_r(f,t)_{W_\kappa,p} := \omega_r^*(f,t)_{p,\kappa}$, as defined in (12.4.1). In fact, let $\psi : \mathbb{T}^d \mapsto \mathbb{B}^d$ be the map defined in (13.1.3); then it follows from Proposition 13.2.6 and (13.1.4) that

$$\omega_r(f;t)_{U_\kappa,p} = \omega_r(f\circ\psi;t)_{W_\kappa,p}. \tag{13.6.1}$$

By (12.4.2), this in turn shows that $\omega_r(f;t)_{U_\kappa,p}$ can be expressed in terms of the modulus of smoothness $\omega_r^*(f,t)_{L^p(h_\kappa,\mathbb{S}^d)}$ on $\mathbb{S}^d$, defined in Definition 10.1.1, for $L^p(h_\kappa^2;\mathbb{S}^d)$, where $h_\kappa^2(x) = \prod_{j=1}^{d+1}|x_j|^{\kappa_j}$.

There is also an equivalent K-functional. Recall the differential operator $\mathscr{D}_{\kappa,\mathbb{T}}$ defined in (13.1.6), which has $\mathscr{V}_n^d(U_\kappa)$ as its eigenspaces. For $r>0$, we define the fractional power $(-\mathscr{D}_{\kappa,\mathbb{T}})^r$ of $\mathscr{D}_{\kappa,\mathbb{T}}$ in a distributional sense by

$$\operatorname{proj}_n(U_\kappa;(-\mathscr{D}_{\kappa,\mathbb{T}})^r f) := (n(n+2\lambda_\kappa))^r \operatorname{proj}_n(U_\kappa;f), \quad n=0,1,2,\ldots.$$

The K-functional is defined in terms of the fractional powers of $\mathscr{D}_{\kappa,\mathbb{T}}$ as follows.

Definition 13.6.2. Let $f\in L^p(U_\kappa;\mathbb{T}^d)$ if $1\le p<\infty$, and $f\in C(\mathbb{T}^d)$ if $p=\infty$. For $r>0$, define the K-functional $K_r(f,t)_{U_\kappa,p}$ by

$$K_r(f,t)_{U_\kappa,p} := \inf_{g\in C^\infty(\mathbb{T}^d)}\left\{\|f-g\|_{U_\kappa,p} + t^r\left\|\left(-\mathscr{D}_{\kappa,\mathbb{T}}\right)^{r/2}g\right\|_{U_\kappa,p}\right\}. \tag{13.6.2}$$

For $f\in L^p(U_\kappa;\mathbb{T}^d)$ if $1\le p<\infty$ and $f\in C(\mathbb{T}^d)$ if $p=\infty$, define

$$E_n(f)_{U_\kappa,p} := \inf_{g\in\Pi_n}\|f-g\|_{U_\kappa,p}.$$

Theorem 13.6.3. *Let $f\in L^p(U_\kappa;\mathbb{T}^d)$ if $1\le p<\infty$, and $f\in C(\mathbb{T}^d)$ if $p=\infty$.*

(i) *If $t\in(0,1)$ and $r>0$, then*

$$\omega_r(f,t)_{U_\kappa,p} \sim K_r(f,t)_{U_\kappa,p}.$$

(ii) *We have the direct inequality*

$$E_n(f)_{U_\kappa,p} := \inf_{g\in\Pi_n}\|f-g\|_{U_\kappa,p} \le c\,\omega_r(f,n^{-1})_{U_\kappa,p}, \quad n=1,2,\ldots,$$

and the weak inverse inequality

$$\omega_r(f,n^{-1})_{U_\kappa,p} \le cn^{-r}\sum_{k=0}^n (k+1)^{r-1}E_k(f)_{U_\kappa,p}.$$

Proof. We denote by $K_r(f,t)_{W_\kappa,p} := K_r^*(f,t)_{p,\kappa}$ the third K-functional, defined by (12.4.4), on the ball $\mathbb{B}^d$. Because of Theorem 12.4.3 and (13.6.1), it suffices for the proof of (i) to show that

$$K_r(f,t)_{U_\kappa,p} = K_r(f\circ\psi, t/2)_{W_\kappa,p}.$$

Using (13.1.4) and (13.1.8), it follows that

$$\left\|(-\mathscr{D}_{\kappa,\mathbb{T}})^{r/2} g\right\|_{U_\kappa,p} = \left\|\left((-\mathscr{D}_{\kappa,\mathbb{T}})^{r/2} g\right)\circ\psi\right\|_{W_\kappa,p} = 2^{-r}\left\|(-\mathscr{D}_{\kappa,\mathbb{B}})^{r/2}(g\circ\psi)\right\|_{W_\kappa,p}.$$

Hence, directly from the definition and (13.1.4),

$$\begin{aligned} K_r(f,t)_{U_\kappa,p} &= \inf_{g\in C^\infty(\mathbb{T}^d)}\left\{\|f\circ\psi - g\circ\psi\|_{W_\kappa,p} + 2^{-r}t^r\|(-\mathscr{D}_{\kappa,\mathbb{B}})^{r/2}(g\circ\psi)\|_{W_\kappa,p}\right\} \\ &= \inf_{g_0}\left\{\|f\circ\psi - g_0\|_{W_\kappa,p} + 2^{-r}t^r\|(-\mathscr{D}_{\kappa,\mathbb{B}})^{r/2}g_0\|_{W_\kappa,p}\right\}, \qquad (13.6.3) \end{aligned}$$

where the infimum is taken over all g_0 such that $g_0 = g\circ\psi \in C^\infty(\mathbb{B}^d)$. Consequently, it follows immediately that $K_r(f,t)_{U_\kappa,p} \ge K_r(f\circ\psi, t/2)_{W_\kappa,p}$. To prove the reverse inequality, for $g\in C^\infty(\mathbb{B}^d)$, we let $g_0(x) = 2^{-d}\sum_{\varepsilon\in\mathbb{Z}_2^d} R(\varepsilon)g(x)$, where $R(\varepsilon)g(x) := g(\varepsilon x)$ for $\varepsilon\in\mathbb{Z}_2^d$. Then g_0 is even in each of its variables. Since $\mathscr{D}_{\kappa,\mathbb{B}}$ is invariant under $\mathbb{Z}_2^d$ by Theorem 11.1.5, we have

$$\|(-\mathscr{D}_{\kappa,\mathbb{B}})^{r/2}g_0\|_{W_\kappa,p} \le 2^{-d}\sum\|(-\mathscr{D}_{\kappa,\mathbb{B}})^{r/2}R(\varepsilon)g\|_{W_\kappa,p} \le \|(-\mathscr{D}_{\kappa,\mathbb{B}})^{r/2}g\|_{W_\kappa,p},$$

and furthermore, we also have

$$\|f\circ\psi - g_0\|_{W_\kappa,p} \le 2^{-d}\sum\|f\circ\psi - R(\varepsilon)g\|_{W_\kappa,p} = \|f\circ\psi - g\|_{W_\kappa,p}.$$

Consequently, since g_0 is even in each of its variables, it follows from (13.6.3) that $K_r(f,t)_{U_\kappa,p} \le K_r(f\circ\psi, t/2)_{W_\kappa,p}$. This completes the proof of (i). The above consideration of taking the invariant part g_0 of g also implies that

$$E_n(f)_{U_\kappa,p} = E_n(f\circ\psi)_{W_\kappa,p},$$

from which the proof of (ii) follows readily from the corresponding result on $\mathbb{B}^d$.

□

13.7 Cubature Formulas on the Simplex

For cubature formulas, the basic results in Sect. 11.6, including Theorem 11.6.2, hold for cubature formulas on the simplex. The lower bound in (11.6.2) is again not sharp in general.

13.7.1 Cubature Formulas on the Simplex and on the Ball

Our main concern in this subsection is a close relation between cubature formulas on the simplex $\mathbb{T}^d$ and those on the ball $\mathbb{B}^d$. Such a relation is easy to be understood in view of Lemma 13.1.2 and the Sobolev theorem for invariant cubature formulas.

Theorem 13.7.1. *If there is a cubature formula of degree n on $\mathbb{T}^d$ given by*

$$\int_{\mathbb{T}^d} f(u)U_\kappa(u)\mathrm{d}u = \sum_{i=1}^{N} \lambda_i f(u_i), \tag{13.7.1}$$

with all $u_i \in \mathbb{T}^d$, then there is a cubature formula of degree $2n+1$ on $\mathbb{B}^d$, invariant under $\mathbb{Z}_2^d$, given by

$$\int_{\mathbb{B}^d} g(x)W_\kappa(x)\mathrm{d}x = \sum_{i=1}^{N} \lambda_i 2^{-k(u_i)} \sum_{\varepsilon\in\mathbb{Z}_2^d} f(\varepsilon_1\sqrt{u_{i,1}},\dots \varepsilon_d\sqrt{u_{i,d}}), \tag{13.7.2}$$

where $k(u)$ denotes the number of nonzero components in u. Furthermore, the relation is reversible; that is, a cubature formula of degree $2n+1$ invariant under $\mathbb{Z}_2^d$ in the form of (13.7.2) implies a cubature formula of degree n in the form of (13.7.1) on $\mathbb{T}^d$.

Proof. Assume that (13.7.1) exists. By (13.1.4), we have then

$$\int_{\mathbb{B}^d} f(u_1^2,\dots,u_d^2)W_\kappa(u)\mathrm{d}u = \sum_{i=1}^{N} \lambda_i f(u_i) = \sum_{i=1}^{N} \lambda_i \frac{1}{2^{k(u_i)}} \sum_{\varepsilon\in\mathbb{Z}_2^d} f(\varepsilon u_i)$$

for all $f \in \Pi_n^d$. By Lemma 13.1.2, this shows that (13.7.2) holds for all $f \in G\Pi_{2n}^d$, that is, it holds for all $\mathbb{Z}_2^d$ invariant polynomials in Π_{2n+1}^d. Hence, by Theorem 11.6.2, (13.7.2) holds for Π_{2n+1}. Evidently, the above proof is reversible. □

Together with the result in Sect. 11.6, we see that the cubature formulas on the simplex are closely related to the cubature formulas on the sphere. In particular, the cubature formulas for $U_0(x) = 1/\sqrt{x_1\cdots x_d(1-|x|)}$ on $\mathbb{T}^d$ correspond to cubature formulas for $W_0(x) = 1/\sqrt{1-\|x\|^2}$ on $\mathbb{B}^d$, which in turn correspond to cubature formulas for $d\sigma$ on $\mathbb{S}^d$.

13.7.2 Positive Cubature Formulas and MZ Inequality

The correspondence in Theorem 13.7.1 allows us to deduce the existence of positive cubature formulas for a maximal separated set of nodes on a simplex.

Definition 13.7.2. Let $\varepsilon > 0$. A subset Λ of $\mathbb{T}^d$ is called ε-separated if $\mathsf{d}_{\mathbb{T}}(x,y) \ge \varepsilon$ for every two distinct points $x, y \in \Lambda$. An ε-separated subset Λ of $\mathbb{T}^d$ is called maximal if $\mathbb{T}^d = \bigcup_{y\in\Lambda} \mathsf{c}_{\mathbb{T}}(y,\varepsilon)$, where

$$\mathsf{c}_{\mathbb{T}}(y,\varepsilon) := \left\{x \in \mathbb{T}^d : \mathsf{d}_{\mathbb{T}}(x,y) \le \varepsilon\right\}.$$

Theorem 13.7.3. *Given a maximal $\frac{\delta}{n}$-separated subset $\Lambda \subset \mathbb{T}^d$ with $\delta \in (0,\delta_0)$ for some small $\delta_0 > 0$, there exist positive numbers λ_y, $y \in \Lambda$ such that $\lambda_y \sim \mathrm{meas}_\kappa^{\mathbb{T}}\left(\mathsf{c}_{\mathbb{T}}(y,\frac{\delta}{n})\right)$ for all $y \in \Lambda$ and*

$$\int_{\mathbb{T}^d} f(x) U_\kappa(x) \mathrm{d}x = \sum_{y\in\Lambda} \lambda_y f(y), \quad f \in \Pi_n^d. \tag{13.7.3}$$

Proof. Given $\Lambda \subset \mathbb{T}^d$, we define $\Lambda_{\mathbb{B}} \subset \mathbb{B}^d$ by

$$\Lambda_{\mathbb{B}} = \bigcup_{\varepsilon\in\mathbb{Z}_2^d} (\Lambda \circ \psi)\varepsilon, \quad (\Lambda\circ\psi)\varepsilon := \{(\varepsilon_1\sqrt{x_1},\ldots,\varepsilon_d\sqrt{x_d}) : x \in \Lambda\}.$$

By (13.3.3), it is easy to see that $\Lambda_{\mathbb{B}}$ is a maximal $\frac{\delta}{n}$-separated subset of $\mathbb{B}^d$. Consequently, by Theorem 11.6.5, there exists a positive cubature formula on the ball in the form of

$$\int_{\mathbb{B}^d} f(x) W_\kappa(x) \mathrm{d}x = \sum_{y\in\Lambda_{\mathbb{B}}} \lambda_y f(y) = \sum_{\varepsilon\in\mathbb{Z}_2^d} \sum_{y\in(\Lambda\circ\psi)\varepsilon} \lambda_y f(y),$$

which implies the existence of the cubature formulas (13.7.3) by Theorem 11.4.3. $\square$

We can also state a Marcinkiewicz–Zygmund inequality for U_κ on $\mathbb{T}^d$. Let

$$U_\kappa\left(\mathsf{c}_{\mathbb{T}}\left(\omega,\frac{\delta}{n}\right)\right) = \int_{\mathsf{c}_{\mathbb{T}}(\omega,\frac{\delta}{n})} U_\kappa(x)\mathrm{d}x.$$

Theorem 13.7.4. *Let Λ be a $\frac{\delta}{n}$-separated subset of $\mathbb{T}^d$ and $\delta \in (0,1]$.*

(*i*) *For all $0 < p < \infty$ and $f \in \Pi_m^d$ with $m \ge n$,*

$$\sum_{y\in\Lambda}\left(\max_{x\in\mathsf{c}_{\mathbb{T}}(y,\frac{\delta}{n})}|f(x)|^p\right)U_\kappa\left(\mathsf{c}_{\mathbb{T}}\left(y,\frac{\delta}{n}\right)\right) \le c_{\kappa,p}\left(\frac{m}{n}\right)^{s_\kappa}\|f\|_{U_\kappa,p}^p, \qquad (13.7.4)$$

where $s_\kappa := d + 2|\kappa| - 2\min\kappa$ and $c_{\kappa,p}$ depends on p when p is close to 0.

(*ii*) *If, in addition, Λ is maximal and $\delta \in (0,\delta_r)$, $\delta > 0$ for some $r \in (0,1)$, then for $f \in \Pi_n^d$, $\|f\|_\infty \sim \max_{y\in\Lambda}|f(y)|$, and for $r \le p < \infty$,*

$$\|f\|_{U_\kappa,p} \sim \left(\sum_{y\in\Lambda}U_\kappa\left(\mathsf{c}_{\mathbb{T}}\left(y,\frac{\delta}{n}\right)\right)\min_{x\in\mathsf{c}_{\mathbb{T}}\left(y,\frac{\delta}{n}\right)}|f(x)|^p\right)^{1/p} \qquad (13.7.5)$$

$$\sim \left(\sum_{y\in\Lambda}U_\kappa\left(\mathsf{c}_{\mathbb{T}}\left(y,\frac{\delta}{n}\right)\right)\max_{x\in\mathsf{c}_{\mathbb{T}}\left(y,\frac{\delta}{n}\right)}|f(x)|^p\right)^{1/p}, \qquad (13.7.6)$$

where the constants of equivalence depend on r when r is close to 0.

Proof. For a given $\Lambda \subset \mathbb{T}^d$, we define $\Lambda_{\mathbb{B}}$ as in the proof of the previous theorem. We then apply the Theorem 11.6.6 to the function $f\circ\psi$ over $\Lambda_{\mathbb{B}}$ for the weight function W_κ and transform the resulting inequalities on $\mathbb{B}^d$ to $\mathbb{T}^d$ by (13.1.4), which gives the stated result. □

13.7.3 Product-Type Cubature Formulas

Product-type cubature formulas on the simplex $\mathbb{T}^d$ can be deduced inductively from the following relation:

$$\int_{\mathbb{T}^d} f(x)\mathrm{d}x = \int_0^1\int_{\mathbb{T}^{d-1}} f\left(x_1,(1-x_1)x'\right)\mathrm{d}x'(1-x_1)^{d-1}\mathrm{d}x_1,$$

where $x' = (x_2,\ldots,x_d) \in \mathbb{T}^{d-1}$. For the integral against $U_\kappa(x)\mathrm{d}x$ on $\mathbb{T}^d$, we can use the above formula and Gaussian quadrature for the Jacobi weight $u_{\alpha,\beta}(t) = t^\beta(1-t)^\alpha$ on $[0,1]$ to deduce a cubature formula of degree $2n-1$. Indeed, since

$$U_\kappa\left(x_1,(1-x_1)x'\right) = U_{\kappa_2,\ldots,\kappa_{d+1}}(x)x_1^{\kappa_1-\frac{1}{2}}(1-x_1)^{|\kappa|-\kappa_1-\frac{d}{2}},$$

if we apply Gaussian quadrature (11.6.14) of degree n to the integral against U_k, then for every $f \in \Pi_{2n-1}^d$, we obtain

$$a_\kappa \int_{\mathbb{T}^d} f(x) U_\kappa(x) \mathrm{d}x = a'_\kappa \int_{\mathbb{T}^{d-1}} \sum_{k=1}^{n} \nu_{k,n}^{(\alpha_1,\beta_1)} f\left(\left(1 - x_{k,n}^{(\alpha_1,\beta_1)}\right) x', x_{k,n}^{(\alpha_1,\beta_1)}\right)$$
$$\times U_{\kappa_2,\ldots,\kappa_{d+1}}(x') \mathrm{d}x',$$

where $\alpha_1 = |\kappa| - \kappa_1 + (d-2)/2$ and $\beta_1 = \kappa_1 - 1/2$ and a'_κ is the normalization constant of the integral over $\mathbb{T}^{d-1}$, which follows since the polynomial $x_d \mapsto f((1-x_d)x', x_d)$ has the same degree $2n-1$ of f, so that the Gaussian quadrature is exact. Evidently, applying Gaussian quadrature repeatedly, we obtain a cubature formula of degree $2n-1$ that has n^d nodes.

13.8 Notes and Further Results

The connection between orthogonal structures on the simplex, the unit ball, and the unit sphere was studied in [179], and the closed formula of the reproducing kernel in (13.1.12) was first established in [180]. These were used for studying orthogonal expansions in [189,190], where the convolution and translation operators were defined. The weight function W_κ is integrable if all κ_i are greater than $-1/2$. The reason that we assume $\kappa_i \ge 0$ can be seen from the closed formulas of the reproducing kernel (13.1.12).

The results on the maximal function and multiplier theorem in Sect. 13.5 were established in [47], and those on the boundedness of projection operators and the Cesàro means in Sect. 13.4 were proved in [48, 49]. The upper estimate of the highly localized kernel in (13.5.2) was given in [91], which could be improved to a subexponential estimate as in (2.7.1) under an additional assumption on the cutoff function, and the estimate was used to establish an upper bound of $\|L_n(U_\kappa; x, \cdot)\|_{U_\kappa,p}$, which is believed to be sharp, but no matching lower bound has been established.

The results on the weighted best approximation in Sect. 13.5 were established in [189, 190]. A natural modulus of smoothness on the simplex is an analogue of the Ditzian–Totik modulus of smoothness. Let e_i denote the usual coordinate vectors and let $e_{i,j} := e_i - e_j$ for $i \ne j$ and $e_{i,i} := e_i$. Define

$$\varphi_{i,j}(x) := \sqrt{x_i x_j}, \quad i \ne j, \quad \text{and} \quad \varphi_{i,i}(x) := \sqrt{x_i(1-|x|)}.$$

Let $\hat{\triangle}^r_{he}$ denote the central difference operator taking in the direction of the vector e. The analogue of the Ditzian–Totik modulus of smoothness then takes the form

$$\omega^r_\varphi(f,t)_p := \sup_{0<h\le t} \max_{1\le i<j\le d} \|\hat{\triangle}^r_{h\varphi_{i,j}e_{ij}} f\|_p, \quad 1 \le p \le \infty.$$

Let $\partial_{i,j} = \partial_i - \partial_j$ for $i \ne j$ and $\partial_{i,i} = \partial_i$. For $r \in \mathbb{N}$, define $\partial^r_{i,j} = \partial^{r-1}_{i,j} \partial_{i,j}$. Then the analogue of the Ditzian–Totik K-functional is defined by

$$K_\varphi^r(f,t)_p := \inf_{g\in C^r(\mathbb{T}^d)} \left\{ \|f-g\|_p + t^r \sum_{1\le i\le j\le d} \|\varphi_{i,j}^r \partial_{i,j}^r g\|_p \right\}.$$

These were defined in [16] and proved to be equivalent: $\omega_\varphi^r(f,t)_p \sim K_\varphi^r(f,t)_p$. They can be used to characterize the best approximation as the Ditzian–Totik pair in one variable.

The relation between cubature formulas on the simplex and on the ball was established in [179], which shows, together with the results in Sect. 11.6, that cubature formulas on the simplex, the ball, and the sphere are closely related. The product-type formulas, however, are essentially the only family of positive cubature formulas on the simplex that exist for all degrees. There are nonpositive cubature formulas on the simplex with relatively smaller numbers of points [80, 84].

Chapter 14
Applications

This chapter contains several topics that can be regarded as applications of what we have developed in the previous chapters. The first topic is tight frames, an active research area that has potential applications in signal processing and sampling theory, among others. In the first section we construct a family of highly localized tight polynomial frames on the unit sphere and show that the frame expansion converges unconditionally in L^p. The second topic is about the node distribution of positive cubature formulas on the sphere. The main result in the second section shows that these nodes are uniformly distributed; in particular, the points of near-minimum n-spherical design on $\mathbb{S}^{d-1}$ are uniformly distributed. The third topic is on positive, and strictly positive, definite functions on the unit sphere, the latter has applications in scattered data interpolation. In the third section, we shall prove the classical characterization of Schoenberg on the positive definite function on the sphere and more recent characterizations of strictly positive definite functions, and include several recent results on such functions. The fourth topic is about the asymptotics of minimal discrete energy points on the sphere, studied in the fourth section, which investigates the asymptotics of the minimum discrete Riesz potential of a set of N points on the sphere as $N \to \infty$. The fifth topic is computerized tomography, one of the most outstanding examples of mathematical applications, which requires an approximation method for recovering a function (or image) from its Radon projections (or x-ray data). We describe, in the fifth section, such a reconstruction algorithm, called OPED, based on orthogonal expansions on the unit disk. The algorithm is an approximation process resulting from discretizing the partial sum operator by Gaussian quadrature; it preserves polynomials of high order, and as a result, is highly accurate and stable.

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14.1 Highly Localized Tight Polynomial Frames on the Sphere

Let H be a Hilbert space with its inner product denoted by $\langle \cdot,\cdot \rangle_H$ and its norm denoted by $\|\cdot\|_H$. If $\{\phi_n\}$ is a complete orthonormal basis in H, then the Parseval identity states that

$$\sum_{n=0}^{\infty} |\langle f, \phi_n \rangle|_H^2 = \|f\|_H^2,$$

which is an example of a tight frame. For many applications, it is desirable to construct a system with this property but without the orthogonality.

Definition 14.1.1. A family of elements $\{f_i : \ i \in I\}$ in a Hilbert space H is called a frame for H if there are constants $0 < A < B < \infty$ such that for all $f \in H$,

$$A\|f\|_H^2 \le \sum_{i \in I} |\langle f, f_i \rangle_H|^2 \le B\|f\|_H^2. \tag{14.1.1}$$

A frame $\{f_i\}_{i\in I}$ is called tight if the constants A and B in Eq. (14.1.1) are equal.

The main aim in this section is to construct a tight polynomial frame on the sphere $\mathbb{S}^{d-1}$ consisting of polynomials that are highly localized. The construction requires two ingredients.

The first is a sequence of polynomials defined through the cutoff functions. To start, we choose a nonnegative function $\phi \in C^\infty(\mathbb{R})$ such that $\operatorname{supp}\phi \subset \{x \in \mathbb{R} : \frac{1}{2} \le |x| \le 2\}$, $\min_{x \in [\frac{3}{4},\frac{3}{2}]} \phi(x) \ge c_0 > 0$, and

$$\sum_{j=-\infty}^{\infty} \left[\phi\left(\frac{x}{2^j}\right)\right]^2 = 1, \quad \text{for all } x \in \mathbb{R} \setminus \{0\}. \tag{14.1.2}$$

It is evident that there exists a function $\tilde\phi$ that satisfies the first two conditions specified above and $\sum_j |\tilde\phi(2^j x)|^2 > 0$ for $x \in \mathbb{R}\setminus\{0\}$. Setting $\phi(x) = \tilde\phi(x)/\sum_j |\tilde\phi(2^j x)|^2$ then gives a ϕ that satisfies Eq. (14.1.2) as well. With ϕ chosen, we define a sequence of polynomials on $[-1,1]$ by

$$G_0(t) = 1, \;\; G_j(t) = \sum_{k=[2^{j-2}]+1}^{2^j} \phi\left(\frac{k}{2^{j-1}}\right) \frac{\lambda + k}{\lambda} C_k^\lambda(t), \quad j \ge 1, \tag{14.1.3}$$

where $\lambda = \frac{d-2}{2}$ and $t \in [-1,1]$. The support set of ϕ shows that we can write the summation over k as from 0 to ∞.

The second ingredient is a sequence of positive cubature formulas. For $j \ge 0$, let $\Lambda_j^d \subset \mathbb{N}_0^d$ be a finite index set. Let $\{x_{j,k} : k \in \Lambda_j^d\}$ be a set of distinct points on $\mathbb{S}^{d-1}$ and $\{\lambda_{j,k} : k \in \Lambda_j^d\}$ a set of positive real numbers in $\mathbb{R}$. In the case of $j = 0$, we set, for convenience, $\Lambda_0^d = \{0\}$, $\lambda_{0,0} = 1$ and take $x_{0,0}$ to be any fixed point on $\mathbb{S}^{d-1}$.

By Theorem 6.3.3, there exists a sequence of positive cubature formulas $\{Q_j\}_{j=1}^\infty$ on $\mathbb{S}^{d-1}$, with $\{x_{j,k}\}$ as its nodes and $\{\lambda_{j,k}\}$ as its weights, that satisfy the following properties:

1. For each $j \in \mathbb{N}$,

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)\,\mathrm{d}\sigma(y) = Q_j f := \sum_{k\in\Lambda_j^d} \lambda_{j,k} f(x_{j,k}), \quad \forall f \in \Pi_{2^{j+6}}^d. \tag{14.1.4}$$

2. The weights $\lambda_{j,k}$ in the cubature formulas (14.1.4) are positive and satisfy

$$\lambda_{j,k} \sim 2^{-j(d-1)}, \ \ \forall j \in \mathbb{N}, \ \ \forall k \in \Lambda_j^d,$$

with the constants of equivalence depending only on d.

With these two ingredients, we can now define elements in our polynomial frame. For $j \in \mathbb{N}_0$ and $k \in \Lambda_j^d$, define

$$\psi_{j,k}(x) := \sqrt{\lambda_{j,k}}\, G_j(x\cdot x_{j,k}), \quad x \in \mathbb{S}^{d-1}, \tag{14.1.5}$$

where in this section, we use $x\cdot y$ to denote the dot product of two vectors x, y in $\mathbb{R}^d$. For convenience, we index these functions by the spherical caps $\mathsf{c}(x_{j,k}, 2^{-j})$, $j \in \mathbb{N}$ and $k \in \Lambda_j^d$; that is, we set

$$\psi_B(x) = \psi_{j,k}(x) \quad \text{when } B = \mathsf{c}(x_{j,k}, 2^{-j}).$$

Furthermore, we shall denote the collections of the spherical caps by

$$\mathscr{B}_j := \left\{\mathsf{c}(x_{j,k}, 2^{-j}) : k \in \Lambda_j^d\right\} \quad \text{and} \quad \mathscr{B} = \bigcup_{j=0}^\infty \mathscr{B}_j.$$

For each spherical cap $B \in \mathscr{B}$, we denote the center and the radius of B by x_B and r_B, respectively. From now on, we will write $\langle\cdot,\cdot\rangle$ for the inner product $\langle\cdot,\cdot\rangle_{\mathbb{S}^{d-1}}$ of $L^2(\mathbb{S}^{d-1})$ given by Eq. (1.1.1).

The heuristic reasoning of the above construction is as follows: By the orthogonality of spherical harmonics,

$$f * G_j = \sum_{k=1}^\infty \phi\left(\frac{k}{2^{j-1}}\right) \operatorname{proj}_k f. \tag{14.1.6}$$

Furthermore, for $f \in L^2(\mathbb{S}^{d-1})$, a further computation using Eq. (1.2.5) shows that by Eq. (14.1.2),

$$f = \sum_{j=0}^\infty (f * G_j) * G_j. \tag{14.1.7}$$

Since $f * G_j$ and $G_j(x \cdot x_{j,k})$ are both polynomials of degree 2^j, we can apply the cubature formula to discretize the integral of $(f * G_j) * G_j$, which is easily seen to be exactly

$$f(x) = \sum_{B \in \mathscr{B}} \langle f, \psi_B \rangle \psi_B(x). \tag{14.1.8}$$

The convergence of the expansion (14.1.8) holds in fact in $L^p(\mathbb{S}^{d-1})$ for $1 < p < \infty$, as will be shown in Theorem 14.1.4.

The main properties of the functions ψ_B defined above are collected in the following theorem. It shows, in particular, that $\{\psi_B\}_{B \in \mathscr{B}}$ forms a highly localized tight polynomial frame for $L^2(\mathbb{S}^{d-1})$. For simplicity, we use throughout this section the notation $|E|$ to denote the Lebesgue measure $\sigma(E)$ of a subset E of $\mathbb{S}^{d-1}$.

Theorem 14.1.2. *The following statements hold:*

(i) For each $B \in \mathscr{B}$, the function ψ_B is highly localized in the spherical cap B in the sense that

$$|\psi_B(x)| \le c_{d,\ell} |B|^{-\frac{1}{2}} \left(1 + \frac{\mathtt{d}(x, x_B)}{r_B}\right)^{-\ell}, \quad \forall \ell > 0, \quad \forall x \in \mathbb{S}^{d-1}. \tag{14.1.9}$$

(ii) For each $B \in \mathscr{B}$, and $r > 0$,

$$c\,\chi_{B/2}(x) \le |B|^{\frac{1}{2}} |\psi_B(x)| \le c_r \left(M(\chi_B)(x)\right)^{\frac{1}{r}}, \quad x \in \mathbb{S}^{d-1}, \tag{14.1.10}$$

where M denotes the Hardy–Littlewood maximal function on $\mathbb{S}^{d-1}$ defined in Sect. 2.3. In particular, this implies that

$$\|\psi_B\|_p \sim |B|^{\frac{1}{p} - \frac{1}{2}}, \quad 0 < p \le \infty. \tag{14.1.11}$$

(iii) For each $f \in L^2(\mathbb{S}^{d-1})$,

$$\|f\|_2^2 = \sum_{B \in \mathscr{B}} |\langle f, \psi_B \rangle|^2; \tag{14.1.12}$$

that is, the sequence $\{\psi_B\}_{B \in \mathscr{B}}$ forms a tight frame for $L^2(\mathbb{S}^{d-1})$.

Proof. The estimate (14.1.9) follows directly from Theorem 2.6.5. To prove Eq. (14.1.10), we assume that $B = \mathtt{c}(x_{j,k}, 2^{-j})$ for some $j \in \mathbb{N}_0$ and $k \in \Lambda_j^d$. When $j = 0$, Eq. (14.1.10) holds trivially. Hence, without loss of generality, we may assume $j \in \mathbb{N}$ and write

$$\psi_B(x) = \sqrt{\lambda_{j,k}} G_j(x \cdot x_{j,k}) = \sqrt{\lambda_{j,k}} \sum_{\nu=0}^{2^j} \phi\left(\frac{\nu}{2^{j-1}}\right) \frac{\nu + \lambda}{\lambda} C_\nu^\lambda(x \cdot x_{j,k}).$$

Since $|C_\nu^\lambda(\cos t)| \le C_\nu^\lambda(1) \sim (\nu + 1)^{d-3}$, it follows that

$$\max_{t \in [0,\pi]} |G_j(\cos t)| = G_j(1) \sim 2^{j(d-1)}.$$

Hence, using Bernstein's inequality for trigonometric polynomials, we obtain

$$G_j(1) - G_j(\cos\theta) = \int_0^\theta \frac{\mathrm{d}}{\mathrm{d}t}\Big(G_j(\cos t)\Big)\,\mathrm{d}t \le 2^j \theta G_j(1),$$

which implies that for $\theta \in [0, 2^{-j-1}]$,

$$G_j(\cos\theta) \ge \frac{1}{2} G_j(1) \ge c 2^{j(d-1)}.$$

Thus, for $x \in \frac{1}{2}B = \mathtt{c}(x_{j,k}, 2^{-j-1})$,

$$\psi_B(x) = \sqrt{\lambda_{j,k}} G_j(x \cdot x_{j,k}) \ge c 2^{j(d-1)/2} \ge c|B|^{-\frac{1}{2}} \chi_{B/2}(x),$$

which proves the first inequality in Eq. (14.1.10).

To prove the second inequality in Eq. (14.1.10), we use Eq. (14.1.9) to obtain

$$|\psi_B(x)| \le c_{d,r} 2^{j(d-1)/2} \min\left\{1, (2^j \mathtt{d}(x, x_{j,k}))^{-\frac{d-1}{r}}\right\}.$$

If $0 \le \mathtt{d}(x, x_{j,k}) \le 2^{-(j+1)}$, then $\frac{1}{2}B = \mathtt{c}(x_{j,k}, 2^{-j-1}) \subset \mathtt{c}(x, 2^{-j})$, and hence

$$\begin{aligned} |\psi_B(x)| \le c_d 2^{j(d-1)/2} &\le c_{d,r} |B|^{-\frac{1}{2}} \left(\frac{1}{|\mathtt{c}(x, 2^{-j})|} \int_{\mathtt{c}(x, 2^{-j})} \chi_{B/2}(y)\,\mathrm{d}\sigma(y) \right)^{\frac{1}{r}} \\ &\le c_{d,r} |B|^{-\frac{1}{2}} (M(\chi_B)(x))^{\frac{1}{r}}. \end{aligned}$$

On the other hand, if $\theta := \mathtt{d}(x, x_{j,k}) > 2^{-j-1}$, then $\frac{1}{2}B \subset \mathtt{c}(x, 2\theta)$, and hence

$$\begin{aligned} |\psi_B(x)| &\le c_{d,r} 2^{j(d-1)/2} (2^j \theta)^{-(d-1)/r} \\ &\le c_{d,r} |B|^{-\frac{1}{2}} \left(\frac{1}{|\mathtt{c}(x, 2\theta)|} \int_{\mathtt{c}(x, 2\theta)} \chi_{B/2}(y)\,\mathrm{d}\sigma(y) \right)^{\frac{1}{r}} \\ &\le c_{d,r} |B|^{-\frac{1}{2}} (M(\chi_B)(x))^{\frac{1}{r}}. \end{aligned}$$

Therefore, in either case, we have the desired upper estimate in Eq. (14.1.10). Finally, Eq. (14.1.11) follows directly from Eq. (14.1.10) with $0 < r < p$.

We now prove (iii). By definition, for each $j \in \mathbb{N}$ and $k \in \Lambda_j^d$,

$$\langle f, \psi_{j,k} \rangle = \sqrt{\lambda_{j,k}} (f * G_j)(x_{j,k}).$$

Since $|f * G_j|^2 \in \Pi_{2^{j+1}}^d$, using the cubature formula (14.1.4), it follows that

$$\sum_{k \in \Lambda_j^d} |\langle f, \psi_{j,k} \rangle|^2 = \sum_{k \in \Lambda_j^d} \lambda_{j,k} |(f * G_j)(x_{j,k})|^2 = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} |(f * G_j)(y)|^2\,\mathrm{d}\sigma(y).$$

From Eq. (14.1.6), it follows that

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} |(f*G_j)(y)|^2\,\mathrm{d}\sigma(y) = \sum_{k=1}^{\infty}\left|\phi\left(\frac{k}{2^{j-1}}\right)\right|^2 \|\operatorname{proj}_k f\|_2^2.$$

Consequently,

$$\begin{aligned}\sum_{B\in\mathscr{B}} |\langle f,\psi_B\rangle|^2 &= |\langle f,\psi_{0,0}\rangle|^2 + \sum_{j=1}^{\infty}\sum_{k\in\Lambda_j^d} |\langle f,\psi_{j,k}\rangle|^2\\ &= \|\operatorname{proj}_0 f\|_2^2 + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left|\phi\left(\frac{k}{2^{j-1}}\right)\right|^2 \|\operatorname{proj}_k f\|_2^2\\ &= \sum_{k=0}^{\infty}\|\operatorname{proj}_k f\|_2^2 = \|f\|_2^2.\end{aligned}$$

This proves the desired identity (14.1.12). □

The infinite series on the right-hand side of the expression (14.1.8) is well defined for every $f \in L^2(\mathbb{S}^{d-1})$. Our next result concerns the unconditional convergence of this series in L^p-norm with $1 < p < \infty$.

Definition 14.1.3. A series $\sum_{n=1}^{\infty} x_n$ in a Banach space B is unconditionally convergent if for every rearrangement $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges.

Theorem 14.1.4. *Let $1 < p < \infty$. The following assertions hold:*

(i) *If $f \in L^p(\mathbb{S}^{d-1})$, then the series $\sum_{B\in\mathscr{B}} \langle f,\psi_B\rangle\psi_B$ converges unconditionally to f in the L^p norm; moreover,*

$$\|f\|_p \sim \left\|\left(\sum_{B\in\mathscr{B}} |\langle f,\psi_B\rangle|^2|\psi_B|^2\right)^{\frac{1}{2}}\right\|_p \sim \left\|\left(\sum_{B\in\mathscr{B}} |\langle f,\psi_B\rangle|^2|B|^{-1}\chi_B\right)^{\frac{1}{2}}\right\|_p, \tag{14.1.13}$$

with the constants of equivalence depending only on d, p, and ϕ.

(ii) *Conversely, if $\{a_B\}_{B\in\mathscr{B}}$ is a sequence of real numbers such that either*

$$\left(\sum_{B\in\mathscr{B}} |a_B|^2|\psi_B|^2\right)^{\frac{1}{2}} \quad or \quad \left(\sum_{B\in\mathscr{B}} |a_B|^2|B|^{-1}\chi_B\right)^{\frac{1}{2}}$$

is in $L^p(\mathbb{S}^{d-1})$, then $\sum_{B\in\mathscr{B}} a_B\psi_B$ converges unconditionally in the L^p-norm to some function $f \in L^p(\mathbb{S}^{d-1})$; moreover,

$$\|f\|_p \le c_1\left\|\left(\sum_{B\in\mathscr{B}} |a_B|^2|\psi_B|^2\right)^{\frac{1}{2}}\right\|_p \le c_2\left\|\left(\sum_{B\in\mathscr{B}} |a_B|^2|B|^{-1}\chi_B\right)^{\frac{1}{2}}\right\|_p, \tag{14.1.14}$$

with the constants c_1 and c_2 depending only on d, p, and ϕ.

Proof. We prove (ii) first. We begin by showing that for every sequence $\{a_B\}_{B\in\mathscr{B}}$ of real numbers, the second inequality of Eq. (14.1.14) holds. Indeed, by the second inequality in Eq. (14.1.10) with $r=1$, $|\psi_B| \le c_d|B|^{-\frac{1}{2}}M(\chi_B)$ for $B\in\mathscr{B}$, which, together with the Fefferman–Stein inequality in Theorem 3.1.4, implies the second inequality of Eq. (14.1.14). Second, we show that for every sequence $\{a_B\}_{B\in\mathscr{B}}$ of real numbers and every finite subset $\mathscr{F}\subset\mathscr{B}$,

$$\left\|\sum_{B\in\mathscr{F}} a_B\psi_B\right\|_p \le c_{p,d}\left\|\left(\sum_{B\in\mathscr{F}}|a_B|^2|\psi_B|^2\right)^{\frac{1}{2}}\right\|_p. \tag{14.1.15}$$

Once Eq. (14.1.15) is proved, it follows that partial sums of any rearrangement of the series $\sum_B a_B\psi_B$ form a Cauchy sequence in $L^p(\mathbb{S}^{d-1})$, which therefore converges, so that the series $\sum_{B\in\mathscr{B}} a_B\psi_B$ converges unconditionally in the space $L^p(\mathbb{S}^{d-1})$. This, together with Eq. (14.1.14), will complete the proof of (ii).

For the proof of Eq. (14.1.15), we define, for a generic $g\in L^1(\mathbb{S}^{d-1})$ and $j=0,1,2,\ldots$,

$$\Delta_j g(x) := g*G_j(x) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} g(y)G_j(x\cdot y)\,\mathrm{d}\sigma(y), \quad x\in\mathbb{S}^{d-1}.$$

If $g\in L^p(\mathbb{S}^{d-1})$, then by Theorem 3.4.2, we have

$$\left\|\left(\sum_{j=0}^{\infty}|\Delta_j(g)|^2\right)^{\frac{1}{2}}\right\|_p \le c_p\|g\|_p, \quad 1<p<\infty.$$

We further define, as in the Definition 5.2.1,

$$\Delta_j^*(g)(x) = \sup_{y\in\mathbb{S}^{d-1}}\frac{|\Delta_j(g)(y)|}{(1+2^j\mathsf{d}(x,y))^{d-1}}, \quad x\in\mathbb{S}^{d-1}.$$

Using Theorem 5.2.2 with $s_w=d-1=\beta$ and $n=2^j$, we see that it is bounded by the maximal function of $\Delta_j g$:

$$\Delta_j^*(g)(x) \le cM(\Delta_j g)(x),\ \ x\in\mathbb{S}^{d-1}. \tag{14.1.16}$$

For convenience, we write $\Delta_B g=\Delta_j g$ for $B\in\mathscr{B}_j$. Set $h=\sum_{B\in\mathscr{F}} a_B\psi_B$. Let $g\in L^q(\mathbb{S}^{d-1})$ be such that $\|g\|_q=1$ and $\|h\|_p=\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} h(x)g(x)\,\mathrm{d}\sigma(x)$, where $q=\frac{p}{p-1}$. We observe that for $B=\mathsf{c}(x_{j,k},2^{-j})\in\mathscr{B}$,

$$\begin{aligned}\left|\langle g,\psi_B\rangle\right| &= \sqrt{\lambda_{j,k}}|\Delta_j(g)(x_{j,k})| \le c_d|B|^{-\frac{1}{2}}\int_{B/2}\Delta_j^*(g)(x)\,\mathrm{d}\sigma(x)\\ &\le c_d|B|^{-\frac{1}{2}}\int_{B/2} M(\Delta_B g)(x)\mathrm{d}\sigma(x).\end{aligned}$$

Therefore, by the definition of h and the choice of g,

$$\|h\|_p = \sum_{B\in\mathscr{F}} a_B\langle\psi_B, g\rangle \le c_d \sum_{B\in\mathscr{F}} |a_B||B|^{-\frac{1}{2}} \int_{B/2} M(\Delta_B g)(x)\mathrm{d}\sigma(x)$$
$$\le c_d \left\| \left(\sum_{B\in\mathscr{F}} |a_B|^2|B|^{-1}\chi_{B/2} \right)^{\frac{1}{2}} \right\|_p \left\| \left(\sum_{B\in\mathscr{B}} |M(\Delta_B g)|^2 \chi_{B/2} \right)^{\frac{1}{2}} \right\|_q.$$

Invoking the Fefferman–Stein inequality, we deduce

$$\left\| \left(\sum_{B\in\mathscr{B}} |M(\Delta_B g)|^2 \chi_{B/2} \right)^{\frac{1}{2}} \right\|_q \le c_p \left\| \left(\sum_{j=0}^{\infty} |\Delta_j(g)|^2 \right)^{\frac{1}{2}} \right\|_q \le c_p,$$

while using the first inequality in Eq. (14.1.10), we have

$$\left(\sum_{B\in\mathscr{F}} |a_B|^2|B|^{-1}\chi_{B/2} \right)^{\frac{1}{2}} \le c_d \left(\sum_{B\in\mathscr{F}} |a_B|^2|\psi_B|^2 \right)^{\frac{1}{2}}.$$

The desired inequality (14.1.15) then follows.

Next, we prove (i). To this end, it suffices to prove that for $f \in L^p$,

$$\left\| \left(\sum_{B\in\mathscr{B}} |\langle f, \psi_B\rangle|^2|B|^{-1}\chi_B \right)^{\frac{1}{2}} \right\|_p \le c_{p,d}\|f\|_p. \tag{14.1.17}$$

In fact, once Eq. (14.1.17) is proved, then by (ii) that we just proved, it follows that for $f \in L^p$, the series $\sum_{B\in\mathscr{B}} \langle f, \psi_B\rangle\psi_B$ is convergent unconditionally in L^p, and by Eq. (14.1.8) and a density argument we must have $f = \sum_{B\in\mathscr{B}} \langle f, \psi_B\rangle\psi_B$. This together with Eqs. (14.1.14), (14.1.15), and (14.1.17) will imply the equivalences

$$\|f\|_p \sim \left\| \left(\sum_{B\in\mathscr{B}} |\langle f, \psi_B\rangle|^2|\psi_B|^2 \right)^{\frac{1}{2}} \right\|_p \sim \left\| \left(\sum_{B\in\mathscr{B}} |\langle f, \psi_B\rangle|^2|B|^{-1}\chi_B \right)^{\frac{1}{2}} \right\|_p,$$

and hence (i). For the proof of Eq. (14.1.17), we first observe that for $B \in \mathscr{B}_j$,

$$|\langle f, \psi_B\rangle| \sim |B|^{\frac{1}{2}}|\Delta_j(f)(x_B)| \le C|B|^{\frac{1}{2}} \inf_{y\in B} \Delta_j^* f(y)$$
$$\le c|B|^{\frac{1}{2}} \inf_{y\in B} M(\Delta_j f)(y),$$

which implies, using the unweighed version of Lemma 5.4.3,

$$\sum_{j=0}^{\infty}\sum_{B\in\mathscr{B}_j}|\langle f,\psi_B\rangle|^2|B|^{-1}\chi_B(x)\le c\sum_{j=0}^{\infty}|M(\Delta_j f)(x)|^2\Big(\sum_{B\in\mathscr{B}_j}\chi_B(x)\Big)$$
$$\le c\sum_{j=0}^{\infty}|M(\Delta_j f)(x)|^2.$$

The desired inequality (14.1.17) then follows from the Fefferman–Stein inequality of Theorem 3.1.4. The proof is complete. □

14.2 Node Distribution of Positive Cubature Formulas on the Sphere

The main result in this section states that the nodes of a positive cubature formula on the sphere must be uniformly distributed for boundary regular domains, a concept that we now define.

Definition 14.2.1. Let D be a closed subset of $\mathbb{S}^{d-1}$ with a nonempty interior and contained in a spherical cap of arc radius less than $3\pi/4$. We call D a boundary regular domain if its boundary, ∂D, satisfies

$$e_\varepsilon(\partial D)\le c\,\varepsilon^{-(d-2)},\quad \forall\varepsilon\in(0,1), \tag{14.2.1}$$

where for a compact subset E of $\mathbb{S}^{d-1}$ and $\varepsilon>0$, $e_\varepsilon(E)$ denotes the minimum number of points $y_1,\dots,y_N$ on $\mathbb{S}^{d-1}$ for which $E\subset\bigcup_{j=1}^N \mathsf{c}(y_j,\varepsilon)$; that is,

$$e_\varepsilon(E):=\min\Big\{N\in\mathbb{N}:\exists y_1,\dots,y_N\in\mathbb{S}^{d-1}\text{ such that }E\subset\bigcup_{j=1}^N\mathsf{c}(y_j,\varepsilon)\Big\}.$$

Let Λ be a finite subset of $\mathbb{S}^{d-1}$ and assume that the positive cubature formula

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}}f(x)\,\mathrm{d}\sigma(x)=\sum_{\eta\in\Lambda}\lambda_\eta f(x_\eta),\quad \text{all }\lambda_\eta>0, \tag{14.2.2}$$

is of degree n, that is, it is exact for $f\in\Pi_n(\mathbb{S}^{d-1})$, for some $n\in\mathbb{N}$.

Theorem 14.2.2. *Let D be a boundary regular domain and let Λ and λ_η be the set of nodes and weights of the positive cubature formula (14.2.2) of degree n. Then*

$$\Big|\sum_{\omega\in D\cap\Lambda}\lambda_\omega-\frac{1}{\omega_d}\sigma(D)\Big|\le cn^{-\frac{1}{2}},\quad n\to\infty, \tag{14.2.3}$$

where the constant c depends only on d.

Recall that a subset $X = \{z_j\}_{j=1}^N$ of $\mathbb{S}^{d-1}$ is called a spherical n-design if the cubature formula

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)\,\mathrm{d}\sigma(y) = \frac{1}{N}\sum_{k=1}^N f(z_k)$$

is exact for every $f \in \Pi_n(\mathbb{S}^{d-1})$. By a recent theorem of [19], stated as Theorem 6.5.1, for every positive integer n, there exists a spherical n-design on $\mathbb{S}^{d-1}$ with number of nodes of asymptotic order n^{d-1} for n sufficiently large. As a consequence of Theorem 14.2.2, we have the following corollary.

Corollary 14.2.3. *Let D be a boundary regular domain of $\mathbb{S}^{d-1}$. Then the set of nodes $X = \{z_j\}_{j=1}^N$ of a spherical n-design with $N \sim n^{d-1}$ satisfies*

$$\left|\frac{1}{N}\#(X\cap D) - \frac{1}{\omega_d}\sigma(D)\right| \le cN^{-\frac{1}{2(d-1)}}, \qquad N \to \infty.$$

The proof of Theorem 14.2.2 is based on one-sided approximation. We need to construct two Lipschitz functions on $\mathbb{S}^{d-1}$ that approximate the characteristic function χ_D of the domain D from above and below, respectively. To be more precise, let us define

$$\mathtt{d}(x,D) := \inf_{y\in D} \mathtt{d}(x,y).$$

For $\alpha \in (0,\pi/2)$, we define

$$D_\alpha^+ := \{x \in \mathbb{S}^{d-1} : \mathtt{d}(x,D) \le \alpha\} \quad \text{and} \quad D_\alpha^- := \{x \in \mathbb{S}^{d-1} : \mathtt{d}(x,D) > \alpha\}.$$

Let $|E|$ denote the Lebesgue measure of a subset E of $\mathbb{S}^{d-1}$. We first consider the approximation to χ_D from above. Define

$$F_\alpha^+(x) := \frac{1}{|\mathtt{c}(x,\alpha)|}\int_{\mathbb{S}^{d-1}} \chi_{D_\alpha^+}(y)\chi_{[\cos\alpha,1]}(\langle x,y\rangle)\,\mathrm{d}\sigma(y), \;\; x \in \mathbb{S}^{d-1}.$$

Lemma 14.2.4. *The function F_α^+ satisfies the following properties:*

(i) $F_\alpha^+(x) = 1$ *for* $x \in D$*, and* $F_\alpha^+(x) = 0$ *whenever* $x \in \mathbb{S}^{d-1}$ *and* $\mathtt{d}(x,D) > 2\alpha$.
(ii) $\chi_D(x) \le F_\alpha^+(x) \le 1$ *for* $x \in \mathbb{S}^{d-1}$.
(iii) $\int_{\mathbb{S}^{d-1}} |F_\alpha^+(x) - \chi_D(x)|\,\mathrm{d}\sigma(x) \le c\alpha$.
(iv) F_α^+ *is a Lipchitz function on* $\mathbb{S}^{d-1}$,

$$|F_\alpha^+(x) - F_\alpha^+(y)| \le c\alpha^{-1}\mathtt{d}(x,y), \;\; x,y \in \mathbb{S}^{d-1}. \tag{14.2.4}$$

Proof. Directly from its definition, we can write

$$F_\alpha^+(x) = \frac{1}{|\mathtt{c}(x,a)|}\sigma\Big\{y \in \mathtt{c}(x,\alpha) : \mathtt{d}(y,D) \le \alpha\Big\} = \frac{1}{|\mathtt{c}(x,a)|}\sigma(\mathtt{c}(x,\alpha)\cap D_\alpha^+). \tag{14.2.5}$$

If $x \in D$, then $\mathtt{d}(y,D) \le \mathtt{d}(y,x) \le \alpha$ whenever $y \in \mathtt{c}(x,\alpha)$, which implies that $\mathtt{c}(x,\alpha) \subset D_\alpha^+$ and hence that $F_\alpha^+(x) = 1$ for $x \in D$. On the other hand, if $x \in \mathbb{S}^{d-1}$ and $\mathtt{d}(x,D) > 2\alpha$, then the triangle inequality of $\mathtt{d}(\cdot,\cdot)$ shows that $\inf_{y\in\mathtt{c}(x,\alpha)} \mathtt{d}(y,D) > \alpha$, which implies $\mathtt{c}(x,\alpha) \cap D_\alpha^+ = \emptyset$ and hence $F_\alpha^+(x) = 0$ by Eq. (14.2.5). This proves (i). Assertion (ii) follows directly from (i) and Eq. (14.2.5).

To prove (iii), we deduce from the already proven (i) and (ii) that

$$\int_{\mathbb{S}^{d-1}} |F_\alpha^+(x) - \chi_D(x)| \,\mathrm{d}\sigma(x) = \int_{\{x\in\mathbb{S}^{d-1}:0<\mathtt{d}(x,D)\le 2\alpha\}} F_\alpha^+(x)\,\mathrm{d}\sigma(x)$$
$$\le \sigma\left\{x \in \mathbb{S}^{d-1} : 0 < \mathtt{d}(x,\partial D) \le 2\alpha\right\}.$$

However, using the condition (14.2.1) and the definition of $e_\varepsilon(\partial D)$, we see that there exist finitely many points $y_1,\ldots,y_N \in \mathbb{S}^{d-1}$ such that $N \le c\alpha^{-d+2}$ and $\partial D \subset \bigcup_{j=1}^N \mathtt{c}(y_j,\alpha)$. Consequently,

$$\sigma\left\{x \in \mathbb{S}^{d-1} : 0 < \mathtt{d}(x,\partial D) \le 2\alpha\right\} \le \sigma\left(\bigcup_{j=1}^N \mathtt{c}(y_j,3\alpha)\right) \le cN\alpha^{d-1} \le c\alpha,$$

which proves assertion (iii).

Finally, we prove (iv). By (ii), we may assume, without loss of generality, that $\mathtt{d}(x,y) \le \frac{1}{2}\alpha$. We then apply Eq. (14.2.5) to obtain

$$F_\alpha^+(x) - F_\alpha^+(y) = \frac{1}{|\mathtt{c}(x,\alpha)|}\left[\sigma(\mathtt{c}(x,\alpha)\cap D_\alpha^+) - \sigma(\mathtt{c}(y,\alpha)\cap D_\alpha^+)\right]$$
$$\le c\alpha^{-d+1}\sigma\left(\mathtt{c}(x,\alpha)\setminus \mathtt{c}(y,\alpha)\right).$$

However, since for every $z \in \mathtt{c}(x,\alpha)\setminus\mathtt{c}(y,\alpha)$, $\alpha \ge \mathtt{d}(z,x) \ge \mathtt{d}(z,y) - \mathtt{d}(y,x) \ge \alpha - \mathtt{d}(y,x) > 0$, it follows that

$$\mathtt{c}(x,\alpha)\setminus\mathtt{c}(y,\alpha) \subset \left\{z \in \mathbb{S}^{d-1} : \alpha - \mathtt{d}(x,y) \le \mathtt{d}(z,x) \le \alpha\right\}.$$

Since $0 < \alpha < \pi/2$, we conclude that

$$F_\alpha^+(x) - F_\alpha^+(y) \le c\alpha^{-d+1}\sigma\left\{z \in \mathbb{S}^{d-1} : \alpha - \mathtt{d}(x,y) \le \mathtt{d}(z,x) \le \alpha\right\}$$
$$\le c\alpha^{-d+1}\alpha^{d-2}\mathtt{d}(x,y) \le c\alpha^{-1}\mathtt{d}(x,y).$$

The Lipchitz condition (14.2.4) then follows by symmetry. □

The function that approximates χ_D from below is defined similarly by

$$F_\alpha^-(x) := \frac{1}{|\mathtt{c}(x,\alpha)|}\int_{\mathbb{S}^{d-1}} \chi_{D_\alpha^-}(y)\chi_{[\cos\alpha,1]}(\langle x,y\rangle)\,\mathrm{d}\sigma(y), \quad x \in \mathbb{S}^{d-1}.$$

Lemma 14.2.5. *The function F_α^- satisfies the following properties:*

(i) $F_\alpha^-(x) = 1$ *if* $x \in D$ *and* $\mathtt{d}(x, \partial D) > 2\alpha$; *and* $F_\alpha^-(x) = 0$ *if* $x \in \mathbb{S}^{d-1} \setminus D$.
(ii) $0 \le F_\alpha^-(x) \le \chi_D(x)$ *for all* $x \in \mathbb{S}^{d-1}$.
(iii) $\int_{\mathbb{S}^{d-1}} |\chi_D(x) - F_\alpha^-(x)| \, \mathrm{d}\sigma(x) \le c\alpha$.
(iv) F_α^- *is a Lipchitz function on* $\mathbb{S}^{d-1}$,

$$|F_\alpha^-(x) - F_\alpha^-(y)| \le c\alpha^{-1}\mathtt{d}(x,y), \quad \forall x, y \in \mathbb{S}^{d-1}. \tag{14.2.6}$$

Proof. The proof follows along the same lines as that of Lemma 14.2.4, so we shall be brief. Directly by its definition,

$$F_\alpha^-(x) = \frac{1}{|\mathtt{c}(x,\alpha)|}\sigma(D_\alpha^- \cap \mathtt{c}(x,\alpha)), \; x \in \mathbb{S}^{d-1}, \tag{14.2.7}$$

from which (i) and (ii) follow. Next, using (i), we have

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} |F_\alpha^-(x) - \chi_D(x)| \, \mathrm{d}\sigma(x) &= \int_{\{x \in D : \mathtt{d}(x, \partial(D)) \le 2\alpha\}} [1 - F_\alpha^+(x)] \, \mathrm{d}\sigma(x) \\ &\le c\sigma\{x \in \mathbb{S}^{d-1} : \mathtt{d}(x, \partial D) \le 2\alpha\} \le c\alpha,\end{aligned}$$

which proves (iii). Finally, for the proof of (iv), without loss of generality, we may assume that $\mathtt{d}(x,y) \le \frac{1}{2}\alpha$. We then obtain from Eq. (14.2.7) that

$$\begin{aligned}F_\alpha^-(x) - F_\alpha^-(y) &= \frac{1}{|\mathtt{c}(x,\alpha)|}\left[\sigma(D_\alpha^- \cap \mathtt{c}(x,\alpha)) - \sigma(D_\alpha^- \cap \mathtt{c}(y,\alpha)\right] \\ &\le c\alpha^{-d+1}\sigma(\mathtt{c}(x,\alpha) \setminus \mathtt{c}(y,\alpha)) \le c\alpha^{-1}\mathtt{d}(x,y).\end{aligned}$$

This completes the proof of Lemma 14.2.5. □

We are now in a position to prove Theorem 14.2.2.

Proof of Theorem 14.2.2. Firstly, by the Jackson inequality in Lemma 4.2.6 and the Lipchitz conditions (14.2.4) and (14.2.6), there exist two spherical polynomials g_n^+ and g_n^- of degree at most n on $\mathbb{S}^{d-1}$ such that

$$\|F_\alpha^+ - g_n^+\|_\infty \le c\alpha^{-1}n^{-1} \quad \text{and} \quad \|F_\alpha^- - g_n^-\|_\infty \le c\alpha^{-1}n^{-1}. \tag{14.2.8}$$

Using the second assertions of Lemmas 14.2.4 and 14.2.5, this implies that

$$g_n^-(x) - c\alpha^{-1}n^{-1} \le \chi_D(x) \le g_n^+(x) + c\alpha^{-1}n^{-1}, \quad \forall x \in D. \tag{14.2.9}$$

Now, by Eq. (14.2.8) and the third assertions in Lemmas 14.2.4 and 14.2.5, we have

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}}\left|g_n^{\pm}(x)-\chi_D(x)\right|\mathrm{d}\sigma(x) \le \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}}\left|g_n^{\pm}(x)-F_\alpha^{\pm}(x)\right|\mathrm{d}\sigma(x)+c\alpha$$
$$\le c\alpha^{-1}n^{-1}+c\alpha,$$

which, in particular, implies that

$$\left|\frac{1}{\omega_d}\sigma(D)-\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}}g_n^{\pm}(x)\,\mathrm{d}\sigma(x)\right| \le c\alpha^{-1}n^{-1}+c\alpha. \tag{14.2.10}$$

Taking the weighted sum of Eq. (14.2.9) over the set Λ with weights λ_η, we obtain, since $\sum_{\eta\in\Lambda}\lambda_\eta = 1$,

$$\sum_{\eta\in\Lambda}\lambda_\eta g_n^-(\eta)-c\alpha^{-1}n^{-1} \le \sum_{\eta\in\Lambda\cap D}\lambda_\eta \le \sum_{\eta\in\Lambda}\lambda_\eta g_n^+(\eta)+c\alpha^{-1}n^{-1}.$$

Applying the positive cubature formula (14.2.2), this implies that

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}}g_n^-(x)\,\mathrm{d}\sigma(x)-c\alpha^{-1}n^{-1} \le \sum_{\eta\in\Lambda\cap D}\lambda_\eta \le \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}}g_n^+(x)\mathrm{d}\sigma(x)+c\alpha^{-1}n^{-1}.$$

Together with Eq. (14.2.10), the above inequality shows that

$$\left|\sum_{\eta\in\Lambda\cap D}\lambda_\eta-\frac{1}{\omega_d}\sigma(D)\right| \le c\alpha+c\alpha^{-1}n^{-1}.$$

Setting $\alpha = n^{-\frac{1}{2}}$ proves the desired estimate (14.2.3). □

14.3 Positive Definite Functions on the Sphere

Let $f:[-1,1]\to\mathbb{R}$ be a continuous function. Associated with $N\in\mathbb{N}$ and a set of points $X_N=\{x_1,\dots,x_N\}$ in $\mathbb{S}^{d-1}$, we denote by $f[X_N]$ the $N\times N$ matrix

$$f[X_N] := \left[f(\cos\mathtt{d}(x_i,x_j))\right]_{i,j=1}^N = \left[f(\langle x_i,x_j\rangle)\right]_{i,j=1}^N.$$

Definition 14.3.1. A continuous function $f:[-1,1]\to\mathbb{R}$ is called positive definite if for every $N\in\mathbb{N}$ and set of N distinct points $X_N=\{x_1,\dots,x_N\}$ in $\mathbb{S}^{d-1}$, the $N\times N$ matrix $f[X_N]$ is Nonnegative definite, and it is said to be strictly positive definite if $f[X_N]$ is positive definite. We denote by Φ_d the space of all positive definite functions on $[-1,1]$ and by $\mathrm{S}\Phi_d$ the space of all strictly positive definite functions on $[-1,1]$.

By the definitions of nonnegative definite and positive definite matrices, f is positive definite if

$$c^T f[X_N]c = \sum_{i=1}^{N}\sum_{j=1}^{N} c_i c_j f\big(\langle x_i, x_j\rangle\big) \ge 0$$

for all $c = (c_1,\ldots,c_N) \in \mathbb{R}^N$ and $N \in \mathbb{N}$, and f is strictly positive definite if $c^T f[X_N]c > 0$ whenever $c \neq 0$.

From the properties of positive definite matrices, we can deduce the following properties of Φ_d:

1. If $f_i \in \Phi_d$ and $c_i \ge 0$ for $i = 1,2,\ldots,n$, then $f = c_1 f_1 + \cdots + c_n f_n \in \Phi_d$.
2. If $f, g \in \Phi_d$ then $fg \in \Phi_d$.

The first statement follows directly from the definition. The second follows from a theorem of Schur, since the matrix $(fg)[X_N]$ is the Hadamard product of the matrices $f[X_N]$ and $g[X_N]$, which preserves nonnegative (positive) definiteness by Schur's theorem. These simple properties also hold for the class $\mathrm{S}\Phi_d$.

Positive definite functions are closely related to spherical harmonics, as can be seen from the fact that the Gegenbauer polynomial $C_n^{(d-2)/2}(t)$ belongs to Φ_d.

Lemma 14.3.2. *For $n \in \mathbb{N}_0$, the polynomial $C_n^\lambda(t)$, $\lambda = \frac{d-2}{2}$, is an element of Φ_d.*

Proof. Let $\{Y_k^n : 0 \le k \le a_n^d\}$, where $a_n^d = \dim \mathscr{H}_n^d$, be an orthonormal basis of the space $\mathscr{H}_n^d$ of spherical harmonics of degree n. By Eq. (1.2.8),

$$C_n^\lambda(\langle x, y\rangle) = \frac{\lambda}{n+\lambda}\sum_{k=1}^{a_n^d} Y_k^n(x) Y_k^n(y). \tag{14.3.1}$$

As a result, the matrix $C_n^\lambda[X_N]$ can be decomposed as a sum of rank-one matrices $\big[Y_k^n(x_i)Y_k^n(x_j)\big]_{i,j=1}^N$. Hence, for every $c \in \mathbb{R}^N$,

$$c^T C_n^\lambda[X_N]c = \frac{\lambda}{n+\lambda}\sum_{k=1}^{a_n^d}\left[\sum_{i=1}^{N} c_i Y_k^n(x_i)\right]^2,$$

which is evidently nonnegative. □

If f is continuous on $[-1,1]$, then it is an element of $L^2(w_\lambda; [-1,1])$, where $w_\lambda(t) = (1-t^2)^{\lambda-\frac{1}{2}}$, so that it can be expanded in the Gegenbauer series

$$f(\cos\theta) = \sum_{n=0}^{\infty} \hat{f}_n C_n^\lambda(\cos\theta), \qquad \lambda := \frac{d-2}{2}, \tag{14.3.2}$$

where $\hat{f}_n$ are given by, according to Eq. (1.2.10),

$$\hat{f}_n := [h_n^\lambda]^{-\frac{1}{2}} c_\lambda \int_0^\pi f(\cos\theta) C_n^\lambda(\cos\theta)(\sin\theta)^{2\lambda}\,\mathrm{d}\theta. \tag{14.3.3}$$

If all $\hat{f}_n$ are nonnegative, then f is a nonnegative sum of functions in Φ_d, so that it is a function in Φ_d itself. A classical result due to Schoenberg states that all positive definite functions are given in this way.

Theorem 14.3.3. *A continuous function $f : [-1,1] \to \mathbb{R}$ is positive definite if and only if $\hat{f}_n \geq 0$ for all n, in which case the series (14.3.2) converges throughout $0 \leq \theta \leq \pi$ absolutely and uniformly to $f(\cos\theta)$. The most general f that is positive definite on the sphere $\mathbb{S}^{d-1}$ is therefore given by the expansion*

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n C_n^\lambda(\cos\theta), \qquad a_n \geq 0, \ \ \forall n \in \mathbb{N}_0.$$

Proof. If all $\hat{f}_n$ are greater than or equal to 0 and the series (14.3.2) converges uniformly, then the function f is continuous, and it is a continuous limit of a sequence of functions in Φ_d, so that it is a function in Φ_d.

In the other direction, let f be a positive definite function. We start with the observation that for every $x \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} f(\langle x,y\rangle)\mathrm{d}\sigma(y) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} f(\langle x,y\rangle)\mathrm{d}\sigma(y)\mathrm{d}\sigma(x) \geq 0.$$

Indeed, equality follows immediately from the rotation-invariance of $\mathrm{d}\sigma$, and applying a positive cubature formula of precision n, we see that the double integral is nonnegative if f is any polynomial of degree at most n; hence it is nonnegative for all continuous functions f by density. Set $e = (0,\ldots,0,1)$. Up to a positive constant $c_n > 0$, the coefficient $\hat{f}_n$ in Eq. (14.3.3) can be written as

$$\hat{f}_n = c_n \int_{\mathbb{S}^{d-1}} f(\langle x,e\rangle) C_n^\lambda(\langle x,e\rangle)\mathrm{d}\sigma(x).$$

Since f and C_n^λ are both in Φ_d, so is their product. Hence, $\hat{f}_n \geq 0$ for all n. The infinite series (14.3.2) is Abel summable for all $\theta \in [0,\pi]$, whence

$$\sum_{n=0}^{m} \left|\hat{f}_n C_n^\lambda(\cos\theta)\right| \leq \sum_{n=0}^{m} \hat{f}_n C_n^\lambda(1) \leq \lim_{r\to 1^-} \sum_{n=0}^{\infty} \hat{f}_n C_n^\lambda(1) r^n = f(1),$$

which shows that the series converges absolutely and uniformly for all θ, and so it is equal to $f(\cos\theta)$. □

Next we consider the strictly positive definite functions. Let f be a strictly positive definite function, $f \in \mathrm{S}\Phi_d$. Since $\mathrm{S}\Phi_d \subset \Phi_d$ by definition, f is of the form (14.3.2) with all $\hat{f}_n$ nonnegative. Directly from the definition,

$$f[X_N] = \sum_{n=0}^{\infty} \hat{f}_n C_n^\lambda[X_N] = \sum_{n\in\mathscr{F}} \hat{f}_n C_n^\lambda[X_N],$$

where we define

$$\mathscr{F} := \{n \in \mathbb{N}_0 : \hat{f}_n > 0\}. \tag{14.3.4}$$

Thus, in order to have $f \in \mathrm{S}\Phi_d$, we need $\mathscr{F}$ to contain enough indices so that if $c^T C_n^\lambda[X_N]c = 0$ for all $n \in \mathscr{F}$, then $c = 0$ for every X_N and $N \in \mathbb{N}_0$.

Theorem 14.3.4. *If f is strictly positive definite, then $\mathscr{F} \subset \mathbb{N}_o$ contains infinitely many even integers and infinitely many odd integers.*

Proof. For a given N, we choose $X_{2N} = \{x_1, \dots, x_{2N}\}$ in such a way that if $x_i \in X_{2N}$, then $-x_i \in X_{2N}$. Splitting f into its even and odd parts, we obtain

$$f = f_e + f_o, \qquad f_e := \sum_{n=0}^{\infty} \hat{f}_{2n} C_{2n}^{\lambda} \quad \text{and} \quad f_o := \sum_{n=0}^{\infty} \hat{f}_{2n+1} C_{2n+1}^{\lambda}.$$

Since C_n^{λ} has the same parity as n, f_e is an even function and f_o is an odd function. Since if $x_\alpha \in X_{2N}$, then there is a $\beta \neq \alpha$ such that $x_\beta = -x_\alpha \in X_{2N}$, it follows that since $\langle x_\alpha, x_j \rangle = -\langle x_\beta, x_j \rangle$, the rows corresponding to α and β are identical in $f_e[X]$ and differ by a sign in $f_o[X_{2N}]$. Consequently,

$$\operatorname{rank} f_e[X_{2N}] \le N \quad \text{and} \quad \operatorname{rank} f_o[X_{2N}] \le N. \tag{14.3.5}$$

Assume now that $\mathscr{F}$ contains finitely many even integers. Let M be the largest integer n for which $\hat{f}_{2n} \neq 0$. Since C_n^{λ} is a polynomial of degree n, we can write

$$f_e = \sum_{n=0}^{M} \hat{f}_{2n} C_{2n}^{\lambda}(t) = \sum_{m=0}^{M} b_m t^{2m}$$

for some b_m with $b_M \neq 0$. Let $G_\ell = (\langle x_i, x_j \rangle^{\ell})_{i,j=1}^{2N}$. For $\ell = 1$, G_ℓ is the Gram matrix, which is nonnegative definite, and since $x_i \in \mathbb{S}^{d-1}$, it is easy to see that $\operatorname{rank} G_1 \le d$. By Schur's theorem, the matrices G_ℓ are also nonnegative definite, and their ranks satisfy the inequality

$$\operatorname{rank} G_\ell \le [\operatorname{rank} G_\ell]^{\ell} \le d^{\ell}.$$

Since $\operatorname{rank}(A+B) \le \operatorname{rank} A + \operatorname{rank} B$, it follows that

$$\operatorname{rank} f_e[X_{2N}] \le \sum_{m=0}^{M} \operatorname{rank} G_{2m} \le \sum_{m=0}^{M} d^{2m} \le M d^{2M} + 1.$$

Together with Eq. (14.3.5), this shows that if $N > M d^{2M} + 1$, then

$$\operatorname{rank} f[X_{2N}] \le \operatorname{rank} f_e[X] + \operatorname{rank} f_o[X] \le (M d^{2M} + 1) + N < 2N.$$

Hence, if $N > M d^{2M} + 1$, then $f[X_{2N}]$ does not have full rank, which implies that f is not strictly positive definite. The case that $\mathscr{F}$ contains finitely many odd integers can be handled similarly. □

The inverse of Theorem 14.3.4 holds as well; namely, that $\mathscr{F} \subset \mathbb{N}_o$ contain infinitely many even integers and infinitely many odd integers is also a sufficient condition for a continuous function to be strictly positive definite. To prove this, we start with a lemma that is of interest in itself.

Lemma 14.3.5. *Let $X_N = \{x_1, \dots, x_N\}$ denote a set of distinct points on $\mathbb{S}^{d-1}$, and let $f \in C[-1,1]$ be positive definite. The matrix $f[X_N]$ is positive definite if and only if the set of functions $\{f(\langle x, x_i\rangle) : 1 \le i \le N\}$ is linearly independent.*

Proof. Let $\{Y_k^n : 1 \le k \le a_n^d\}$ denote an orthonormal basis of $\mathscr{H}_n^d$. It is convenient to introduce the notation of $\mathbb{Y}_n(x) := (Y_1^n(x), \dots, Y_{a_n^d}^n(x))$ as a column vector. By the definition of f and Eq. (14.3.1),

$$\sum_{j=1}^N c_j f(\langle x, x_j\rangle) = \sum_{n=0}^\infty \hat{f}_n \sum_{j=1}^N c_j C_n^\lambda(\langle x, x_j\rangle) = \sum_{n=0}^\infty \frac{\lambda \hat{f}_n}{n+\lambda} \left[\sum_{j=1}^N c_j \mathbb{Y}_n(x_j)\right]^T \mathbb{Y}_n(x).$$

In particular, evaluating at x_i and summing up over i shows that

$$c^T f[X_N] c = \sum_{i=1}^N \sum_{j=1}^N c_i c_j f(\langle x_i, x_j\rangle) = \sum_{n=0}^\infty \frac{\lambda \hat{f}_n}{n+\lambda} \left[\sum_{j=1}^N c_j \mathbb{Y}_n(x_j)\right]^T \sum_{i=1}^N c_i \mathbb{Y}_n(x_i).$$

Thus, by the first identity, $\sum_{j=1}^N c_j f(\langle x, x_j\rangle) = 0$ if and only if $\sum_{j=1}^N c_j \mathbb{Y}_n(x_j) = 0$ for all $n \in \mathscr{F}$ with $\mathscr{F}$ as given in Eq. (14.3.4) and, by the second identity, if and only if $c^T f[X_N] c = 0$. □

For the proof of the sufficiency part, we need the addition formula for the Gegenbauer polynomials. For $x, y \in \mathbb{S}^{d-1}$, we write

$$x = (\sin\theta\, x', \cos\theta) \quad \text{and} \quad y = (\sin\phi\, y', \cos\phi), \quad 0 \le \theta, \phi \le \pi, \quad x', y' \in \mathbb{S}^{d-2}.$$

Using the orthonormal basis (1.5.6) of $\mathscr{H}_n^d$, which has a product structure, and Eq. (14.3.1) (or using the addition formula of the Gegenbauer polynomials), we see that

$$C_n^\lambda(\langle x, y\rangle) = \sum_{k=0}^n b_{k,n}^\lambda Q_{k,n}^\lambda(\theta) Q_{k,n}^\lambda(\phi) C_k^{\lambda-\frac{1}{2}}(\langle x', y'\rangle), \tag{14.3.6}$$

where the coefficients $b_{k,n}^\lambda > 0$ are positive constants and

$$Q_{k,n}^\lambda(\theta) := (\sin\theta)^k C_{n-k}^{\lambda+k}(\cos\theta), \qquad 0 \le k \le n.$$

Theorem 14.3.6. *Let $d \ge 3$ and f be positive definite. If $\mathscr{F} \subset \mathbb{N}_0$ contains infinitely many even indices and infinitely many odd indices, then f is strictly positive definite on $\mathbb{S}^{d-1}$.*

Proof. For a given set of N distinct points X_N on $\mathbb{S}^{d-1}$, we choose a point p (the "pole") such that $\langle p, x_i\rangle$ form a set of N distinct numbers in the open interval $(-1,1)$. It is elementary to prove that such a point exists. Without loss of generality, we may assume that $p = e_d := (0, \dots, 0, 1) \in \mathbb{S}^{d-1}$. Each point x_i then has a representation in the form

$$x_i = (x_i' \sin\theta_i, \cos\theta_i), \quad x_i' \in \mathbb{S}^{d-2}, \quad \cos\theta_i = \langle p, x_i\rangle.$$

As we have seen in the proof of Theorem 14.3.4, the case that there exist x_i and x_j in X_N such that $x_i = -x_j$ deserves special attention. Such a pair is called antipodal. If an antipodal pair exists, then $\cos\theta_i = -\cos\theta_j$, so that $\theta_i = \pi - \theta_j$, and consequently, $\sin\theta_i = \sin\theta_j$ and $x'_i = -x'_j$. Thus, x_i and x_j are antipodal if and only if $\sin\theta_i = \sin\theta_j$. By Eq. (14.3.6) and the expression of f,

$$\sum_{j=1}^{N} c_j f(\langle x, x_j\rangle) = \sum_{n=0}^{\infty} \hat{f}_n \sum_{k=0}^{n} b_{k,n}^{\lambda} \left[\sum_{j=1}^{N} c_j Q_{k,n}^{\lambda}(\theta_j) C_k^{\lambda-\frac{1}{2}}(\langle x'_j, x'\rangle)\right] Q_{k,n}^{\lambda}(\theta).$$

Hence, by Lemma 14.3.5, it is sufficient to prove the following claim: if

$$\sum_{j=1}^{N} c_j Q_{k,n}^{\lambda}(\theta_j) C_k^{\lambda-\frac{1}{2}}(\langle x'_j, x'\rangle) = 0, \quad 0 \le k \le n, \quad n \in \mathscr{F}, \tag{14.3.7}$$

then $c_1 = \cdots = c_N = 0$. Note that $Q_{n,n}^{\lambda}(\theta) = (\sin\theta)^n$.

The proof uses induction on N, and it is sufficient to consider Eq. (14.3.7) for $k = n$. The claim is obviously true when $N = 1$, and it is also true when $N = 2$ and the two points are antipodal. We now prove the claim for X_N with N distinguished points, assuming that it has been established for X_M with $M < N$. Let $1 \le j_0 \le N$ be such that $\sin\theta_{j_0} = \max_{1\le j\le N} \sin\theta_j$. We need to consider two cases:

Case 1. The set $X_N \setminus \{x_{j_0}\}$ does not contain the antipodal point of x_{j_0}. In this case, $\sin\theta_{j_0} > \sin\theta_j$ for $j \ne j_0$. Hence, evaluating Eq. (14.3.7) with $k = n$ at x_{j_0} and dividing by the coefficient of c_{j_0}, we obtain

$$\sum_{j\ne j_0} c_j \left(\frac{\sin\theta_j}{\sin\theta_{j_0}}\right)^n \frac{C_n^{\lambda-\frac{1}{2}}(\langle x'_j, x'\rangle)}{C_n^{\lambda-\frac{1}{2}}(1)} + c_{j_0} = 0.$$

Since $\mathscr{F}$ is an infinite set, we can let $n \to \infty$ through a subset of integers in $\mathscr{F}$. Since $|C_n^{\lambda-\frac{1}{2}}(\langle x'_j, x'\rangle)| \le C_n^{\lambda-\frac{1}{2}}(1)$ and $\sin\theta_j/\sin\theta_{j_0} < 1$ for all $j \ne j_0$, we conclude that $c_{j_0} = 0$. Consequently, we can remove one point from Eq. (14.3.7), so that we can invoke the induction hypothesis to conclude that $c_j = 0$ for all $j \ne j_0$.

Case 2. The set $X_N \setminus \{x_{j_0}\}$ contains the antipodal point of x_{j_0}. Let $x_{j_1} = -x_{j_0} \in X_N$ for $j_1 \ne j_0$. Then $\sin\theta_{j_0} = \sin\theta_{j_1} > \sin\theta_j$ for $j \ne j_0, j_1$. Since $C_n^{\mu}(-1) = (-1)^n C_n^{\mu}(1)$, evaluating Eq. (14.3.7) with $k = n$ at x_{j_0} and dividing by the coefficient of c_{j_0}, we obtain

$$\sum_{j\ne j_0, j_1} c_j \left(\frac{\sin\theta_j}{\sin\theta_{j_0}}\right)^n \frac{C_n^{\lambda-\frac{1}{2}}(\langle x'_j, x'\rangle)}{C_n^{\lambda-\frac{1}{2}}(1)} + c_{j_0} + (-1)^n c_{j_1} = 0.$$

By the assumption on $\mathscr{F}$, we can let n goes to infinity through a sequence of even integers and a sequence of odd integers and obtain $c_{j_0} + c_{j_1} = 0$ and $c_{j_0} - c_{j_1} = 0$, respectively. Consequently, $c_{j_0} = c_{j_1} = 0$. We can again invoke the induction hypothesis to conclude that $c_j = 0$ for $j \ne j_0, j_1$. □

A positive definite function on $\mathbb{S}^{d-1}$, as a univariate function defined on $[-1,1]$, could also be positive definite on $\mathbb{S}^m$ for some $m \neq d$. Our next result shows an inclusion relation.

Theorem 14.3.7. *For $d \geq 1$, we have $\Phi_{d+1} \subset \Phi_d$ and $\mathrm{S}\Phi_{d+1} \subset \mathrm{S}\Phi_d$.*

Proof. By the connection formula (B.2.10) of the Gegenbauer polynomials,

$$C_n^\lambda(t) = \sum_{0 \leq k \leq n/2} a_{k,n}^\lambda C_{n-2k}^{\lambda-\frac{1}{2}}(t)$$

for some constants $a_{k,n}^\lambda > 0$, from which the stated result follows readily from Theorems 14.3.3 and 14.3.6. □

The strictly positive definite functions can be used to interpolate scattered data on the sphere. Suppose that numerical values $\lambda_1, \ldots, \lambda_N$ are associated with prescribed points $X_N = \{x_1, \ldots, x_N\}$ in $\mathbb{S}^{d-1}$. If f is a strictly positive definite function, then we can find a function of the form

$$F(x) = \sum_{j=1}^N c_j f(\langle x, x_j \rangle), \qquad x \in \mathbb{S}^{d-1},$$

that interpolates the data. Indeed, the interpolation conditions mean that

$$\lambda_i = F(x_i) = \sum_{j=1}^N c_j f(\langle x_i, x_j \rangle), \qquad 1 \leq i \leq N,$$

which is a linear system that has the coefficient matrix $f[X_N]$. Since $f[X_N]$ is positive definite, the system has a unique solution.

For the purpose of interpolation, it is often desirable to choose a function f that has compact support. We give an example of such a function. Define $(x)_+$ by $(x)_+ = x$ if $x \geq 0$ and $(x)_+ = 0$ if $x < 0$. For $0 < \theta < \pi$ and $\delta > 0$, define

$$f_{\theta,\delta}(t) := (\theta - \arccos t)_+^\delta, \quad t \in [-1,1]. \tag{14.3.8}$$

By its definition, the function $y \mapsto f_{\theta,\delta}(\langle x, y \rangle) = (\theta - \mathrm{d}(x,y))_+^\delta$ has compact support, and its support set is the spherical cap $\mathsf{c}(x,\theta)$.

Proposition 14.3.8. *If $\delta \geq 2$, then $f_{\theta,\delta}$ is strictly positive definite on $\mathbb{S}^3$ and $\mathbb{S}^2$.*

Proof. Since $\Phi_4 \subset \Phi_3$, we need only to prove $f_{\theta,\delta} \in \Phi_4$. For $d = 4$, $\lambda = 1$ and $C_n^\lambda = U_n$, the Chebyshev polynomial of the second kind. The coefficients of $f := f_{\theta,\delta}$ are then given by, up to a positive constant,

$$\hat{f}_n = \frac{1}{\pi} \int_{-1}^1 f_{\theta,\delta}(t) U_n(t) \sqrt{1-t^2}\, \mathrm{d}t = \frac{1}{\pi} \int_0^\theta (\theta - \phi)^\delta (\sin \phi)^2 \mathrm{d}\phi.$$

Since the fractional integral

$$\mathscr{L}^{\mu} g := \frac{1}{\Gamma(\mu)} \int_0^t (t-\theta)^{\mu-1} g(\theta) \mathrm{d}\theta$$

satisfies $\mathscr{L}^{\mu+\delta} = \mathscr{L}^{\delta}\mathscr{L}^{\mu}$, it suffices to show that $\hat{f}_n > 0$ for $\delta = 2$, which we need to establish for all $n \geq 0$ and $\theta \in (0,\pi)$. The case $n = 0$ is trivial. Assume now $n > 0$. Since $U_n(\cos t) = \sin(n+1)t/\sin t$, we have

$$\pi \hat{f}_n = \int_0^{\theta} (\theta-\phi)^2 \Big(\sin(n+1)\phi\Big) \sin\phi \mathrm{d}\phi = I(\theta,n) - I(\theta,n+2), \tag{14.3.9}$$

where, on using $2\sin(n+1)\phi\sin\phi = \cos n\phi - \cos(n+2)\phi$,

$$I(\theta,n) := \frac{1}{2}\int_0^{\theta} (\theta-\phi)^2 \cos n\phi \mathrm{d}\phi.$$

Integrating by parts shows that

$$I(\theta,n) = \theta^3 h(n\theta) \quad \text{with} \quad h(u) = \frac{u-\sin u}{u^3}, \quad u > 0.$$

Thus, it is enough to show that h is a strictly decreasing function on $(0,\infty)$. Indeed, a straightforward computation shows that

$$h'(u) = -\frac{2u + u\cos u - 3\sin u}{u^4}. \tag{14.3.10}$$

By the trivial inequalities $\cos u \geq -1$ and $-\sin u \geq -1$, we see that

$$g(u) := 2u + u\cos u - 3\sin u \geq 2u - u - 3 = u - 3 \geq 0 \quad \text{for } u \geq 3.$$

On the other hand, the Taylor expansion of $g(u)$ takes the form

$$g(u) = 2\sum_{k=2}^{\infty} (-1)^k \frac{(k-1)u^{2k-3}}{(2k+1)!} = \frac{2u^5}{5!} - \frac{4u^7}{7!} + \frac{6u^9}{9!} - \cdots,$$

which is an alternating series, hence positive, if $u^2 < 21$, which clearly covers $u \in [0,3]$. Consequently, $g(u) \geq 0$ for all $u \geq 0$. This implies that $h(u)$ is a strictly decreasing function of $u \in (0,\infty)$. □

For a given function f, checking the signs of all the Gegenbauer coefficients can be an arduous, or impossible, task. In this regard, the following theorem is of interest.

Theorem 14.3.9. *Let* $d \in \{3, 4, \ldots, 8\}$ *and* $m = \lceil \frac{d-2}{2} \rceil$. *Let* $f : [-1, 1] \mapsto \mathbb{R}$ *and assume that* $g(\cdot) = f(\cos \cdot)$ *satisfies the following conditions:*

(1) $g \in C^m[0, \pi]$, *and the right derivative* $g^{(m+1)}(0+)$ *exists and is finite;*
(2) $supp(g) \subset [0, \pi)$*;*
(3) $(-1)^m g^{(m)}$ *is convex.*

Then $f \in \Phi_d$. *If, in addition,* $g^{(m)}$*, restricted to* $(0, \pi)$*, does not reduce to a linear polynomial, then* $f \in S\Phi_d$.

The sufficient condition given in this theorem is an analogue of the Pólya criterion for functions to have nonnegative Fourier transform. We do not give the proof of this theorem. See Sect. 14.6 for references and further discussions.

14.4 Asymptotics for Minimal Discrete Energy on the Sphere

The goal in this section is to determine the asymptotics for minimal discrete energy on the sphere. The problem is of interest in physics, chemistry, and computer science.

Definition 14.4.1. For a given $s > 0$, the discrete s-energy associated with a finite subset $\Lambda_N = \{x_1, \ldots, x_N\}$ of distinct points on $\mathbb{S}^{d-1}$ is

$$E_s(\mathbb{S}^{d-1}, \Lambda_N) := \sum_{1 \le i < j \le N} \|x_i - x_j\|^{-s}.$$

The minimal s-energy for N points on the sphere is defined by

$$\mathscr{E}_s(\mathbb{S}^{d-1}, N) := \inf_{\Lambda_N} E_s(\mathbb{S}^{d-1}, \Lambda_N), \tag{14.4.1}$$

where the infimum is taken over all N-point subsets of $\mathbb{S}^{d-1}$. A subset Λ_N of N points on $\mathbb{S}^{d-1}$ for which the infimum in Eq. (14.4.1) is attained is called an s-extremal configuration.

For a given N, s-extremal configurations are known in only a handful of cases, when N is small. The limit of the minimal s-energy as $N \to \infty$, however, can be determined. To illustrate the main idea, we consider the continuous case of the energy integral first.

Let $\mathscr{M}(\mathbb{S}^{d-1})$ denote the collection of all probability measures on the Borel σ-algebra of $\mathbb{S}^{d-1}$. For convenience, we denote by $\sigma_0 := \sigma/\omega_d$ the surface measure on $\mathbb{S}^{d-1}$ normalized so that $\sigma_0(\mathbb{S}^{d-1}) = \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma_0 = 1$.

Definition 14.4.2. For $0 < s < d-1$, the energy integral with respect to a probability measure μ on the Borel σ-algebra of $\mathbb{S}^{d-1}$ is defined by

$$I_{d,s}(\mu) := \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \|x-y\|^{-s} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y).$$

Clearly, $I_{d,s}(\sigma_0)$ is well defined and is finite for every $0 < s < d-1$.

Theorem 14.4.3. *For* $0 < s < d-1$,

$$\min_{\mu \in \mathscr{M}(\mathbb{S}^{d-1})} I_{d,s}(\mu) = I_{d,s}(\sigma_0) = \frac{\Gamma(\frac{d}{2})\Gamma(d-1-s)}{\Gamma(\frac{d-s}{2})\Gamma(d-1-\frac{s}{2})}.$$

Proof. For $x, y \in \mathbb{S}^{d-1}$, $\|x-y\|^2 = 2 - 2\langle x, y \rangle$. We define, for $\varepsilon > 0$,

$$K_{\varepsilon,s}(t) := (2 - 2t + \varepsilon)^{-s/2}, \quad t \in [-1, 1].$$

Using the Rodrigues formula for the Gegenbauer polynomials, Eqs. (B.1.2) and (B.2.1),

$$(1-t^2)^{\lambda - \frac{1}{2}} C_n^\lambda(t) = (-1)^n c_n \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n (1-t^2)^{n+\lambda-\frac{1}{2}},$$

where c_n is a positive constant, we deduce by integration by parts that

$$K_{\varepsilon,s}(t) = \sum_{n=0}^{\infty} a_n(\varepsilon, s) \frac{n+\lambda}{\lambda} C_n^\lambda(t), \qquad \lambda := \frac{d-2}{2},$$

with $a_n(\varepsilon, s) = \frac{\lambda}{n+\lambda} (\hat{K}_{\varepsilon,s})_n > 0$ for all $n \in \mathbb{Z}_+$. Since $K_{\varepsilon,s}(t)$ is real analytic on $[-1, 1]$, the series converges uniformly on $[-1, 1]$. Since for every $x, y \in \mathbb{S}^{d-1}$ and $s > 0$,

$$\|x-y\|^{-s} = (2 - 2\langle x, y \rangle)^{-s/2} \ge K_{\varepsilon,s}(\langle x, y \rangle),$$

it follows by Eq. (14.3.1) and Fubini's theorem that

$$\begin{aligned} I_{d,s}(\mu) &\ge \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} K_{\varepsilon,s}(\langle x, y \rangle) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\ &= \sum_{n=0}^{\infty} a_n(\varepsilon, s) \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \sum_{j=1}^{a_n^d} Y_{n,j}(x) Y_{n,j}(y) \, \mathrm{d}\mu(x) \mathrm{d}\mu(y) \\ &= \sum_{n=0}^{\infty} a_n(\varepsilon, s) \left| \sum_{j=1}^{a_n^d} \int_{\mathbb{S}^{d-1}} Y_{n,j}(x) \, \mathrm{d}\mu(x) \right|^2 \ge a_0(\varepsilon, s), \end{aligned} \tag{14.4.2}$$

which implies, by the definition of $(\hat{K}_{\varepsilon,s})_0$, that for every $\varepsilon > 0$,

$$I_{d,s}(\mu) \ge a_0(\varepsilon, s) = c_\lambda \int_{-1}^{1} (2 - 2t + \varepsilon)^{-s/2} (1-t^2)^{\lambda - \frac{1}{2}} \mathrm{d}t.$$

Letting $\varepsilon \to 0+$, we deduce by the dominated convergence theorem that

$$I_{d,s}(\mu) \geq c_\lambda \int_{-1}^{1} (2-2t)^{-\frac{s}{2}}(1-t^2)^{\lambda-\frac{1}{2}}\mathrm{d}t = \frac{\Gamma(d/2)\Gamma(d-1-s)}{\Gamma((d-s)/2)\Gamma(d-1-s/2)}.$$

Finally, since $K_{\varepsilon,s}(\langle x,y\rangle)$ is a zonal function and $\mathrm{d}\sigma_0$ is rotation-invariant,

$$I_{d,s}(\sigma_0) = \int_{\mathbb{S}^{d-1}} (2-2\langle x,y\rangle)^{-s/2}\mathrm{d}\sigma_0(y) = c_\lambda \int_{-1}^{1}(2-2t)^{-s/2}(1-t^2)^{\lambda-\frac{1}{2}}\mathrm{d}t,$$

so that $I_{d,s}(\mu) \geq I_{d,s}(\sigma_0)$. The proof is complete. □

The proof of Theorem 14.4.3 with slight modifications yields the following lower estimates of $\mathscr{E}_s(\mathbb{S}^{d-1},N)$.

Theorem 14.4.4. *Let $d \geq 3$. Then as $N \to \infty$,*

$$\mathscr{E}_s(\mathbb{S}^{d-1},N) \geq \frac{1}{2}N^2 I_{d,s}(\sigma_0) - c\begin{cases} N^{1+\frac{s}{d-1}}, & d-3 < s < d-1, \\ N^{1+\frac{s}{2+s}}, & 0 < s \leq d-3, \end{cases} \tag{14.4.3}$$

where c is an absolute positive constant depending only on d and s.

Proof. Let $\Lambda_N := \{x_1,\dots,x_N\}$ be any given subset of $\mathbb{S}^{d-1}$ and let μ be a measure in $\mathscr{M}(\mathbb{S}^{d-1})$ that satisfies

$$\mu(\{x_j\}) = \frac{1}{N}, \quad 1 \leq j \leq N.$$

Then Eq. (14.4.2) for this μ ensures that

$$\frac{2}{N^2}\sum_{1\leq i<j\leq N}(2-2\langle x_i,x_j\rangle+\varepsilon)^{-s/2} + \frac{1}{N}\varepsilon^{-s/2} \geq a_0(\varepsilon,s).$$

Since $\|x-y\|^{-s} = (2-2\langle x,y\rangle)^{-s/2} \geq (2-2\langle x,y\rangle+\varepsilon)^{-s/2}$ for any $x,y \in \mathbb{S}^{d-1}$, this implies that

$$\mathscr{E}_s(\mathbb{S}^{d-1},N) \geq \frac{1}{2}N^2 a_0(\varepsilon,s) - \frac{1}{2}N\varepsilon^{-s/2} \tag{14.4.4}$$

for all $\varepsilon > 0$. Changing variables $t \mapsto (1-t)/2$ and rearranging, we obtain

$$a_0(\varepsilon,s) = 2^{2\lambda-s}c_\lambda \int_0^1 \left(1+\frac{\varepsilon}{4t}\right)^{-\frac{s}{2}} t^{\frac{d-3}{2}-\frac{s}{2}}(1-t)^{\frac{d-3}{2}}\,\mathrm{d}t. \tag{14.4.5}$$

Since $a_0(0,s) = I_{d,s}(\sigma_0)$, an integration by parts shows that

$$a_0(\varepsilon,s) = \left(1+\frac{\varepsilon}{4}\right)^{-s/2} I_{d,s}(\sigma_0) - 2^{2\lambda-s-3}s\varepsilon c_\lambda$$
$$\times \int_0^1 \left(1+\frac{\varepsilon}{4t}\right)^{-\frac{s}{2}-1} t^{-2}\left[\int_0^t u^{\frac{d-3}{2}-\frac{s}{2}}(1-u)^{\frac{d-3}{2}}\,\mathrm{d}u\right]\mathrm{d}t$$
$$\geq I_{d,s}(\sigma_0) - c_s\varepsilon - c_s\varepsilon\int_0^1 (4t+\varepsilon)^{-\frac{s}{2}-1}t^{\frac{d-3}{2}}\,\mathrm{d}t.$$

For $0 < s < d-3$, the last integral is easily seen to be bounded by a constant, whereas for $d-3 < s < d-1$, it is bounded by $c\varepsilon^{\frac{d-3}{2}-\frac{\varepsilon}{2}}$, as can be seen by splitting the integral into two integrals over $[0,\varepsilon]$ and $[\varepsilon,1]$. To complete the proof, we then apply Eq. (14.4.4) with $\varepsilon = N^{-\frac{2}{d-1}}$ if $d-3 < s < d-1$ and with $\varepsilon = N^{-\frac{2}{2+s}}$ if $0 < s < d-3$. □

The following theorem gives an upper estimate of $\mathscr{E}_s(\mathbb{S}^{d-1},N)$.

Theorem 14.4.5. *If* $d \geq 3$ *and* $0 < s < d-1$*, then*

$$\mathscr{E}_s(\mathbb{S}^{d-1},N) \leq \frac{1}{2}I_{d,s}(\sigma_0)N^2 - cN^{1+\frac{s}{d-1}}. \tag{14.4.6}$$

In particular, for $d-3 < s < d-1$ *and* $d \geq 3$*,*

$$\mathscr{E}_s(\mathbb{S}^{d-1},N) = \frac{1}{2}I_{d,s}(\sigma_0)N^2 - O(1)N^{1+\frac{s}{d-1}}. \tag{14.4.7}$$

Proof. By Theorem 6.4.2, for each positive integer N, there exists an area-regular partition $\mathscr{P}_N = \{R_1,\ldots,R_N\}$ of $\mathbb{S}^{d-1}$ such that

$$\sigma_0(R_j) = \frac{1}{N} \quad \text{and} \quad \operatorname{diam}(R_j) \leq cN^{-\frac{1}{d-1}}.$$

Let σ_j^* denote the restriction of the measure $N\sigma_0$ to R_j. Then

$$\mathscr{E}_s(\mathbb{S}^{d-1},N) \leq \int_{R_1}\cdots\int_{R_N}\sum_{1\leq i<j\leq N}\|x_i-x_j\|^{-s}\,\mathrm{d}\sigma_1^*(x_1)\cdots\mathrm{d}\sigma_N^*(x_N)$$
$$= \frac{1}{2}\sum_{i=1}^N\int_{R_i}\sum_{j\neq i}\int_{R_j}\|x_i-x_j\|^{-s}\,\mathrm{d}\sigma_j^*(x_j)\mathrm{d}\sigma_i^*(x_i)$$
$$= \frac{1}{2}N^2\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\|x-y\|^{-s}\,\mathrm{d}\sigma_0(x)\mathrm{d}\sigma_0(y)$$
$$-\frac{1}{2}\sum_{i=1}^N\int_{R_i}\int_{R_i}\|x-y\|^{-s}\,\mathrm{d}\sigma_i^*(x)\,\mathrm{d}\sigma_i^*(y),$$

which is bounded above by

$$\frac{1}{2}N^2 I_{d,s}(\sigma_0) - \frac{N}{2}(\mathrm{diam}(R_i))^{-s} \le \frac{1}{2}N^2 I_{d,s}(\sigma_0) - cN^{1+\frac{s}{d-1}}.$$

This proves Eq. (14.4.6). Finally, Eq. (14.4.7) follows directly from Eqs. (14.4.6) and (14.4.3). □

Theorem 14.4.6. *If $d \ge 3$ and $s > d-1$, then*

$$c_1 N^{1+\frac{s}{d-1}} \le \mathscr{E}_s(\mathbb{S}^{d-1}, N) \le c_2 N^{1+\frac{s}{d-1}}.$$

Proof. We first prove the lower bound. Let $\Lambda_N := \{x_1, \dots, x_N\}$ be any configuration of N points on $\mathbb{S}^{d-1}$. For each $1 \le i \le N$, define

$$r_i := \min_{j \ne i} \|x_i - x_j\| = \mathrm{dist}\,(x_i, \Lambda_N \setminus \{x_i\}).$$

Clearly, for each $1 \le i \ne j \le N$, $\|x_i - x_j\| \ge \max\{r_i, r_j\}$. Thus, the spherical caps $\mathsf{c}(x_i, r_i/2)$, $1 \le i \le N$, are disjoint, which implies that

$$c\sum_{i=1}^N r_i^{d-1} \le \sum_{i=1}^N \sigma_0(\mathsf{c}(x_i, r_i/2)) \le \sigma_0(\mathbb{S}^{d-1}) = 1.$$

On the other hand, using Hölder's inequality with $p = \frac{s}{d-1+s} \in (0,1)$ and $p' = \frac{p}{p-1} = -\frac{s}{d-1}$, we obtain

$$\sum_{j=1}^N r_j^{d-1} \ge N^{\frac{1}{p}} \left(\sum_{j=1}^N r_j^{(d-1)p'}\right)^{\frac{1}{p'}} = N^{1+\frac{d-1}{s}} \left(\sum_{j=1}^N r_j^{-s}\right)^{-\frac{d-1}{s}}.$$

Together, the above two displayed inequalities yield

$$\sum_{j=1}^N r_j^{-s} \ge c_{s,d} N^{(1+\frac{d-1}{s})\frac{s}{d-1}} = c_{s,d} N^{1+\frac{s}{d-1}}. \tag{14.4.8}$$

Now, for each $1 \le i \le N$, letting $j_i \in \{1, \dots, N\}$ be such that $r_i = \|x_i - x_{j_i}\|$, we then obtain from Eq. (14.4.8) that

$$\begin{aligned} E_s\left(\mathbb{S}^{d-1}, \Lambda_N\right) &= \frac{1}{2}\sum_{i=1}^N \sum_{j \ne i} \|x_i - x_j\|^{-s} \ge \frac{1}{2}\sum_{i=1}^N \|x_i - x_{j_i}\|^{-s} \\ &= \frac{1}{2}\sum_{j=1}^N r_j^{-s} \ge \frac{1}{2} c_{s,d} N^{1+\frac{s}{d-1}}, \end{aligned}$$

which proves the desired lower bound.

To prove the upper bound, let $\Lambda_N := \{x_1,\dots,x_N\}$ be a configuration of N points on $\mathbb{S}^{d-1}$ that minimizes the s-energy. For each i, let $D_i = \mathbb{S}^{d-1}\setminus \mathsf{c}(x_i, N^{-\frac{1}{d-1}})$ and put $D = \cap_{i=1}^N D_i$. Then

$$\sigma_0\left(\mathbb{S}^{d-1}\setminus D\right) \le \sum_{j=1}^N \sigma_0(\mathbb{S}^{d-1}\setminus D_i) = N\sigma_0(\mathsf{c}(x_1, N^{-\frac{1}{d-1}})$$
$$\le N\frac{\omega_{d-1}}{\omega_d}\int_0^{N^{-\frac{1}{d-1}}} \theta^{d-2}\mathrm{d}\theta = \frac{\omega_{d-1}}{\omega_d(d-1)} < 1,$$

which implies that

$$\sigma_0(D) = 1-\sigma_0\left(\mathbb{S}^{d-1}\setminus D\right) \ge 1-\frac{\omega_{d-1}}{\omega_d(d-1)} > 0. \tag{14.4.9}$$

Next, we define, for a given index i, the function

$$U_i^s(x) := \sum_{1\le j\ne i\le N} \|x-x_j\|^{-s}, \qquad x\in\mathbb{S}^{d-1}. \tag{14.4.10}$$

Then, by the definition of the set D,

$$\int_D U_i^s(x)\,\mathrm{d}\sigma_0(x) = \sum_{j\ne i}\int_D \|x-x_j\|^{-s}\,\mathrm{d}\sigma_0(x) \le \sum_{j\ne i}\int_{D_j} \|x-x_j\|^{-s}\,\mathrm{d}\sigma_0(x)$$
$$\le c\sum_{j\ne i}\int_{N^{-\frac{1}{d-1}}}^{\pi} \theta^{d-2-s}\mathrm{d}\theta \le cN^{\frac{s}{d-1}}.$$

Since Λ_N minimizes the s-energy, the function U_i^s attains its minimum at the point x_i. Therefore, by Eq. (14.4.9) and the above inequality,

$$U_i^s(x_i) \le \frac{1}{\sigma_0(D)}\int_D U_i^s(x)\,\mathrm{d}\sigma_0(x) \le cN^{\frac{s}{d-1}}. \tag{14.4.11}$$

It follows that

$$\mathscr{E}_s(\mathbb{S}^{d-1},N) = \sum_{1\le i<j\le N}\|x_i-x_j\|^{-s} = \frac{1}{2}\sum_{i=1}^N\sum_{j\ne i}\|x_i-x_j\|^{-s}$$
$$= \frac{1}{2}\sum_{i=1}^N U_i^s(x_i) \le cN^{1+\frac{s}{d-1}},$$

which proves the desired upper bound. □

Corollary 14.4.7. *Let $d\ge 3$ and $s > d-1$. Let $\Lambda_N := \{x_1,\dots,x_N\}$ be a configuration of points on $\mathbb{S}^{d-1}$ that minimizes the s-energy. Then there is a constant c, depending only on s and d, such that*

$$\min_{1\le i\ne j\le N}\|x_i-x_j\| \ge cN^{-\frac{1}{d-1}}.$$

Proof. By Eq. (14.4.11), we have

$$\sum_{j:1\le j\ne i\le N} \|x_i - x_j\|^{-s} = U_i^s(x_i) \le cN^{\frac{s}{d-1}}, \quad 1\le i\le N,$$

which implies, in particular, that

$$\max_{1\le i<j\le N} \|x_i - x_j\|^{-s} \le cN^{\frac{s}{d-1}}.$$

The desired lower estimate then follows because $s>0$. □

We conclude this section with the following theorem for $s=d-1$.

Theorem 14.4.8. *For $d\ge 3$,*

$$\lim_{N\to\infty} (N^2\log N)^{-1}\mathscr{E}_{d-1}(\mathbb{S}^{d-1},N) = \frac{\omega_{d-1}}{2(d-1)\omega_d}. \tag{14.4.12}$$

Proof. Let $\gamma_d := \omega_{d-1}/\omega_d$. First, we prove the lower bound:

$$\liminf_{N\to\infty}(N^2\log N)^{-1}\mathscr{E}_{d-1}(\mathbb{S}^{d-1},N) \ge \frac{\gamma_d}{2(d-1)}. \tag{14.4.13}$$

Let $a_0(\varepsilon,s)$ be defined by Eq. (14.4.2), and for simplicity, we write $a_0(\varepsilon) := a_0(\varepsilon,d-1)$. From Eq. (14.4.5), a straightforward computation shows that for $\varepsilon\to 0$,

$$a_0(\varepsilon) = 2^{-1}\gamma_d\int_1^{4\varepsilon^{-1}} (1+u^{-1})^{-\frac{d-1}{2}}u^{-1}\,\mathrm{d}u + O(1) = 2^{-1}\gamma_d|\log\varepsilon| + O(1).$$

Thus, invoking Eq. (14.4.4) with $s=d-1$, we obtain

$$\mathscr{E}_{d-1}(\mathbb{S}^{d-1},N) \ge \frac{1}{2}\gamma_d N^2|\log\varepsilon| - N\varepsilon^{-\frac{d-1}{2}} - O(1)N^2.$$

Setting $\varepsilon = N^{-2/(d-1)}$, we then deduce that

$$\mathscr{E}_{d-1}(\mathbb{S}^{d-1},N) \ge \frac{\gamma_d}{d-1}N^2\log N - O(1)N^2,$$

which proves the desired lower estimate (14.4.13).

To complete the proof, it remains to prove the upper estimate,

$$\limsup_{N\to\infty}(N^2\log N)^{-1}\mathscr{E}_{d-1}(\mathbb{S}^{d-1},N) \le \frac{1}{2(d-1)}\gamma_d. \tag{14.4.14}$$

Let $\Lambda_N := \{x_1,\dots,x_N\}$ be a configuration that minimizes the $(d-1)$-energy. For $r>0$, set

$$D_i(r) = \mathbb{S}^{d-1} \setminus \mathsf{c}\left(x_i, rN^{-1/(d-1)}\right), \quad 1 \le i \le N, \quad \text{and} \quad D(r) := \bigcap_{i=1}^{N} D_i(r).$$

A straightforward computation as in Eq. (14.4.9) shows that

$$\sigma_0(D(r)) \ge 1 - \frac{\gamma_d}{d-1} r^{d-1}. \tag{14.4.15}$$

Let $U_i^s(x)$ be defined as in Eq. (14.4.10) and set $U_i := U_i^{d-1}$. For a fixed $r > 0$, a straightforward computation shows that as $N \to \infty$,

$$\int_{D(r)} U_i(x)\,\mathrm{d}\sigma_0(x) \le \sum_{j \ne i} \int_{D_j(r)} \|x - x_j\|^{-d+1}\,\mathrm{d}\sigma_0(x)$$
$$= \gamma_d N\left[-\log(rN^{-1/(d-1)})\right] + O(N) = \frac{\gamma_d}{d-1} N \log N + O(N).$$

Since Λ_N minimizes the $(d-1)$-energy, the function U_i attains its minimum at the point x_i. Therefore, using Eq. (14.4.15), we obtain

$$U_i(x_i) \le \frac{1}{\sigma_0(D(r))} \int_{D(r)} U_i(x)\,\mathrm{d}\sigma_0(x) \le \frac{1}{1 - c_d r^{d-1}} \frac{\gamma_d}{d-1} N \log N + O(N),$$

where $c_d := \frac{\gamma_d}{d-1}$, which implies that

$$\mathscr{E}_{d-1}(\mathbb{S}^{d-1}, N) = \frac{1}{2} \sum_{i=1}^{N} U_i(x_i) \le \frac{1}{1 - c_d r^{d-1}} \frac{\gamma_d}{2(d-1)} N^2 \log N + O(N^2).$$

Letting $r \to 0^+$ yields the desired upper estimate (14.4.14). $\square$

14.5 Computerized Tomography

Computerized tomography (CT) offers a noninvasive method for 2D cross-sectional or 3D imaging of an object; it has a wide range of applications, including diagnostic medicine and industrial material testing and inspection. A typical CT application consists of two steps. The first step is acquisition of data. The energy of an x-ray will diminish when the x-ray passes through an object being irradiated; the measurement of the amount by which the energy diminishes is the x-ray data, which depends on the density of the object that the x-ray encounters. The second step is reconstruction, which means applying an algorithm to the x-ray data to recover the density, or the image, of the object. Mathematically, each x-ray is represented by a Radon projection of a function f, representing the density of the object, defined by

$$\mathscr{R}f(\theta, t) := \int_{I(\theta,t)} f(x,y)\,\mathrm{d}\ell, \qquad 0 \le \theta \le 2\pi, \quad -1 \le t \le 1, \tag{14.5.1}$$

where f is scaled so that the object is within the disk $\mathbb{B}^2$, $I(\theta,t) = \{(x,y) : x\cos\theta + y\sin\theta = t\} \cap \mathbb{B}^2$ is a line segment inside $\mathbb{B}^2$, and $d\ell$ denotes the Lebesgue measure on the line. Thus, the image reconstruction means solving the inverse problem of recovering a function f from its Radon projections. If continuous data are given, that is, if projections for all t and θ are known, then the solution to the inverse problem of finding f from its Radon projections was solved by Radon in 1917. In practice, however, only finitely many projections can be measured. Hence, the essential problem of CT is to find an effective algorithm that produces a good approximation to f based on a finite number of Radon projections.

In this section we discuss an effective algorithm, called OPED, based on orthogonal polynomial expansions on the disk. We start with the following simple relation.

Lemma 14.5.1. *For $f \in L^1(\mathbb{B}^2)$, $g \in C[-1,1]$, and $\xi = (\cos\phi, \sin\phi)$,*

$$\int_{\mathbb{B}^2} f(x) g(\langle x, \xi\rangle) \mathrm{d}x = \int_{-1}^{1} \mathscr{R}_\phi(f;t) g(t) \mathrm{d}t. \tag{14.5.2}$$

Proof. The points on the line segment $I(\theta,t)$ can be represented by

$$x_1 = t\cos\theta - s\sin\theta, \quad x_2 = t\sin\theta + s\cos\theta,$$

for $s \in \left[-\sqrt{1-t^2}, \sqrt{1-t^2}\right]$, so that the Radon projection can be written as

$$\mathscr{R}_\theta(f;t) = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta) \mathrm{d}s. \tag{14.5.3}$$

The change of variables $x_1 = t\cos\phi - s\sin\phi$ and $x_2 = t\sin\phi + s\cos\phi$ amounts to a rotation, which leads to

$$\begin{aligned} \int_{\mathbb{B}^2} f(x) g(\langle x, \xi\rangle) \mathrm{d}x &= \int_{\mathbb{B}^2} f(t\cos\phi - s\sin\phi, t\sin\phi + s\cos\phi) g(t) \mathrm{d}t \mathrm{d}s \\ &= \int_{-1}^{1} \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\phi - s\sin\phi, t\sin\phi + s\cos\phi) \mathrm{d}s g(t) \mathrm{d}t. \end{aligned}$$

The inner integral is precisely $\mathscr{R}_\phi(f;t)$. □

Recall that the Chebyshev polynomial of the second kind, $U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}$, $t = \cos\theta$, satisfies the orthonormal relation

$$\frac{2}{\pi} \int_{-1}^{1} U_m(x) U_n(x) \sqrt{1-x^2} \mathrm{d}x = \delta_{m,n}, \quad m,n \in \mathbb{N}_0.$$

The following lemma gives the Radon projection of polynomials in $\mathscr{V}_k(\mathbb{B}^2)$, the space of orthogonal polynomials with respect to the constant weight on the disk.

Lemma 14.5.2. *If $P \in \mathscr{V}_k(\mathbb{B}^2)$, then for each $t \in (-1,1)$, $0 \le \theta \le 2\pi$,*

$$\mathscr{R}_\theta(P;t) = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)P(\cos\theta,\sin\theta). \tag{14.5.4}$$

Proof. A change of variables in Eq. (14.5.3) shows that

$$\mathscr{R}_\theta(P;t) = \sqrt{1-t^2}\int_{-1}^{1} P\left(t\cos\theta - s\sqrt{1-t^2}\sin\theta, t\sin\theta + s\sqrt{1-t^2}\cos\theta\right)\mathrm{d}s.$$

The integral is a polynomial in t, since an odd power of $\sqrt{1-t}$ in the integrand is always accompanied by an odd power of s, which has integral zero. Therefore, $Q(t) := \mathscr{R}_\theta(P;t)/\sqrt{1-t^2}$ is a polynomial of degree k in t for every θ. Furthermore, the integral also shows that $Q(1) = P(\cos\theta,\sin\theta)$. By Eq. (14.5.2),

$$\int_{-1}^{1}\frac{\mathscr{R}_\theta(P;t)}{\sqrt{1-t^2}}U_j(t)\sqrt{1-t^2}\mathrm{d}t = \int_{\mathbb{B}^2}P(x)U_j(x_1\cos\theta + x_2\sin\theta)\mathrm{d}x = 0,$$

for $j = 0,1,\dots,k-1$, since $P \in \mathscr{V}_k(\mathbb{B}^2)$. Since Q is of degree k, we conclude that $Q(t) = cU_k(t)$ for some constant independent of t. Setting $t = 1$ and using the fact that $U_k(1) = k+1$, we have $c = P(\cos\theta,\sin\theta)/(k+1)$. □

As shown in the identity (14.5.4), the Chebyshev polynomial U_k plays an important role in our discussion below. For convenience, we define

$$U_k(\theta;x) := U_k(x_1\cos\theta + x_2\sin\theta), \quad 0 \le \theta \le \pi, \quad x = (x_1,x_2) \in \mathbb{B}^2.$$

Setting $f(x) = U_k(x_1\cos\theta + x_2\sin\theta)$ in Eq. (14.5.3) and using Eq. (14.5.4), we derive from the orthogonality of the Chebyshev polynomials that

$$\frac{1}{\pi}\int_{\mathbb{B}^2}U_k(\theta;x)U_k(\phi;x)\mathrm{d}x = \frac{1}{k+1}U_k(\cos(\phi-\theta)), \tag{14.5.5}$$

which is a special case of Eq. (11.1.18). Recall that the zeros of U_k are $\cos j\pi/(k+1)$, $1 \le j \le k$. The above identity implies that

$$\left\{U_k\left(\tfrac{j\pi}{k+1};x\right) : 0 \le j \le k\right\}$$

is an orthonormal basis of $V_k(\mathbb{B}^2)$, which has already appeared in Theorem 11.1.11. Using this orthonormal basis, the reproducing kernel $P_k(\cdot,\cdot)$ of $\mathscr{V}_k(\mathbb{B}^2)$ can be written as

$$P_k(x,y) = \sum_{j=0}^{k}U_k(\theta_{j,k};x)U_k(\theta_{j,k};y) \qquad \theta_{j,k} = \tfrac{j\pi}{k+1}. \tag{14.5.6}$$

This kernel satisfies an integral expression given in Eq. (11.1.16). For our purpose, however, the following formula of this kernel is more useful.

Lemma 14.5.3. *The reproducing kernel $P_k(\cdot,\cdot)$ of $\mathscr{V}_k(\mathbb{B}^2)$ satisfies*

$$P_k(x,y) = \frac{k+1}{2\pi}\int_{\mathbb{S}^1} U_k(\langle x,\xi\rangle)U_k(\langle y,\xi\rangle)\mathrm{d}\sigma(\xi).$$

Proof. From the elementary identity $\sin(k+1)\theta - \sin(k-1)\theta = 2\cos k\theta\sin\theta$, it follows readily that $U_k(\cos\theta) = U_{k-2}(\cos\theta) + 2\cos k\theta$, which implies

$$U_k(\cos\theta) = 2\sum_{0\le j\le (k-1)/2} \cos(k-2j)\theta + \tau_k, \tag{14.5.7}$$

where $\tau_k = 1$ if k is even and $\tau_k = 0$ if k is odd. Using the addition formula of the cosine function, a simple computation shows that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos i\theta\cos j(\theta-\phi)\mathrm{d}\theta = \delta_{i,j}\cos j\phi.$$

Expanding both $U_k(\cos\theta)$ and $U_k(\cos(\theta-\theta_{j,k}))$ according to Eq. (14.5.7) and using the above integral relation, it is not hard to verify that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U_k(\cos\theta)U_k(\cos(\phi-\theta))\mathrm{d}\theta = U_k(\cos\phi).$$

In particular, by the periodicity and the definition of $\theta_{j,k}$, it then follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U_k(\cos(\theta-\theta_{i,k}))U_k(\cos(\theta-\theta_{j,k}))\mathrm{d}\theta = (k+1)\delta_{i,j}. \tag{14.5.8}$$

Now, by Eq. (11.1.16) with $\mu = 1/2$ and $d = 2$, for $\xi \in \mathbb{S}^1$, the polynomial $x \mapsto U_k(\langle x,\xi\rangle)$ is an element in $\mathscr{V}_k(\mathbb{B}^2)$, so that it can be written as a linear combination of the orthonormal basis $\{U_k(\theta_{j,k};\xi) : 0 \le j \le k\}$ of $\mathscr{V}_k(\mathbb{B}^2)$, which gives, by Eq. (14.5.5),

$$U_k(\langle x,\xi\rangle) = \frac{1}{k+1}\sum_{j=0}^{k} U_k(\theta_{j,k};\xi)U_k(\theta_{j,k};x).$$

Writing $\xi = (\cos\theta,\sin\theta) \in \mathbb{S}^1$, we have $U_k(\theta_{j,k};\xi) = U_k(\cos(\theta-\theta_{j,k}))$. Consequently, by Eq. (14.5.8), we conclude that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{S}^1} U_k(\langle x,\xi\rangle)U_k(\langle y,\xi\rangle)\mathrm{d}\sigma(\xi)\\
&\quad= \frac{1}{(k+1)^2}\sum_{j=0}^{k}\sum_{i=0}^{n} U_n(\theta_{j,k};x)U_n(\theta_{i,k};y)\frac{1}{2\pi}\int_{\mathbb{S}^1} U_k(\theta_{j,k};\xi)U_k(\theta_{i,k};\xi)\mathrm{d}\sigma(\xi)\\
&\quad= \frac{1}{k+1}\sum_{j=0}^{k} U_n(\theta_{j,k};x)U_k(\theta_{j,k};y) = P_k(x,y),
\end{aligned}$$

where the last equal sign follows from Eq. (14.5.6). □

Recall that the reproducing kernel $P_k(x,y)$ is the kernel function of $\operatorname{proj}_k f$. The following lemma establishes the connection between the Radon projections and the orthogonal expansions.

Theorem 14.5.4. *For $f \in L^1(\mathbb{B}^2)$ and $n = 0,1,2,\ldots$,*

$$\operatorname{proj}_k f(x) = \frac{k+1}{2\pi^2} \int_{\mathbb{S}^1} \int_{-1}^{1} \mathscr{R}_\theta(f,t) U_k(t) \mathrm{d}t \, U_k(\langle x, \xi \rangle) \mathrm{d}\sigma(\xi).$$

Proof. By the integral representation of $\operatorname{proj}_n f$ and Lemma 14.5.3, we obtain, on changing the order of integration,

$$\begin{aligned} \operatorname{proj}_k f(x) &= \frac{1}{\pi} \int_{\mathbb{B}^2} f(y) P_k(x,y) \mathrm{d}y \\ &= \frac{k+1}{2\pi^2} \int_{-\pi}^{\pi} \left[\int_{\mathbb{B}^2} f(y) U_k(\langle y, \xi \rangle) \mathrm{d}y \right] U_k(\langle x, \xi \rangle) \mathrm{d}\sigma(\xi). \end{aligned}$$

Applying Eq. (14.5.2) to the inner integral gives the desired formula. □

Recall that the partial sum operator of the orthogonal expansion is given by

$$S_n f(x) = \sum_{k=0}^{n} \operatorname{proj}_k f(x).$$

We deduce immediately the following expression for $S_n f$.

Corollary 14.5.5. *For $n = 0,1,2,\ldots$, the partial sum $S_n f$ can be written as*

$$S_n f(x) = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-1}^{1} \mathscr{R}_\theta(f,t) \Phi_n(t; x_1 \cos\theta + x_2 \sin\theta) \mathrm{d}t \mathrm{d}\theta,$$

where the function Φ_n is defined by

$$\Phi_n(t;u) = \sum_{k=0}^{n} (k+1) U_k(t) U_k(u).$$

Thus, the partial sum $S_n f$ can be expressed as a double integral of the Radon projections. More interesting is the following corollary, which expresses $S_n f$ in terms of semidiscrete Radon projections.

Corollary 14.5.6. *For $n \in \mathbb{N}_0$, let N be a positive integer such that $N \geq 2n$. Then*

$$S_n f(x) = \frac{1}{\pi N} \sum_{k=0}^{N-1} \int_{-1}^{1} \mathscr{R}_{\frac{2k\pi}{N}}(f,t) \Phi_n\left(t; x_1 \cos\tfrac{2k\pi}{N} + x_2 \sin\tfrac{2k\pi}{N}\right) \mathrm{d}t. \tag{14.5.9}$$

Proof. Let $\xi = (\cos\theta, \sin\theta)$. By Eq. (14.5.2), for each fixed x, the integral

$$\int_{-1}^{1} \mathscr{R}_\theta(f,t) \Phi_n(t; x_1 \cos\theta + x_2 \sin\theta) \mathrm{d}t = \int_{\mathbb{B}^2} f(y) \Phi_n(\langle y, \xi \rangle; \langle x, \xi \rangle) \mathrm{d}y$$

is a trigonometric polynomial of degree $2n$ in the variable θ, so that the integral of this polynomial over $[-\pi,\pi]$ can be replaced by the quadrature formula (6.1.6) of N points whenever $N \ge 2n$, since the quadrature formula is exact for all trigonometric polynomials of degree N. Hence, Eq. (14.5.9) follows from the expression for $S_n f$ in Corollary 14.5.5. □

In the case that n is an even integer, we can substantially reduce the number of Radon projections in the representation (14.5.9), as in the following theorem.

Theorem 14.5.7. *For $n \in \mathbb{N}$, let $\phi_{\nu,n} := \frac{2\pi\nu}{n+1}$ for $0 \le \nu \le n$. If n is an even integer, then*

$$S_n f(x) = \frac{1}{\pi(n+1)} \sum_{\nu=0}^{n} \int_{-1}^{1} \mathscr{R}_{\phi_{\nu,n}}(f,t) \Phi_n\left(t; x_1 \cos\phi_{\nu,n} + x_2 \sin\phi_{\nu,n}\right) \mathrm{d}t. \quad (14.5.10)$$

Proof. This comes from the observation that if $\xi = (\cos\theta, \sin\theta)$, then the function $U_k(\langle x,\xi\rangle)U_k(\langle y,\xi\rangle)$ is an even trigonometric polynomial of degree $2k$ in θ modulo a trigonometric polynomial of degree k. Indeed, if $x = r(\cos\phi, \sin\phi)$, then it is easy to see that

$$\begin{aligned} U_k(\langle x,\xi\rangle) = U_k(r\cos(\theta-\phi)) &= \sum_{0\le j\le k/2} b_j r^{k-2j} (\cos(\theta-\phi))^{k-2j} \\ &= \sum_{0\le j\le (k-1)/2} b_j(r) \cos(k-2j)(\theta-\phi) + \tau_k(r), \end{aligned}$$

where $\tau_k(r) = 0$ if k is odd, which implies, by the product formula of the cosine, that $U_k(\langle x,\xi\rangle)U_k(\langle y,\xi\rangle)$ has the desired property. If n is even, then the quadrature formula (6.1.6) with $n+1$ nodes is exact for all trigonometric polynomials $P(\theta)$ that are even in θ, in addition to all trigonometric polynomials of degree $n+1$, as the proof of Proposition 6.1.5 shows. Applying this quadrature to the representation of the reproducing kernel $P_k(x,y)$ in Lemma 14.5.3, we conclude that

$$P_k(x,y) = \frac{k+1}{n+1} \sum_{\nu=0}^{n} U_k(x_1\cos\phi_{\nu,n} + x_2 \sin\phi_{\nu,n}) U_k(y_1 \cos\phi_{\nu,n} + y_2 \sin\phi_{\nu,n})$$

for $0 \le k \le n$, from which Eq. (14.5.10) follows as in the proof of Theorem 14.5.4. □

By its definition, $S_n f$ is the best approximation to f from Π_n^2 in the $L^2(\mathbb{B}^2)$ metric. We have just proved that it can be expressed in terms of Radon projections in finite directions. For a reconstruction algorithm, we need an approximation process based on the finite Radon data. We can discretize the integral over $[-1,1]$ by the Gaussian quadrature formula for the Chebyshev weight function of the second kind, given in Proposition 11.6.7 with $\alpha = \beta = 1/2$, which states that

$$\frac{2}{\pi} \int_{-1}^{1} g(t)\sqrt{1-t^2}\mathrm{d}t = \frac{1}{M+1} \sum_{j=1}^{M} \sin^2 \frac{j\pi}{M+1} g\left(\cos\frac{j\pi}{M+1}\right) \quad (14.5.11)$$

for all polynomials g of degree at most $2M-1$. We state in the following definition a particular choice of $M=n$ that results in an approximation process $\mathscr{A}_n f$, based on the finite Radon data.

Definition 14.5.8. Let $n \in \mathbb{N}_0$ be an even integer. Let $\phi_{k,n} = \frac{2\pi k}{n+1}$ and $\psi_{j,n} := \frac{j\pi}{n+1}$. Define

$$\mathscr{A}_n f(x) := \frac{1}{n+1} \sum_{\nu=0}^{n} \sum_{k=0}^{n} \lambda_{k,\nu} (k+1) U_k(x\cos\phi_{\nu,n} + y\sin\phi_{\nu,n}), \tag{14.5.12}$$

where

$$\lambda_{k,\nu} := \frac{1}{2(n+1)} \sum_{j=1}^{n} \sin((k+1)\psi_{j,n}) \mathscr{R}_{\phi_{\nu,n}}(f, \cos\psi_{j,n}). \tag{14.5.13}$$

The function $\mathscr{A}_n f$ is a discretization of $S_n f$ and can be used to reconstruct the function f from the Radon data. This is the basis of the OPED reconstruction algorithm. It has the following remarkable property.

Theorem 14.5.9. *The operator $\mathscr{A}_n$ in the OPED algorithm preserves polynomials of degree $n-1$. More precisely, $\mathscr{A}_n(f) = f$ whenever $f \in \Pi^2_{n-1}$.*

Proof. This result comes from the construction of $\mathscr{A}_n f$. If f is a polynomial of degree at most $n-1$, then so is $\mathscr{R}_{\phi_\nu}(f;t)/\sqrt{1-t^2}$ for every ν by Lemma 14.5.2. Furthermore, the polynomial $\Phi_\nu(t;x)$ is a polynomial of degree n in t. The product of the two then has degree $2n-1$. Applying the quadrature formula (14.5.11) with $M=n$ to the product of these two polynomial functions in Eq. (14.5.10), we obtain a discretization of $S_n f$, which is, after rearrangement, $\mathscr{A}_n f$. The discretization is exact if f is a polynomial of degree at most $n-1$, so that $\mathscr{A}_n f(x) = S_n f(x) = f(x)$ in this case. □

If we choose $M=n+1$ in the quadrature formula (14.5.11), so that it is exact for all polynomials of degree $2n+1$, we would end up with a slightly different operator $\mathscr{A}_n$ that will preserve all polynomials of degree up to n instead of $n-1$. By choosing $M=n$, however, the angle ϕ_ν and the angle $\frac{j\pi}{2n+1}$ of t_j have the same denominator, which can be used to facilitate the computation. Another reason for choosing $M=n$ lies in the scanning geometry, which stands for the layout of the x-ray lines used in the reconstruction algorithm, since it is determined by the design of the scanner used for data acquisition. The fan geometry, in which the x-ray source emit x-rays in fans, is preferred for faster data collection, where parallel geometry, in which the x-rays are groups of parallel rays, is often called for in the reconstruction algorithm. The data set in the OPED algorithm in Definition 14.5.8 can be collected via fan data, and it can be reordered into parallel data, as seen in Fig. 14.1, in which the black bullets on the circumference denote the positions where the x-ray source emits the x-rays, and the small circles on the circumference denote the positions of the detectors that collect the data. The thick lines denote one group of parallel rays.

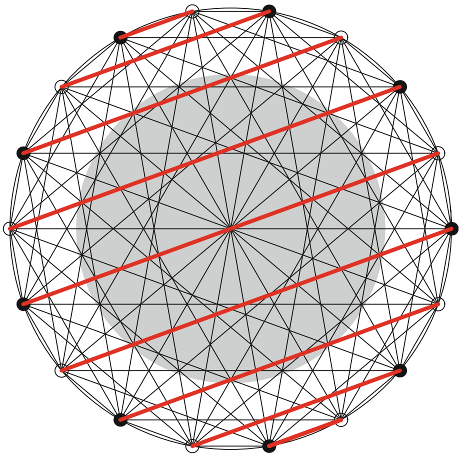

Fig. 14.1 Scanning geometry for OPED data with $n = 8$

The fact that $\mathscr{A}_n f$ preserves polynomials of degree $n-1$ means that if the image is represented by a polynomial of degree $n-1$, then $\mathscr{A}_n f$ recovers the image exactly. Since the algorithm is often used for $n = 512$ or more, this suggests that $\mathscr{A}_n f$ should have a favorable approximation behavior. Indeed, we have the following quantitative result.

Theorem 14.5.10. *The operator norm* $\|\mathscr{A}_n\|_\infty$ *of* $\mathscr{A}_n : C(\mathbb{B}^2) \mapsto C(\mathbb{B}^2)$ *satisfies*

$$\|\mathscr{A}_n\|_\infty \sim n \log n. \tag{14.5.14}$$

The proof of this result is quite involved, and we refer the reader to the original paper; see Sect. 14.6. The estimate (14.5.14) should be compared with the fact that $\|S_n\|_\infty \sim n$, a special case of Theorem 11.4.1, which shows that discretizing with Gaussian quadrature increases the norm by only a benign $\log n$ factor.

Corollary 14.5.11. *If* $f \in C^r(\mathbb{B}^2)$, *then* $\mathscr{A}_n f$ *in the OPED algorithm satisfies*

$$\|f - \mathscr{A}_n f\|_\infty \le c n \log n E_{n-1}(f)_\infty \le c_f \frac{\log n}{n^{r-1}}.$$

Proof. Since $A_n f$ preserves polynomials, the estimate follows from the triangle inequality and Corollary 12.2.13. □

To apply the OPED algorithm to reconstruct an image, we need to evaluate $\mathscr{A}_n f(x)$ on a grid of $N \times N$ points, for example, $N = 256$ or $N = 512$. Using the fast Fourier transform (FFT), the matrix of $\lambda_{j,\nu}$, $0 \le j, \nu \le n$, can be evaluated at the cost of $n^2 \log n$ operations, which can be computed independently. The double sum in Eq. (14.5.12) costs about n^2 operations. Hence, evaluation of $A_n f$ over an $N \times N$ grid requires about $n^2 \times N^2$ operations, which is about n^4 operations if $N \approx n$. There is, however, a fast implementation that can reduce the operation to about n^3

computations; see the notes at the end of the chapter. Numerical computation has shown that the OPED algorithm is an efficient, stable, and accurate algorithm for image reconstruction.

14.6 Notes and Further Results

Section 14.1: The theory of frames and tight frames attracted considerable attention with the spread of wavelet theory. For the theory of frames and recent developments and applications, we refer to [31,79] and the references therein. The main references for this section are [39, 129]. Many function spaces, such as L^p, H^p, the Besov spaces, and the Triebel–Lizorkin spaces on the sphere, can be characterized in terms of the coefficients in frame expansions (see [39, 128]). Nonlinear m-term approximation by polynomial frames was also studied in [39,128]. We refer also to [121] for polynomial frames on $\mathbb{S}^{d-1}$. The elements of the tight frame in Eq. (14.1.5) are called *needlets* in [129] in view of their highly localized construction.

Tight polynomial frames can be constructed in many other domains. In fact, the construction works in a fairly general setup. Below, we give an outline, in which notation that is not explicitly defined is mostly, we believe, self-explanatory. Let (E,μ) be a measure space and assume that there is an orthogonal decomposition $L^2(E,\mu) = \bigoplus_{n=0}^{\infty}\mathscr{V}_n$, where $\mathscr{V}_n$ are finite-dimensional subspaces. Let P_n be the kernel of the orthogonal projection $\mathrm{proj}_n : L^2(E,\mu) \to \mathscr{V}_n$, i.e.,

$$(\mathrm{proj}_n f)(x) = \int_E P_n(x,y) f(y)\mathrm{d}\mu(y), \quad f \in L^2(E,\mu).$$

Define the analogue of Eq. (14.1.3) in terms of the kernel P_n,

$$G_0(x,y) := P_0(x,y) \quad \text{and} \quad G_j(x,y) := \sum_{\nu=0}^{\infty} \phi\Big(\frac{\nu}{2^{j-1}}\Big) P_\nu(x,y), \quad j = 1,2,\ldots,$$

and define $(L_j * f)(x) := \int_E L_j(x,y) f(y)\mathrm{d}\mu(y)$ for brevity. Then the following decomposition follows readily from the conditions on ϕ:

$$f = \sum_{j=0}^{\infty} L_j * (L_j * f) \quad \text{for } f \in L^2(E,\mu). \tag{14.6.1}$$

If there exists a cubature formula with nodes $x_{j,k}$ and coefficients $\lambda_{j,k}$ for the integral $\int_E f\mathrm{d}\mu$ and functions $f = gh$, where $g,h \in \bigoplus_{m=0}^{2^{j+6}}\mathscr{V}_m$, then we can discretize the right-hand side of Eq. (14.6.1) to write

$$f(x) = \sum_{j=0}^{\infty} \sum_{k\in\Lambda_j^d} \sqrt{\lambda_{j,k}} L_j(x_{j,k},x) \int_E f(y)\sqrt{\lambda_{j,k}} L_j(x_{j,k},y)\mathrm{d}y.$$

Thus, if we define $\psi_{j,k}(x) := \sqrt{\lambda_{j,k}} L_j(x_{j,k}, x)$, then it readily follows that

$$f = \sum_{j=0}^{\infty} \sum_{k \in \Lambda_j^d} \langle \psi_{j,k}, f \rangle \psi_{j,k} \quad \text{and} \quad \|f\|_{L^2(E,\mu)} = \Big(\sum_{j=0}^{\infty} \sum_{k \in \Lambda_j^d} |\langle f, \psi_{j,k} \rangle|^2 \Big)^{\frac{1}{2}}.$$

Following the above outline, tight polynomial frames were constructed and studied in several other domains, including the unit ball and the simplex with the Jacobi weight functions, as well as $\mathbb{R}^d$ with the Hermite weight and $\mathbb{R}_+^d$ with the Laguerre weight; see [91, 138, 139] and the references in [91].

Section 14.2: Materials in this section were selected from the paper [172], which contains several other interesting results on the distribution of points of a spherical design. It is shown in [73, 170, 172] that if Λ is a spherical n-design on $\mathbb{S}^{d-1}$, then

$$\mathbb{S}^{d-1} \subset \bigcup_{\eta \in \Lambda} \mathsf{c}(\eta, \arccos t_n),$$

where t_n is the largest root of the following algebraic polynomial on $[-1, 1]$:

$$Q_n(t) := \begin{cases} P_k^{(\frac{d-3}{2}, \frac{d-3}{2})}(t), & \text{if } n = 2k-1, \\ P_k^{(\frac{d-3}{2}, \frac{d-1}{2})}(t), & \text{if } n = 2k, \end{cases}$$

where $P_k^{(\alpha,\beta)}$ is the Jacobi polynomial. The same conclusion remains true if Λ is the set of points of an arbitrarily given positive cubature formula of degree n on $\mathbb{S}^{d-1}$.

Section 14.3: The study of positive definite functions on the sphere was initiated by I.J. Schoenberg in his classical paper [150], which contains Theorem 14.3.3 and several other results.

The strictly positive definite functions were first studied in [193], motivated by the problem of interpolation on the sphere, where the addition formula of Eq. (14.3.6) was used to show that f is strictly positive definite if all $\hat{f}_n$ are positive. The requirement that $\mathscr{F}$ contain infinitely many even and infinitely many odd integers was first recognized in [118], where the necessity in Theorem 14.3.4 was established. The sufficiency was established in [32].

The strict positive definiteness of $f_{\theta,\delta}$, defined in Eq. (14.3.8), was established in [12] for $d = 3, 4, \ldots, 8$, while Proposition 14.3.8 contains the case $d = 3, 4$, and it was conjectured to hold for all $d \geq 3$. A stronger conjecture is as follows: For $\delta, \lambda > 0$ and $n \in \mathbb{N}_0$, define

$$F_n^{\lambda,\delta}(\theta) = \int_0^{\theta} (\theta - \phi)^{\delta} C_n^{\lambda}(\cos\phi)(\sin\phi)^{2\lambda} \, d\phi, \quad 0 < \theta < \pi. \tag{14.6.2}$$

Then $F_n^{\lambda,\delta}(\theta) > 0$ for all θ in $(0, \pi]$ if and only if $\delta \geq \lambda + 1$. The function $f_{\theta,\delta}$ plays a critical role in the proof of Theorem 14.3.9, which was established in [12] and was shown to hold for higher dimensions whenever the conjecture on $f_{\theta,\delta}$ is affirmative. In the case $d = 2$, the circle, an analogous result was given in [12]; see also [76]. For further results in this direction, see the recent survey [77]. The conditions in Theorem 14.3.9 that warrant the positive (strictly positive) definiteness of a function f are called Pólya's criterion, because of their similarity to the criterion that Pólya developed for functions with nonnegative Fourier transforms.

The strictly positive definite functions can be used for scattered data interpolation. The error of such interpolation has been studied by many authors; see, for example, [92, 127]. Such interpolation is closely related to interpolation by radial basis functions on $\mathbb{R}^d$, for which see [25, 175]. There are other tools for scattered data interpolation on the sphere; see, for example, [72].

Section 14.4: Most of the material in this section was selected from [99]. An improvement of Theorem 14.4.6 was obtained in [81], where it was shown that if $s > d-1$, then the limit

$$\lim_{N\to\infty} \frac{\mathscr{E}_s(\mathbb{S}^{d-1},N)}{N^{1+\frac{s}{d-1}}}$$

exists, and any extremal s-energy configurations are asymptotically uniformly distributed on $\mathbb{S}^{d-1}$, whereas it remains open what this limit in fact is. For the sphere $\mathbb{S}^2$, the following lower estimate is proved in [99] for $s > 2$:

$$\limsup_{N\to\infty} \frac{\mathscr{E}_s(\mathbb{S}^2,N)}{N^{1+s/2}} \le \frac{1}{2}\left(\frac{\sqrt{3}}{8\pi}\right)^{s/2} \zeta_L(s), \tag{14.6.3}$$

where

$$\zeta_L(s) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m^2+mn+n^2)^{-s/2}, s > 2,$$

which is the zeta function for the hexagonal lattice L consisting of points of the form $m(1,0)+n(1/2,\sqrt{3}/2)$ for $m,n \in \mathbb{Z}$. It is conjectured in [99] that equality holds in Eq. (14.6.3). It is shown in [100] that the minimum s-energy points on $\mathbb{S}^{d-1}$ are well separated for the case $d-2 \le s < d-1$ as well. Namely, the conclusion of Corollary 14.4.7 remains true when $d-2 \le s < d-1$. For further results, see the recent survey [22].

Section 14.5: Computerized tomography has a wide range of applications and is covered by many books and articles, from theoretical to computational to practical. For further discussion on Radon transforms, we refer to [52, 83], the first of which contains a translation of Radon's 1917 paper that started the investigation into the transform that bears his name. For the aspect of reconstruction of images from x-ray data, we refer to [86, 93].

The most commonly used reconstruction algorithm is the FBP (filtered back-projection) algorithm, based on the relationship between the Fourier transform and the Radon transform, which has been under intense study for many decades. The OPED algorithm was formulated much more recently in [191]. It is based on orthogonal expansions and has a neat mathematical formulation. The use of orthogonal expansion for reconstruction appeared already in the landmark paper of Cormack [35] that started the era of CT. It was also used in [110, 113, 114] in connection with tomography. In particular, Lemma 14.5.2 was proved in [114] and was used in [110] for a reconstruction method that uses orthogonal expansion on the disk. That the partial sum operator can be expressed as semidiscrete Radon data appeared more recently in [191], where the OPED algorithm was proposed and Theorem 14.5.10 and the convergence of the OPED algorithm were established. The fast implementation was proposed and analyzed in [194] and further accelerated in [195]. For further features of this algorithm and its implementations, see [168] and the references therein.

There is a $\mathbb{B}^d$ version of the integral formula in Lemma 14.5.3; see [136, 192]. However, it is difficult to use the formula to derive a 3D reconstruction algorithm on the ball $\mathbb{B}^3$, since it requires a good cubature formula for integrals on $\mathbb{S}^2$. For 3D imaging, it is much easier to consider a cylindrical domain, as proposed in [191]; the analogue of Theorem 14.5.10 and the convergence theorem on the cylinder were established in [173]. There is a close relationship between singular value decomposition (SVD) of the Radon transform and orthogonal expansions, and the truncated SVD can be effectively implemented by the OPED algorithm; see [192].

Appendix A
Distance, Difference and Integral Formulas

A.1 Distance on Spheres, Balls and Simplexes

The distance in $\mathbb{R}^d$ is the Euclidean distance denoted as usual by $\|x-y\|$. We define the distance functions for spheres, balls and simplexes below.

Distance on the sphere: For the sphere $\mathbb{S}^{d-1}$, it is more convenient to use the geodesic distance defined by

$$\mathtt{d}(x,y) := \arccos \langle x,y \rangle, \qquad x,y \in \mathbb{S}^{d-1},$$

which is the distance between x and y on the largest circle on $\mathbb{S}^{d-1}$ that passes through x and y. Evidently, $0 \le \mathtt{d}(x,y) \le \pi$. For $x,y \in \mathbb{S}^{d-1}$, the two distances are comparable. Indeed, if $x,y \in \mathbb{S}^{d-1}$, then

$$\|x-y\| = \sqrt{2-2\langle x,y \rangle} = \sqrt{2-2\cos \mathtt{d}(x,y)} = 2\sin \frac{\mathtt{d}(x,y)}{2},$$

which implies, by an elementary inequality of the sine function, that

$$\frac{2}{\pi}\mathtt{d}(x,y) \le \|x-y\| \le \mathtt{d}(x,y). \tag{A.1.1}$$

Distance on the ball: For the unit ball $\mathbb{B}^d$ of $\mathbb{R}^d$, the distance is defined by

$$\mathtt{d}_{\mathbb{B}}(x,y) := \arccos \left\{ \langle x,y \rangle + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2} \right\}.$$

This distance is deduced from the geodesic distance on the hemisphere $\mathbb{S}^d_+ := \{x \in \mathbb{S}^d : x_{d+1} \ge 0\}$ of $\mathbb{R}^{d+1}$ by the bijection

$$x \in \mathbb{B}^d \mapsto x' := \left(x, \sqrt{1-\|x\|^2} \right) \in \mathbb{S}^d_+, \tag{A.1.2}$$

F. Dai and Y. Xu, *Approximation Theory and Harmonic Analysis on Spheres and Balls*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-6660-4,

and hence it is a true distance on $\mathbb{B}^d$. It is a more suitable distance for the unit ball than the Euclidean distance, since it takes into account the difference between the points inside the ball and those near the boundary.

The following lemma provides an important relation between $\mathtt{d}_{\mathbb{B}}(\cdot,\cdot)$ and the Euclidean norm $\|\cdot\|$ in $\mathbb{B}^d$.

Lemma A.1.1. *For $x,y \in \mathbb{B}^d$, we have*

$$\big|\|x\| - \|y\|\big| \le \frac{1}{\sqrt{2}}\mathtt{d}_{\mathbb{B}}(x,y)\left(\sqrt{1-\|x\|^2} + \sqrt{1-\|y\|^2}\right) \tag{A.1.3}$$

and

$$\left|\sqrt{1-\|x\|^2} - \sqrt{1-\|y\|^2}\right| \le \mathtt{d}_{\mathbb{B}}(x,y). \tag{A.1.4}$$

Proof. Let $0 \le \alpha, \beta \le \pi/2$ be defined from $\|x\| = \cos\alpha$ and $\|y\| = \cos\beta$. Using spherical–polar coordinates $x = \|x\|\xi$ and $y = \|y\|\zeta$, where $\xi, \zeta \in \mathbb{S}^{d-1}$, we see that

$$\mathtt{d}_{\mathbb{B}}(x,y) = \arccos\left(\cos\alpha\cos\beta\langle\xi,\zeta\rangle + \sin\alpha\sin\beta\right) \ge \arccos(\cos(\alpha-\beta)),$$

which yields $\mathtt{d}_{\mathbb{B}}(x,y) \ge |\alpha - \beta|$. On the other hand, since $0 \le \alpha, \beta \le \pi/2$, we have $\cos\frac{\alpha-\beta}{2} \ge \cos(\pi/4) = \sqrt{2}/2$, and consequently,

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \ge \sqrt{2}\sin\frac{\alpha+\beta}{2}.$$

Using the above, we obtain

$$\begin{aligned}\big|\|x\| - \|y\|\big| &= |\cos\alpha - \cos\beta| = 2\sin\frac{|\alpha-\beta|}{2}\sin\frac{\alpha+\beta}{2}\\ &\le \frac{1}{\sqrt{2}}|\alpha-\beta|(\sin\alpha + \sin\beta) \le \frac{1}{\sqrt{2}}\mathtt{d}_{\mathbb{B}}(x,y)(\sqrt{1-\|x\|^2} + \sqrt{1-\|y\|^2}).\end{aligned}$$

Thus (A.1.3) is established. The estimate (A.1.4) follows immediately from (A.1.1). □

Distance on the simplex: The simplex $\mathbb{T}^d$ is another domain that has boundary, on which the distance is defined by

$$\mathtt{d}_{\mathbb{T}}(x,y) := \arccos\left\{\sqrt{x_1y_1} + \cdots + \sqrt{x_dy_d} + \sqrt{x_{d+1}y_{d+1}}\right\},$$

where $x_{d+1} = 1 - x_1 - \cdots - x_d$ and $y_{d+1} = 1 - y_1 - \cdots - y_d$. This distance is deduced from the distance on $\mathbb{B}^d_+ := \{x \in \mathbb{B}^d : x_1 \ge 0, \ldots, x_d \ge 0\}$ under the mapping $x \in \mathbb{B}^d \mapsto (x_1^2 \ldots, x_d^2) \in \mathbb{T}^d$. It satisfies the following property:

Lemma A.1.2. *For $x,y \in \mathbb{T}^d$,*

$$|\sqrt{x_j} - \sqrt{y_j}| \le \mathtt{d}_{\mathbb{T}}(x,y), \quad 1 \le j \le d+1. \tag{A.1.5}$$

Proof. Given $x, y \in \mathbb{T}^d$, set $a_j := \sqrt{x_j} =: \cos\theta_j$ and $b_j := \sqrt{y_j} =: \cos\phi_j$, where $0 \le \theta_j, \phi_j \le \pi/2$. Applying the Cauchy–Schwarz inequality we get, for $1 \le j \le d$,

$$\sum_{i=1}^{d} a_i b_i + \sqrt{1 - a_1^2 - \cdots - a_d^2}\sqrt{1 - b_1^2 - \cdots - b_d^2} \le a_j b_j + \sqrt{1 - a_j^2}\sqrt{1 - b_j^2},$$

as can be seen by moving $a_j b_j$ to the left-hand side first. Hence,

$$\mathtt{d}_{\mathbb{T}}(x,y) \ge \arccos(\cos\theta_j \cos\phi_j + \sin\theta_j \sin\phi_j) = \arccos(\cos(\theta_j - \phi_j)),$$

which yields $\mathtt{d}_{\mathbb{T}}(x,y) \ge |\theta_j - \phi_j|$, and as a consequence, Eq. (A.1.5), on using the trigonometric identity $\cos\theta - \cos\phi = 2\sin\frac{\theta-\phi}{2}\sin\frac{\theta+\phi}{2}$ and an obvious estimate. □

A.2 Euler Angles and Rotations

For $d = 2$, a rotation from (x_1, x_2) to (x_1', x_2') by an angle θ is given by

$$x_1' = x_1\cos\theta + x_2\sin\theta, \quad x_2' = -x_1\sin\theta + x_2\cos\theta.$$

Writing in terms of elements in SO(2), this is given by

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

For $\mathbb{S}^2$, a rotation can be decomposed in terms of Euler angles. Let us define

$$g_{1,2}(\theta) := \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g_{2,3}(\theta) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

Lemma A.2.1. *Every $g \in SO(3)$ can be decomposed as $g = g_{1,2}(\theta)g_{2,3}(\phi)g_{1,2}(\psi)$, where θ, ϕ, ψ, called Euler angles of g, satisfy $0 \le \theta, \phi < 2\pi$, $0 \le \psi \le \pi$.*

Proof. For $x \in \mathbb{S}^3$, we write the spherical coordinates of x as

$$x_1 = \sin\phi\sin\theta, \quad x_2 = \sin\phi\cos\theta \quad x_3 = \cos\phi,$$

where $0 \le \theta < 2\pi$ and $0 \le \phi \le \pi$, which is equivalent to $x = g_{1,2}(\theta)g_{2,3}(\phi)e_3$ for $e_3 = (0,0,1)$. Thus, every $g \in SO(3)$ satisfies, setting $x = ge_3$, the equation $ge_3 = g_{1,2}(\theta)g_{2,3}(\phi)e_3$ for some θ, ϕ. This shows that $g_{2,3}(\theta)^{-1}g_{1,2}(\phi)^{-1}g \in \mathrm{SO}(3)$, or $g = g_{1,2}(\theta)g_{2,3}(\phi)h$ for some $h \in SO(3)$ satisfying $he_3 = e_3$. Since h is a rotation in (x_1, x_2), it must be of the form $h = g_{1,2}(\psi)$ for some ψ with $0 \le \psi \le 2\pi$. □

Euler angles form a coordinate system of $SO(3)$. Every element in g can be uniquely expressed in Euler angles as long as $\phi \neq 0, \pi$, as can be seen from the above proof.

The concept of Euler angles can be extended to higher dimensions. Let $g_j(\theta) \in SO(d)$ denote a rotation by an angle θ in the (x_j, x_{j+1})-plane while all other coordinates are kept fixed. As a matrix, $g_j(\theta)$ differs from the identity matrix by a 2×2 rotation matrix on its $(j, j+1)$ main minor.

Lemma A.2.2. *Every rotation $g \in SO(d)$ can be represented in the form*

$$g = g^{(d-1)} \cdots g^{(1)} \quad \textit{with} \quad g^{(k)} = g_1(\theta_{k,k}) \cdots g_k(\theta_{1,k}).$$

Proof. The proof uses induction. The cases $d = 2$ and $d = 3$ have already been given. Assume the statement for $SO(d-1)$ and consider $SO(d)$. Let $g \in SO(d)$ and let $e_d = (0, \ldots, 0, 1)$. Then, as in the case of $d = 3$, the spherical coordinates of $x = ge_d$ in Eq. (1.5.1) can be written as $ge_d = g_1(\theta_{d-1,d-1}) \cdots g_{d-1}(\theta_{1,d-1}) e_d = g^{(d-1)} e_d$. Consequently, the rotation $[g^{(d-1)}]^{-1} g$ belongs to $SO(d-1)$, a subgroup of $SO(d)$ that fixes e_d. Thus, $g = g^{(d-1)} h$ with $h \in SO(d-1)$, which completes the induction. □

There are altogether $\binom{d}{2}$ Euler angles. The construction shows that the Euler angles satisfy $0 \le \theta_{k,k} < 2\pi$ and $0 \le \theta_{j,k} \le \pi$, $1 \le j < k$. The decomposition is unique except when one of $\theta_{j,k}$, $1 \le j \le k-1$, is either 0 or π.

A.3 Basic Properties of Difference Operators

Let f be a function defined on $\mathbb{R}$. For $r = 1, 2, \ldots$, we define the difference operator $\triangle^r$ by

$$\triangle^0 = I, \quad \triangle f(x) = f(x) - f(x+1), \qquad \triangle^r = \triangle^{r-1}(\triangle f(x)), \tag{A.3.1}$$

where I denotes the identity operator. By induction, it is easy to see that

$$\triangle^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x+k). \tag{A.3.2}$$

For a sequence $\{a_k\}_{k=0}^{\infty}$, the difference operator $\triangle^r a_k$ is defined as $\triangle^r f(k)$ with $f(k) = a_k$, $k \in \mathbb{N}_0$.

Some of the basic properties of the difference operators are collected in the following proposition.

Proposition A.3.1. *Let $r \in \mathbb{N}$. Then:*

(*i*) *The action $\triangle^r f(x)$ satisfies*

$$\triangle^r (f(x)g(x)) = \sum_{k=0}^{r} \binom{r}{k} \triangle^k f(x) \triangle^{r-k} g(x+k). \tag{A.3.3}$$

(*ii*) *If* $f \in C^r(\mathbb{R})$, *then*

$$\triangle^r f(x) = (-1)^r \int_{[0,1]^r} f^{(r)}(x+u_1+\cdots+u_r)\,\mathrm{d}u_1\cdots\mathrm{d}u_r. \tag{A.3.4}$$

(*iii*) *If* $\min_{k\geq 0}|a_k| \geq \delta$ *for some absolute constant* $\delta \in (0,1)$, *then*

$$\left|\triangle^r\left(\frac{1}{a_k}\right)\right| \leq c_{\delta,r}\triangle_*^r a_k, \tag{A.3.5}$$

where with $\Lambda_{m,r} := \{(i_1,\dots,i_m) : i_1,\dots,i_m \in \mathbb{N},\, i_1+\cdots+i_m = r\}$,

$$\triangle_*^r a_k := \max_{\substack{(i_1,\dots,i_m)\in\Lambda_{m,r},1\leq m\leq r\\ k_1,\dots,k_m\in\{k,k+1,\dots,k+r\}}} \left[|\triangle^{i_1}a_{k_1}||\triangle^{i_2}a_{k_2}|\cdots|\triangle^{i_m}a_{k_m}|\right]. \tag{A.3.6}$$

All three assertions in Proposition A.3.1 can be verified by induction, and we leave the proofs to the interested reader. The first two properties are classical and can be found in numerous books on approximation theory and numerical analysis. Other useful properties of the difference operators are stated in the following two lemmas.

Lemma A.3.2. *If* $\sup_{k\in\mathbb{N}_0}|a_k| \leq 1$ *and* $m,r \in \mathbb{N}$, *then*

$$|\triangle^r a_k^m| \leq c_r m^r\left(\triangle_*^r a_k\right)\left(\triangle_{*,r}^0 a_k\right)^{\max\{m-r,0\}}, \tag{A.3.7}$$

where $\triangle_{*,r}^0 a_k := \max_{k\leq j\leq k+r}|a_j|$, *and it is agreed that* $0^0 = 1$. *If, in addition,* $m \geq r$, *then*

$$\triangle_*^r\left[(a_k)^m\right] \leq c_r m^r\left(\triangle_{*,r}^0 a_k\right)^{m-r}\triangle_*^r a_k. \tag{A.3.8}$$

Proof. For the proof of Eq. (A.3.7), we use induction on $m+r$. If $m+r=2$, then $m=r=1$, and (A.3.7) holds trivially. Now assuming that Eq. (A.3.7) is true for $m+r\leq s$ and some positive integer $s\geq 2$, we deduce the assertion for the case of $m+r=s+1$ as follows. If $m=1$ or $r=1$, Eq. (A.3.7) can be verified directly. Hence, without loss of generality, we may assume that $m,r\geq 2$. It then follows by Eq. (A.3.3) that

$$\triangle^r a_k^m = \triangle^{r-1}\left(\left(\triangle a_k^{m-1}\right)a_{k+1}\right) + \triangle^{r-1}\left(a_k^{m-1}\triangle a_k\right) := J_1 + J_2.$$

For the first term, J_1, we use Eq. (A.3.3) and the induction hypothesis to deduce

$$\begin{aligned}
|J_1| \leq{}& c_r\sum_{v=0}^{r-2} m^{v+1}\left(\triangle_*^{v+1}a_k\right)\left(\triangle_{*,v+1}^0 a_k\right)^{\max\{0,m-v-2\}}\left|\triangle^{r-1-v}a_{k+1+v}\right| \\
&+\left|\triangle^r a_k^{m-1}\right||a_{k+r}| \\
\leq{}& c_r m^r\left(\triangle_*^r a_k\right)\left(\triangle_{*,r}^0 a_k\right)^{\max\{m-r,0\}}.
\end{aligned}$$

An almost identical argument applies to the second term, and it shows that

$$|J_2| \le c_r m^r \left(\triangle_*^r a_k\right)\left(\triangle_{*,r}^0 a_k\right)^{\max\{m-r,0\}}.$$

Together, the two estimates complete the induction process. Finally, Eq. (A.3.8) follows directly from Eq. (A.3.7). □

Lemma A.3.3. *Let s be a given nonzero real number. If there exists an absolute constant $\delta \in (0,1)$ such that $\delta \le a_k \le \delta^{-1}$ for all $k \in \mathbb{N}_0$, then for every nonnegative integer r,*

$$|\triangle^r (a_k)^s| \le c_{\delta,r,s} \triangle_*^r a_k, \tag{A.3.9}$$

where $c_{\delta,r,s}$ depends only on r, δ, and s.

Proof. We use induction on r. Equation (A.3.9) holds trivially if $r = 0$ or $s = 0$. Now assume that Eq. (A.3.9) holds for $r \ge n$ for some integer n. For the case $r = n+1$ and $s > 0$, observe that

$$\begin{aligned}\triangle^{n+1}(a_k^s) &= s\triangle^n\left(\triangle a_k \int_0^1 (a_{k+1} + x\triangle a_k)^{s-1}\,\mathrm{d}x\right)\\ &= s\sum_{v=0}^n \binom{n}{v}\left(\triangle^{n+1-v} a_{k+v}\right)\int_0^1 \triangle^v\left((a_{k+1}+x\triangle a_k)^{s-1}\right)\mathrm{d}x.\end{aligned}$$

Since for $x \in (0,1)$ and $k \ge 0$, $\delta \le a_{k+1} + x\triangle a_k \le \delta^{-1}$, using the induction hypothesis for $r \le n$, we obtain that for $0 \le v \le n$,

$$\left|\triangle^v\left((a_{k+1}+x\triangle a_k)^{s-1}\right)\right| \le c_{s,n,\delta}\left(\triangle_*^v a_k + \triangle_*^v a_{k+1}\right).$$

Combining the last two displayed formulas proves that

$$|\triangle^{n+1} a_k^s| \le C_{n,s,\delta} \triangle_*^{n+1} a_k, \quad s > 0,$$

which is Eq. (A.3.9) for $s > 0$ and proves the lemma. □

A.4 Cesàro Means and Difference Operators

For $\delta \in \mathbb{R}$, the Cesàro (C,δ) means of the sequence $\{a_k\}_{k=0}^\infty$ are defined by

$$s_n^\delta := \frac{1}{A_n^\delta}\sum_{k=0}^n A_{n-k}^\delta a_k, \quad n = 0,1,\ldots, \tag{A.4.1}$$

where

$$A_k^\delta = \binom{k+\delta}{k} = \frac{(\delta+k)(\delta+k-1)\ldots(\delta+1)}{k!}. \tag{A.4.2}$$

Directly from the definition, A_k^δ and s_n^δ can be seen to satisfy, for all $\delta, \tau \in \mathbb{R}$,

$$(1-r)^{-\delta-1} = \sum_{n=0}^{\infty} A_n^\delta r^n, \qquad A_n^{\delta+\tau} = \sum_{k=0}^{n} A_{n-k}^{\tau-1} A_k^\delta, \tag{A.4.3}$$

$$(1-r)^{-\delta-1} \sum_{n=0}^{\infty} a_n r^n = \sum_{n=0}^{\infty} A_n^\delta s_n^\delta r^n, \qquad s_n^{\delta+\tau} = \frac{1}{A_n^{\delta+\tau}} \sum_{k=0}^{n} A_{n-k}^{\tau-1} A_k^\delta s_k^\delta. \tag{A.4.4}$$

The numbers A_k^δ are defined for all $\delta \in \mathbb{R}$. Several of their properties that can be easily verified (see [197, p. 77]), are listed below:

$$A_n^\delta = 0, \qquad \delta = -1, -2, \ldots, \quad n = 1, 2, \ldots; \tag{A.4.5}$$

$$A_n^\delta = \frac{n^\delta}{\Gamma(\delta+1)}(1 + O(n^{-1})), \qquad \delta \neq -1, -2, \ldots; \tag{A.4.6}$$

$$A_n^\delta - A_{n-1}^\delta = A_n^{\delta-1}, \quad \sum_{k=0}^{n} A_k^\delta = A_n^{\delta+1}. \tag{A.4.7}$$

For $r = 1, 2, \ldots,$ the summation by parts formula is given by

$$\sum_{k=0}^{\infty} a_k b_k = \sum_{k=0}^{\infty} \triangle^{r+1} b_k \sum_{j=0}^{k} A_{k-j}^r a_j = \sum_{k=0}^{\infty} \triangle^{r+1} b_k A_k^r s_k^r, \tag{A.4.8}$$

where the s_k^δ are the (C, δ) means of $\{a_k\}_{k=0}^n$.

Lemma A.4.1. *If $\{a_j\}_{j=0}^\infty$ is a bounded sequence of complex numbers satisfying $\sum_{j=1}^\infty |\triangle^{\ell+1} a_j| j^\ell < \infty$ for some nonnegative integer ℓ, then a_n converges, and if $L := \lim_{n\to\infty} a_n$, then $\sum_{j=0}^\infty (a_j - L)$ converges and satisfies*

$$\sum_{j=0}^{\infty} (a_j - L) = \sum_{j=0}^{\infty} \left(\triangle^{\ell+1} a_j\right) A_j^\ell (s_j^\ell - L).$$

Proof. We first claim that for each positive integer r,

$$\sum_{j=0}^{\infty} |\triangle^r a_j| A_j^{r-1} \leq \sum_{j=0}^{\infty} \left|\triangle^{r+1} a_j\right| A_j^r. \tag{A.4.9}$$

We may assume that the infinite sum on the right-hand side of Eq. (A.4.9) is finite. Then for each $k, m \in \mathbb{N}$,

$$|\triangle^r a_k - \triangle^r a_{k+m}| = \left| \sum_{j=k}^{k+m-1} \triangle^{r+1} a_j \right| \leq \sum_{j=k}^{k+m-1} \left|\triangle^{r+1} a_j\right| A_j^r \to 0, \text{ as } k \to \infty,$$

which shows that $\{\triangle^r a_k\}$ is a Cauchy sequence in $\mathbb{C}$ and therefore is convergent. Furthermore,

$$\left|\sum_{j=k}^{2k-1} \triangle^r a_j\right| = \left|\triangle^{r-1} a_k - \triangle^{r-1} a_{2k}\right| \le c_r \sup_j |a_j| < \infty,$$

and we must have $\lim_{n\to\infty} \triangle^r a_n = 0$, which further implies $\triangle^r a_n = \sum_{j=n}^{\infty} \triangle^{r+1} a_j$. The claim (A.4.9) then follows.

Now applying Eq. (A.4.9) ℓ times yields $\sum_{j=0}^{\infty} |\triangle a_j| \le \sum_{k=0}^{\infty} |\triangle^{\ell+1} a_k| A_k^{\ell} < \infty$, which implies, in particular, that $\lim_{n\to\infty} a_n = L$ exists and that $\lim_{n\to\infty} \triangle^r a_n = 0$ for all positive integers r. Thus, applying summation by parts $\ell+1$ times to the partial sums $s_n := \sum_{j=0}^{n}(a_j - L)$ and letting $n \to \infty$, we complete the proof. □

Lemma A.4.2. *For* $1 \le m \le n$, $r \in (0,1)$ *and* $\delta \in \mathbb{R}$,

$$\left|\sum_{k=0}^{m} A_{m-k}^{\delta} A_k^{-\delta-2} r^{m-k}\right| \le c(1-r)\left(1 + (m(1-r))^k\right). \tag{A.4.10}$$

Proof. By Eqs. (A.4.3) and (A.4.5), $\sum_{k=0}^{m} A_{m-k}^{\delta} A_k^{-\delta-2} = A_m^{-1} = 0$. We shall now prove that

$$\left|\sum_{k=0}^{m} A_{m-k}^{\delta} A_k^{-\delta-2}(1 - r^{m-k})\right| \le c(1-r)\left(1 + (m(1-r))^k\right).$$

To this end, let $\eta \in C^{\infty}(\mathbb{R})$ be such that $\eta(x) = 1$ for $|x| \le \frac{1}{4}$, and $\eta(x) = 0$ for $|x| \ge \frac{1}{2}$. We split the sum in Eq. (A.4.10) into two parts, $\Sigma_1 + \Sigma_2$, where

$$\Sigma_1 = \sum_{j=0}^{m} \eta\left(\tfrac{j}{m}\right) A_{m-j}^{\delta} A_j^{-\delta-2}(1 - r^{m-j}),$$

$$\Sigma_2 = \sum_{j=0}^{m} \left(1 - \eta\left(\tfrac{j}{m}\right)\right) A_{m-j}^{\delta} A_j^{-\delta-2}(1 - r^{m-j}).$$

The estimate of the second term follows from a direct computation using Eq. (A.4.6),

$$|\Sigma_2| \le c(1-r) \sum_{m/4 \le j \le m} (m-j+1)^{\delta+1} j^{-\delta-2} \le c(1-r).$$

To estimate Σ_1, we let k be a positive integer, $k \ge \delta$. Performing summation by parts k times and using Eq. (A.4.7), we obtain

$$\Sigma_1 = \sum_{0 \le j \le m/2} \triangle^k \left(A_{m-j}^{\delta} \eta\left(\tfrac{j}{m}\right)(1 - r^{m-j})\right) A_j^{-\delta-2+k},$$

where, and throughout this proof, the difference operator, defined in A.3.1, acts on the variable j. Since $\triangle^p A^\delta_{m-j} = A^{\delta-p}_{m-j}$ and $\triangle^p(1-r^{m-j}) = (-1)^p(1-r)^p r^{m-j-p}$, by Eq. (A.4.7) and induction, it follows from the product formula of the difference operator (A.4.8), (A.4.6) and $(1-r^m) \le m(1-r)$ that

$$\begin{aligned}\left|\triangle^\ell\left(A^\delta_{m-j}(1-r^{m-j})\right)\right| &= \left|\sum_{p=0}^{\ell}\binom{\ell}{p}A^{\delta-(\ell-p)}_{m-j}\triangle^p(1-r^{m-j})\right| \\ &\le m^{\delta-\ell+1}(1-r) + cm^{\delta-\ell}\sum_{p=1}^{\ell}\binom{\ell}{p}m^p(1-r)^p \\ &\le cm^{\delta-\ell+1}(1-r)\left(1+(m(1-r))^{\ell-1}\right)\end{aligned}$$

for $1 \le \ell \le k$, whereas for $\ell = 0$, the last estimate is replaced by $cm^{\delta-k+1}(1-r)$. Consequently, since $|\triangle^p \eta(\frac{j}{m})| = m^{-p}|\eta^{(p)}(\xi)| \le cm^{-p}$, using the product formula of the difference operator one more time gives

$$\begin{aligned}\left|\triangle^k\left(A^\delta_{m-j}\eta\left(\tfrac{j}{m}\right)(1-r^{m-j})\right)\right| &\le c(1-r)m^{\delta-k+1}\left[1+\sum_{\ell=1}^{k}\left(1+(m(1-r))^{\ell-1}\right)\right] \\ &\le c(1-r)m^{\delta-k+1}\left(1+(m(1-r))^k\right).\end{aligned}$$

Consequently, using $\sum_j |A_j^{-\delta-2}| \sim \sum_j (j+1)^{-\delta-2} \le c$, by Eq. (A.4.6), we conclude that

$$|\Sigma_1| \le c(1-r)\left(1+(m(1-r))^k\right).$$

Putting the above together proves Eq. (A.4.10). □

A.5 Integrals over Spheres and Balls

This section contains several integral identities that are used in the text. We start with the connection between integration on $\mathbb{S}^{d-1}$ and the special orthogonal group $SO(d)$, which consists of orthogonal matrices with determinant 1.

Lemma A.5.1. *Let* dh *be the Haar measure on* $SO(d)$. *Then*

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(x)\,\mathrm{d}\sigma(x) = \int_{SO(d)} f(hz)\,\mathrm{d}h, \quad \forall x \in \mathbb{S}^{d-1}.$$

Proof. Since $\mathrm{d}\sigma$ is invariant under the action of $SO(d)$, it follows that

$$\int_{\mathbb{S}^{d-1}} f(x)\,\mathrm{d}\sigma(x) = \int_{SO(d)}\int_{\mathbb{S}^{d-1}} f(x)\,\mathrm{d}\sigma(x)\,\mathrm{d}h = \int_{\mathbb{S}^{d-1}}\int_{SO(d)} f(hx)\,\mathrm{d}h\,\mathrm{d}\sigma(x).$$

For $x, y \in \mathbb{S}^{d-1}$, there is a $g \in SO(d)$ such that $x = gy$, so that

$$\int_{SO(d)} f(hx)\,\mathrm{d}h = \int_{SO(d)} f(hgy)\,\mathrm{d}h = \int_{SO(d)} f(hy)\,\mathrm{d}h$$

by the invariance of dh over $SO(d)$. Together, these two equations give the stated result. □

Lemma A.5.2. *For $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{S}^{d-1}} f(\langle x,y\rangle)\mathrm{d}\sigma(y) = \omega_{d-1}\int_{-1}^{1} f(\|x\|t)(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t. \tag{A.5.1}$$

Proof. Let $Q \in SO(d)$ be a rotation such that $Qx = \|x\|e_d$. Since dσ is invariant under $SO(d)$,

$$\int_{\mathbb{S}^{d-1}} f(\langle x,y\rangle)\mathrm{d}\sigma(y) = \int_{\mathbb{S}^{d-1}} f(\langle Qx,y\rangle)\mathrm{d}\sigma(y) = \int_{\mathbb{S}^{d-1}} f(\|x\|y_d)\mathrm{d}\sigma(y).$$

Parameterizing $\mathbb{S}^{d-1}$ by $x = (\xi\sin\theta, \cos\theta)$, $\xi \in \mathbb{S}^{d-2}$, $0 \le \theta \le \pi$, we see that

$$\int_{\mathbb{S}^{d-1}} f(\|x\|y_d)\mathrm{d}\sigma(y) = \omega_{d-1}\int_0^{\pi} f(\|x\|\cos\theta)(\sin\theta)^{d-2}\mathrm{d}\theta.$$

Changing variables $t \to \cos\theta$ then proves Eq. (A.5.1). □

Corollary A.5.3. *For $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{B}^d} f(\langle x,y\rangle)(1-\|y\|^2)^{\mu}\mathrm{d}y = c_\mu\int_{-1}^{1} f(\|x\|t)(1-t^2)^{\mu+\frac{d-1}{2}}\mathrm{d}t, \tag{A.5.2}$$

where the value of c_μ can be determined by setting $f(t) = 1$.

Proof. Using polar coordinates and Eq. (A.5.1),

$$\begin{aligned}\int_{\mathbb{B}^d} f(\langle x,y\rangle)(1-\|y\|^2)^{\mu}\mathrm{d}y &= \int_0^1 r^{d-1}(1-r^2)^{\mu}\int_{\mathbb{S}^{d-1}} f(r\langle x,\xi\rangle)\mathrm{d}\sigma(\xi)\mathrm{d}r\\ &= \omega_{d-1}\int_0^1 r^{d-1}(1-r^2)^{\mu}\int_{-1}^{1} f(r\|x\|t)(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t\mathrm{d}r.\end{aligned}$$

Changing variables $rt \mapsto s$ and exchanging the order of integration, we obtain

$$\begin{aligned}\int_{\mathbb{B}^d} f(\langle x,y\rangle)(1-\|y\|^2)^{\mu}\mathrm{d}y &= \omega_{d-1}\int_{-1}^{1} f(s\|x\|)\int_{|s|}^{1} r(1-r^2)^{\mu}(r^2-s^2)^{\frac{d-3}{2}}\mathrm{d}r\mathrm{d}s\\ &= c\,\omega_{d-1}\int_{-1}^{1} f(s\|x\|)(1-s^2)^{\mu+\frac{d-1}{2}}\mathrm{d}s,\end{aligned}$$

on changing variables $t = (r^2 - s^2)/(1 - s^2)$ in the integral against dr, where the constant c is equal to $\frac{1}{2}\int_0^1 t^{\frac{d-3}{2}}(1-t)^{\mu+\frac{d-1}{2}}\mathrm{d}t$. □

Lemma A.5.4. *Let d and m be positive integers. If m ≥ 2, then*

$$\int_{\mathbb{S}^{d+m-1}} f(y)\mathrm{d}\sigma = \int_{\mathbb{B}^d} (1-\|x\|^2)^{\frac{m-2}{2}} \left[\int_{\mathbb{S}^{m-1}} f(x, \sqrt{1-\|x\|^2}\,\xi)\mathrm{d}\sigma(\xi)\right] \mathrm{d}x, \quad \text{(A.5.3)}$$

whereas if m = 1, then

$$\int_{\mathbb{S}^d} f(y)\mathrm{d}\sigma = \int_{\mathbb{B}^d} \left[f(x, \sqrt{1-\|x\|^2}) + f(x, -\sqrt{1-\|x\|^2})\right] \frac{\mathrm{d}x}{\sqrt{1-\|x\|^2}}. \quad \text{(A.5.4)}$$

Proof. For $m \geq 2$, making a change of variables $y \mapsto (x, \sqrt{1-\|x\|^2}\xi)$, $x \in \mathbb{B}^d$ and $\xi \in \mathbb{S}^{m-1}$ in the integral over $\mathbb{S}^{d+m-1}$ yields

$$\mathrm{d}\sigma_{d+m}(y) = (1-\|x\|^2)^{\frac{m-2}{2}} \mathrm{d}x\mathrm{d}\sigma_m(\xi),$$

from which Eq. (A.5.3) follows immediately. In the case of $m = 1$, we write, for $y \in \mathbb{S}^d$, $y = (\sqrt{1-t^2}x, t)$, where $x \in \mathbb{S}^{d-1}$ and $-1 \leq t \leq 1$. It follows that

$$\mathrm{d}\sigma_{d+1}(y) = (1-t^2)^{(d-2)/2}\mathrm{d}t\ \mathrm{d}\sigma_d(x).$$

Changing variables $y \mapsto (\sqrt{1-t^2}x, t)$ gives

$$\begin{aligned}
\int_{\mathbb{S}^d} f(y)\mathrm{d}\sigma &= \int_{-1}^1 \int_{\mathbb{S}^{d-1}} f\left(\sqrt{1-t^2}x, t\right) \mathrm{d}\sigma_d(x)(1-t^2)^{\frac{d-2}{2}}\mathrm{d}t \\
&= \int_0^1 \int_{\mathbb{S}^{d-1}} \left[f\left(\sqrt{1-t^2}x, t\right) + f\left(\sqrt{1-t^2}x, -t\right)\right] \mathrm{d}\sigma_d(x)(1-t^2)^{\frac{d-2}{2}}\mathrm{d}t \\
&= \int_0^1 \int_{\mathbb{S}^{d-1}} \left[f(rx, \sqrt{1-r^2}) + f\left(rx, -\sqrt{1-r^2}\right)\right] \mathrm{d}\sigma_d(x) r^{d-1} \frac{\mathrm{d}r}{\sqrt{1-r^2}},
\end{aligned}$$

from which Eq. (A.5.4) follows from Eq. (A.5.1). □

Appendix B
Jacobi and Related Orthogonal Polynomials

In this appendix we collect formulas and properties of the Jacobi polynomials and Gegenbauer polynomials that are needed in the book. Most of the properties are stated without proof. Our main reference is the classical treatise by Szegő [162].

B.1 Jacobi Polynomials

For parameters $\alpha, \beta > -1$, the Jacobi weight function is defined by

$$w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad -1 < x < 1. \tag{B.1.1}$$

The normalization constant $c_{\alpha,\beta}$ of the weight function is given by

$$c_{\alpha,\beta}^{-1} := \int_{-1}^{1} w_{\alpha,\beta}(x)\mathrm{d}x = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

For $n \ge 0$, the Jacobi polynomials are defined by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right) \\ &= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right), \end{aligned} \tag{B.1.2}$$

where the hypergeometric function ${}_2F_1$ is defined by

They are normalized so that

$$P_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!}. \tag{B.1.3}$$

The Jacobi polynomials are orthogonal with respect to $w_{\alpha,\beta}$: for $n, m \ge 0$,

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$$c_{\alpha,\beta}\int_{-1}^{1}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)w_{\alpha,\beta}(x)\mathrm{d}x = h_n^{(\alpha,\beta)}\delta_{n,m}, \tag{B.1.4}$$

$$h_n^{\alpha,\beta} := \frac{(\alpha+1)_n(\beta+1)_n(\alpha+\beta+n+1)}{n!(\alpha+\beta+2)_n(\alpha+\beta+2n+1)}.$$

Some properties of Jacobi polynomials are listed below:

1. The leading coefficient of $P_n^{(\alpha,\beta)}$ is $a_n = \dfrac{(n+\alpha+\beta+1)_n}{2^n n!}$.
2. $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$.
3. $P_n^{(\alpha,\beta)}(x)$ satisfies the differential equation
$$(1-x^2)y'' - (\alpha-\beta+(\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0.$$
4. The three-term relation holds: setting $P_{-1}^{(\alpha,\beta)}(x) = 0$, then $P_0^{(\alpha,\beta)}(x) = 1$ and
$$\begin{aligned}P_{n+1}^{(\alpha,\beta)}(x) &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}xP_n^{(\alpha,\beta)}(x)\\ &\quad+\frac{(2n+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}P_n^{(\alpha,\beta)}(x)\\ &\quad-\frac{(\alpha+n)(\beta+n)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}P_{n-1}^{(\alpha,\beta)}(x).\end{aligned}$$
5. For $n \ge 1$,
$$\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{B.1.5}$$
6. The Dirichlet–Mehler formula [162, (4.10.12)] holds: for $\mu > 0$,
$$\frac{P_n^{(\alpha-\mu,\beta+\mu)}(x)}{P_n^{(\beta+\mu,\alpha-\mu)}(1)} = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)}\int_0^1 \frac{P_n^{(\alpha,\beta)}((1+x)t-1)}{P_n^{(\beta,\alpha)}(1)}t^{\beta}(1-t)^{\mu-1}\mathrm{d}t. \tag{B.1.6}$$

The Jacobi polynomials also have the following additional properties:
For arbitrary $\alpha,\beta\in\mathbb{R}$ [162, (7.32.5) and (4.1.3)],

$$\left|P_n^{(\alpha,\beta)}(\cos\theta)\right| \le cn^{-\frac{1}{2}}(n^{-1}+\theta)^{-\alpha-\frac{1}{2}}(n^{-1}+\pi-\theta)^{-\beta-\frac{1}{2}}. \tag{B.1.7}$$

For $\alpha,\beta,\mu > -1$ and $p > 0$ [162, p. 391],

$$\int_0^1\left|P_n^{(\alpha,\beta)}(t)\right|^p(1-t)^{\mu}\mathrm{d}t \sim \begin{cases} n^{\alpha p-2\mu-2}, & p > p_{\alpha,\mu},\\ n^{-\frac{p}{2}}\log n, & p = p_{\alpha,\mu},\\ n^{-\frac{p}{2}}, & p < p_{\alpha,\mu},\end{cases}\quad p_{\alpha,\mu} := \frac{2\mu+2}{\alpha+\frac{1}{2}}. \tag{B.1.8}$$

Lemma B.1.1. *([162, p. 198]) For $\alpha,\beta > -1$ and $n^{-1} \le \theta \le \pi - n^{-1}$,*

$$P_n^{(\alpha,\beta)}(\cos\theta) = \pi^{-\frac{1}{2}} n^{-\frac{1}{2}} (\sin\tfrac{\theta}{2})^{-\alpha-\frac{1}{2}} (\cos\tfrac{\theta}{2})^{-\beta-\frac{1}{2}} \left[\cos(N\theta+\tau_\alpha) + \mathscr{O}(1)(n\sin\theta)^{-1}\right],$$

where $N = n + \frac{\alpha+\beta+1}{2}$ and $\tau_\alpha = -\frac{\pi}{2}(\alpha+\frac{1}{2})$.

For $n \ge 0$, let $k_n(w_{\alpha,\beta};x,y)$ denote the kernel of the Fourier expansion in the Jacobi polynomials; that is,

$$k_n(w_{\alpha,\beta};x,y) := \sum_{j=0}^{n} (h_j^{\alpha,\beta})^{-1} P_j^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y).$$

The kernel satisfies the Christoffel–Darboux formula [162, (4.5.2)]. In particular,

$$k_n(w_{\alpha,\beta};x,1) = 2^{-\alpha-\beta-1} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} P_n^{(\alpha+1,\beta)}(x). \tag{B.1.9}$$

As a consequence, [162, (4.5.3)],

$$P_n^{(\alpha+1,\beta)}(x) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \times \sum_{j=0}^{n} \frac{(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)} P_j^{(\alpha,\beta)}(x). \tag{B.1.10}$$

For $\delta \ge 0$, let $k_n^\delta(w_{\alpha,\beta};x,y)$ denote the kernel of the Cesàro (C,δ) means of the Jacobi series; that is,

$$k_n^\delta(w_{\alpha,\beta};x,y) := \frac{1}{A_n^\delta} \sum_{j=0}^{n} A_{n-j}^\delta (h_j^{\alpha,\beta})^{-1} P_j^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(y).$$

The estimate of the kernel $k_n^\delta(w_{\alpha,\beta};x,1)$ is established in [18, Theorem 2.1] and [34, Theorem 3.9], based on fundamental relations in [162, (9.4.4) and (9.41.14)].

Lemma B.1.2. *Let $\alpha,\beta \ge -1/2$ and $u \in [-1,1]$. If $0 \le \delta \le \alpha+3/2$, then*

$$|k_n^\delta(w_{\alpha,\beta},1,u)| \le cn^{\alpha+1/2-\delta}\Big[(1-u+n^{-2})^{-(\delta+\alpha+3/2)/2} + (1+u+n^{-2})^{-(\beta+1/2)/2}\Big]. \tag{B.1.11}$$

If $\alpha+3/2 \le \delta \le \alpha+\beta+2$,

$$|k_n^\delta(w_{\alpha,\beta},1,u)| \le cn^{-1}\Big[(1-u+n^{-2})^{-(\alpha+3/2)} + (1+u+n^{-2})^{-(\alpha+\beta+2-\delta)/2}\Big]. \tag{B.1.12}$$

If $\delta \ge \alpha+\beta+2$,

$$0 \le k_n^\delta(w_{\alpha,\beta},1,u) \le cn^{-1}(1-u+n^{-2})^{-(\alpha+3/2)}. \tag{B.1.13}$$

B.2 Gegenbauer Polynomials

For $\lambda > -1/2$, the Gegenbauer weight function is defined by

$$w_\lambda(x) := (1-x^2)^{\lambda-1/2}, \quad -1 < x < 1,$$

which is a special case of the Jacobi weight, and its normalization constant is given by $c_\lambda = c_{\lambda-1/2,\lambda-1/2}$. The Gegenbauer polynomials are defined by

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x), \tag{B.2.1}$$

and they satisfy

$$C_n^\lambda(1) = \frac{(2\lambda)_n}{n!}. \tag{B.2.2}$$

The Gegenbauer polynomials satisfy the orthogonal relation

$$c_\lambda \int_{-1}^1 C_n^\lambda(x) C_m^\lambda(x) w_\lambda(x) \mathrm{d}x = \frac{\lambda}{(n+\lambda)} C_n^\lambda(1) \delta_{n,m}. \tag{B.2.3}$$

There is another connection to the Jacobi polynomials, the quadratic transform,

$$C_{2n}^\lambda(x) := \frac{(\lambda)_n}{\left(\frac{1}{2}\right)_n} P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2x^2-1). \tag{B.2.4}$$

Furthermore, there is one more representation in the following hypergeometric formula:

$$C_n^\lambda(x) = \frac{(\lambda)_n 2^n}{n!} x^n {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ 1-n-\lambda \end{matrix}; \frac{1}{x^2}\right). \tag{B.2.5}$$

The Gegenbauer polynomials satisfy the following properties:

1. The leading coefficient of $C_n^\lambda(t)$ is $\dfrac{(\lambda)_n 2^n}{n!}$.
2. C_n^λ satisfies the differential equation

$$(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0.$$

3. The three-term relation holds: setting $C_{-1}^\lambda(t) = 0$, then $C_0^\lambda(t) = 1$ and

$$C_{n+1}^\lambda(x) = \frac{2(n+\lambda)}{n+1} x C_n^\lambda(x) - \frac{n+2\lambda-1}{n+1} C_{n-1}^\lambda(x).$$

4. The following differentiation relation holds:

$$\frac{\mathrm{d}}{\mathrm{d}x} C_n^{\lambda}(x) = 2\lambda C_{n-1}^{\lambda+1}(x). \tag{B.2.6}$$

5. We have the following recurrence relation in λ:

$$(n+\lambda) C_n^{\lambda}(x) = \lambda \left(C_{n+1}^{\lambda+1}(x) - C_{n-1}^{\lambda+1}(x) \right). \tag{B.2.7}$$

The Gegenbauer polynomials also satisfy the following additional properties: The Poisson formula: for $0 \le r < 1$,

$$\frac{1-r^2}{(1-2xr+r^2)^{\lambda+1}} = \sum_{n=0}^{\infty} \frac{n+\lambda}{\lambda} C_n^{\lambda}(x) r^n. \tag{B.2.8}$$

The product formula:

$$\frac{C_n^{\lambda}(x) C_n^{\lambda}(y)}{C_n^{\lambda}(1)} = c_\lambda \int_{-1}^{1} C_n^{\lambda}(xy + \sqrt{1-x^2}\sqrt{1-y^2}t)(1-t^2)^{\lambda-1} \mathrm{d}t. \tag{B.2.9}$$

The connection formula: for $\lambda \ge \mu$,

$$C_n^{\lambda}(x) = \sum_{0 \le k \le n/2} a_{k,n} C_{n-2k}^{\mu}(x), \tag{B.2.10}$$

where

$$a_{k,n} := \frac{(n-2k+\mu)(\lambda-\mu)_k (n-k)_\lambda}{(n-k+\mu)(n-k)_\mu}.$$

In particular, the Gegenbauer polynomials can be expanded as

$$C_n^{\lambda}(\cos\theta) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{k,n} \cos(n-2k)\theta, \tag{B.2.11}$$

where $a_k = \alpha_k \alpha_{n-k}$ for $0 \le k \le \lfloor \frac{n}{2} \rfloor$, except that $a_{\lfloor \frac{n}{2} \rfloor} = \alpha^2_{\lfloor \frac{n}{2} \rfloor}$ when n is even, and

$$\alpha_n = \binom{n+\lambda-1}{n}.$$

The special case $\lambda = 0$ is the Chebyshev polynomial of the first kind, denoted by $T_n(x)$, and it satisfies

$$\lim_{\lambda \to 0} \frac{1}{\lambda} C_n^{\lambda}(x) = T_n(x) = \cos n\theta, \quad x = \cos\theta.$$

The case $\lambda = 1$ is a Chebyshev polynomial of the second kind, denoted by $U_n(x)$,

$$U_n(x) = C_n^1(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta.$$

The case $\lambda = \frac{1}{2}$ is a Legendre polynomial, often denoted by

$$P_n(x) = C_n^{1/2}(x),$$

which are orthogonal for dx on $-1 \le x \le 1$.

B.3 Generalized Gegenbauer Polynomials

The generalized Gegenbauer polynomials are orthogonal with respect to the weight function:

$$v_{\lambda,\mu}(x) = |x|^{2\lambda}(1-x^2)^{\mu-1/2}, \quad x \in [-1,1].$$

We define the generalized Gegenbauer polynomials as

$$\begin{aligned} C_{2n}^{(\lambda,\mu)}(x) &:= \frac{(\lambda+\mu)_n}{\left(\mu+\frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\mu-1/2)}(2x^2-1), \\ C_{2n+1}^{(\lambda,\mu)}(x) &:= \frac{(\lambda+\mu)_{n+1}}{\left(\mu+\frac{1}{2}\right)_{n+1}} x P_n^{(\lambda-1/2,\mu+1/2)}(2x^2-1), \end{aligned} \tag{B.3.1}$$

which become C_n^λ when $\mu = 0$. They satisfy the relation

$$C_{2n}^{(\lambda,\mu)}(1) = \frac{(\lambda+\mu)_n\left(\lambda+\frac{1}{2}\right)_n}{n!\left(\mu+\frac{1}{2}\right)_n}, \quad C_{2n+1}^{(\lambda,\mu)}(1) = \frac{(\lambda+\mu)_{n+1}\left(\lambda+\frac{1}{2}\right)_n}{n!\left(\mu+\frac{1}{2}\right)_{n+1}}. \tag{B.3.2}$$

Their orthogonality relation is given by

$$b_{\lambda,\mu}\int_{-1}^1 C_n^{(\lambda,\mu)}(t)C_m^{(\lambda,\mu)}(t)v_{\lambda,\mu}(t)\mathrm{d}t = \frac{\lambda+\mu}{\lambda+\mu+n}C_n^{(\lambda,\mu)}(1)\delta_{m,n}. \tag{B.3.3}$$

These polynomials are closely related to the Gegenbauer polynomials. Indeed, for $\lambda > -1/2$, $\mu > 0$ and $n \ge 0$,

$$C_n^{(\lambda,\mu)}(x) = c_\mu \int_{-1}^1 C_n^{\lambda+\mu}(xt)(1+t)(1-t^2)^{\mu-1}\mathrm{d}t. \tag{B.3.4}$$

More generally, they satisfy

$$\frac{C_n^{(\lambda,\mu)}(x)C_n^{(\lambda,\mu)}(y)}{C_n^{(\lambda,\mu)}(1)} = c_{\lambda,\mu}\int_{-1}^{1} C_n^{\lambda+\mu}(txy + s\sqrt{1-x^2}\sqrt{1-y^2}) \times (1+t)(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}\mathrm{d}s\mathrm{d}t. \tag{B.3.5}$$

B.4 Associated Legendre Polynomials

The associated Legendre polynomials, denoted by $P_n^k(x)$ for $n \neq k$, are defined by

$$P_n^k(x) = \frac{(-1)^k}{2^n n!}(1-x^2)^{k/2}\frac{\mathrm{d}^{k+n}}{\mathrm{d}x^{k+n}}(1-x^2)^n \tag{B.4.1}$$

for $-n \le k \le n$. For $k \le n$, they can be written in terms of derivatives of Legendre polynomials and further by Gegenbauer polynomials:

$$P_n^k(x) = (-1)^n(1-x^2)^{k/2}\frac{\mathrm{d}^k}{\mathrm{d}x^k}P_n(x) = (2k-1)!!(-1)^n(1-x^2)^{k/2}C_{n-k}^{k+1/2}(x).$$

The case $k < 0$ can be expressed by $k > 0$ as

$$P_n^{-k}(x) = (-1)^k\frac{(n-k)!}{(n+k)!}P_n^k(x).$$

These polynomials satisfy the orthogonality relation

$$\frac{1}{2}\int_{-1}^{1} P_n^k(x)P_m^k(x)\mathrm{d}x = \frac{(n+k)!}{(2n+1)(n-k)!}\delta_{n,m}.$$

B.5 Estimates of Normalized Jacobi Polynomials

In Chap. 10, we need estimates on the differences of Jacobi polynomials, for which we work with the normalized Jacobi polynomials

$$R_k^{(\alpha,\beta)}(t) := \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\alpha,\beta)}(1)}, \qquad n = 0,1,2,\ldots.$$

First we restate several properties of the Jacobi polynomials in terms of $R_k^{(\alpha,\beta)}$ $\alpha \ge$ for $\beta > -\frac{1}{2}$:

$$\max_{t\in[-1,1]}|R_k^{(\alpha,\beta)}(t)| = R_k^{(\alpha,\beta)}(1) = 1, \tag{B.5.1}$$

$$R_k^{(\alpha,\beta)}(t) - R_{k+1}^{(\alpha,\beta)}(t) = (1-t)\frac{2k+\alpha+\beta+2}{2(\alpha+1)}R_k^{(\alpha+1,\beta)}(t), \tag{B.5.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}R_k^{(\alpha,\beta)}(x) = \frac{k(k+\alpha+\beta+1)}{2(\alpha+1)}R_{k-1}^{(\alpha+1,\beta+1)}(x). \tag{B.5.3}$$

Furthermore, for $\mu > 0$ and $\theta \in [0,\pi]$,

$$\begin{aligned} R_k^{(\alpha,\beta)}(\cos\theta) = {} & \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\mu)\Gamma(\mu)(1-\cos\theta)^{\alpha}} \\ & \times \int_0^{\theta} R_k^{(\alpha-\mu,\beta+\mu)}(\cos t)(\cos t - \cos\theta)^{\mu-1} \\ & (1-\cos t)^{\alpha-\mu}\sin t\,\mathrm{d}t, \end{aligned} \tag{B.5.4}$$

which is a Direchlet–Mehler formula for the Jacobi polynomials [162, (4.10.11)].

In the rest of this section, the difference operator $\triangle^j$ is acting on the sequence $k \mapsto \phi(k) = R_k^{(\alpha,\beta)}(\cos\theta)$ whenever the notation $\triangle^j\left(\phi(k) = R_k^{(\alpha,\beta)}(\cos\theta)\right)$ is involved, and we assume $\alpha \ge \beta > -\frac{1}{2}$.

Lemma B.5.1. *For $j,k \in \mathbb{N}_0$ and $\theta \in (0,\frac{\pi}{2}]$,*

$$\left|\triangle^j\left(R_k^{(\alpha,\beta)}(\cos\theta)\right)\right| \le c(\alpha,\beta,j)\theta^j \min\left\{1,(k\theta)^{-\alpha-\frac{1}{2}}\right\}. \tag{B.5.5}$$

Proof. We use induction on the integer j. Inequality (B.5.5) for the case of $j=0$ is a simple consequence of Eq. (B.1.7). Now assuming that Eq. (B.5.5) is true for $j \le \ell$ for some ℓ, we use Eqs. (B.5.2) and (A.3.3) to obtain

$$\begin{aligned} \triangle^{\ell+1}R_k^{(\alpha,\beta)}(\cos\theta) &= \frac{1-\cos\theta}{\alpha+1}\triangle^{\ell}\left(\left(k+\frac{\alpha+\beta}{2}+1\right)R_k^{(\alpha+1,\beta)}(\cos\theta)\right) \\ &= \frac{1-\cos\theta}{\alpha+1}\left(\left(k+\frac{\alpha+\beta}{2}+1\right)\right. \\ &\qquad \left.\triangle^{\ell}R_k^{(\alpha+1,\beta)}(\cos\theta) - \ell\triangle^{\ell-1}R_{k+1}^{(\alpha+1,\beta)}(\cos\theta)\right), \end{aligned}$$

which, using the induction hypothesis, yields the estimate Eq. (B.5.5) for the case $j=\ell+1$. □

Lemma B.5.2. *Let $a>1$ be a fixed number. If $\theta \in (0,\frac{\pi}{2}]$ and $k\theta \le a$, then*

$$0 < c_{\alpha,\beta,a} \le \frac{1-R_k^{(\alpha,\beta)}(\cos\theta)}{k(k+\alpha+\beta+1)\theta^2} \le c'_{\alpha,\beta,a}. \tag{B.5.6}$$

Furthermore, let $b>1$ be a fixed number. Then for $j \in \mathbb{N}_0$ and $0 \le k \le b\theta^{-1}$,

$$\left|\triangle^j\left(\frac{1-R_k^{(\alpha,\beta)}(\cos\theta)}{k(k+\alpha+\beta+1)\theta^2}\right)\right| \le c_{\alpha,\beta,j,b}\theta^j. \tag{B.5.7}$$

Proof. Using Bernstein's inequality for trigonometric polynomials, we have, for $j \ge 0$,

$$\begin{aligned}\left|R_j^{(\alpha+1,\beta+1)}(\cos t)-1\right| &= \left|R_j^{(\alpha+1,\beta+1)}(\cos t)-R_j^{(\alpha+1,\beta+1)}(\cos 0)\right|\\ &\le jt\left\|R_j^{(\alpha+1,\beta+1)}\right\|_{L^\infty[-1,1]} = jt.\end{aligned}$$

It follows that for $0 \le t \le \frac{1}{2j}$,

$$\frac{1}{2} \le R_j^{(\alpha+1,\beta+1)}(\cos t) \le 1. \tag{B.5.8}$$

Using the identity (B.5.3), we then obtain

$$1-R_k^{(\alpha,\beta)}(\cos\theta) = \frac{k(k+\alpha+\beta+1)}{2(\alpha+1)}\int_{\cos\theta}^{1} R_{k-1}^{(\alpha+1,\beta+1)}(t)\,\mathrm{d}t, \tag{B.5.9}$$

which, using Eq. (B.5.8), implies that

$$\frac{1}{4(\alpha+1)} \le \frac{1-R_k^{(\alpha,\beta)}(\cos\theta)}{k(k+\alpha+\beta+1)(1-\cos\theta)} \le \frac{1}{2(\alpha+1)}, \quad 0<\theta<\frac{1}{2k}. \tag{B.5.10}$$

This proves Eq. (B.5.6) for $0<\theta<\frac{1}{2k}$. Inequality (B.5.6) for the case $\frac{1}{2} \le k\theta \le a$ follows directly from Lemma B.5.3 below, which implies, in particular, that $|R_k^{(\alpha,\beta)}(\cos\theta)| \le \gamma_{\alpha,\beta} < 1$ for some absolute constant $\gamma_{\alpha,\beta} \in (0,1)$ when $k\theta \ge \frac{1}{2}$.

Finally, we use B.5.9 to obtain

$$\left|\triangle^j\left(\frac{1-R_k^{(\alpha,\beta)}(\cos\theta)}{k(k+\alpha+\beta+1)\theta^2}\right)\right| \le \frac{1}{2\theta^2(\alpha+1)}\int_0^\theta \left|\triangle^j\left(R_{k-1}^{(\alpha+1,\beta+1)}(\cos u)\right)\right|\sin u\,\mathrm{d}u,$$

which is bounded from above by $c_{\alpha,\beta,j,b}\theta^j$ on using Eq. (B.5.5), which proves Eq. (B.5.7). □

Finally, we state an estimate for the Jacobi polynomial itself. What distinguishes this estimate is that the constant in front of the main term is 1.

Lemma B.5.3. *If $\theta \in (0,\frac{\pi}{2}]$, then*

$$\left|R_k^{(\alpha,\beta)}(\cos\theta)\right| \le (1+c_{\alpha,\beta}k^2\theta^2)^{-\frac{\alpha}{2}-\frac{1}{4}}, \tag{B.5.11}$$

for some positive constant $c_{\alpha,\beta}$ that depends only on α and β.

Proof. We first deal with the case $k\theta \ge a := \sqrt{2}(c(\alpha,\beta,0))^{2/(2\alpha+1)}$, where $c(\alpha,\beta,j)$ is the same constant as in Eq. (B.5.5). A straightforward computation shows that in this case,

$$(1+a^{-2}k^2\theta^2)^{-\frac{\alpha}{2}-\frac{1}{4}} \geq c(\alpha,\beta,0)(k\theta)^{-\alpha-\frac{1}{2}},$$

which, using Lemma B.5.1 with $j=0$, implies Eq. (B.5.11), with $c_{\alpha,\beta}$ any constant satisfying $0 < c_{\alpha,\beta} < a^{-2}$. Next, we treat the case $0 \leq k\theta \leq \frac{1}{2}$. From the proof of Lemma B.5.2, it follows that if $0 \leq k\theta \leq \frac{1}{2}$, then

$$\frac{1}{2} \leq R_k^{(\alpha,\beta)}(\cos\theta) \leq 1 - \frac{k(k+\alpha+\beta+1)}{4(\alpha+1)}(1-\cos\theta).$$

However, if $c_\alpha = \frac{4}{\pi^2(2\alpha+1)(\alpha+1)}$, then

$$1 - \frac{k(k+\alpha+\beta+1)}{4(\alpha+1)}(1-\cos\theta) \leq 1 - \frac{(k\theta)^2}{\pi^2(\alpha+1)} \leq (1+c_\alpha k^2\theta^2)^{-\frac{\alpha}{2}-\frac{1}{4}}. \quad \text{(B.5.12)}$$

Combining the last two inequalities, we have proved Eq. (B.5.11) with $0 < c_{\alpha,\beta} \leq c_\alpha$ for the case $0 \leq k\theta \leq \frac{1}{2}$. Finally, we consider the case $\frac{1}{2} \leq k\theta \leq a+1$, which is more difficult to deal with. We will use the formula (B.5.4) with μ satisfying $0 < \mu < \alpha + \frac{1}{2}$. Let

$$M(t,\theta) := (\cos t - \cos\theta)^{\mu-1}(1-\cos t)^{\alpha-\mu}, \quad 0 \leq t \leq \theta.$$

Then Eq. (B.5.4) with $n=0$ shows that

$$\frac{c_{\mu,\alpha}}{(1-\cos\theta)^\alpha}\int_0^\theta M(t,\theta)\sin t\, dt = 1 \quad \text{with} \quad c_{\mu,\alpha} := \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\mu)\Gamma(\mu)}.$$

It then follows by Eqs. (B.5.4) and (B.5.1) that

$$\begin{aligned}
1 - R_k^{(\alpha,\beta)}(\cos\theta) &= \frac{C(\mu,\alpha)}{(1-\cos\theta)^\alpha}\int_0^\theta \Big(1 - R_k^{(\alpha-\mu,\beta+\mu)}(\cos t)\Big) M(t,\theta)\sin t\,\mathrm{d}t \\
&\geq \frac{C(\mu,\alpha)}{(1-\cos\theta)^\alpha}\int_0^{\frac{1}{2k}} \Big(1 - R_k^{(\alpha-\mu,\beta+\mu)}(\cos t)\Big) M(t,\theta)\sin t\,\mathrm{d}t \\
&\geq \frac{c_{\alpha,\beta}}{\theta^{2\alpha}}\int_0^{\frac{1}{2k}} k^2t^2 M(t,\theta)\sin t\,\mathrm{d}t \geq c > 0,
\end{aligned}$$

where we have used Eq. (B.5.10), $1/2 \leq k\theta \leq a+1$, and that $M(t,\theta) \geq 0$. Similarly, using Eqs. (B.5.8) and (B.5.1), one has

$$1 + R_k^{(\alpha,\beta)}(\cos\theta) \geq \frac{3}{2}\frac{c(\mu,\alpha)}{(1-\cos\theta)^\alpha}\int_0^{\frac{1}{2k}} M(t,\theta)\sin t\,\mathrm{d}t \geq c > 0.$$

Combining the last two inequalities, we conclude that

$$\left|R_k^{(\alpha,\beta)}(\cos\theta)\right| \leq 1-\gamma < 1 \quad \text{with} \quad \gamma = \min\{c'_{\alpha,\beta}, c''_{\alpha,\beta}\}.$$

This proves Eq. (B.5.11) in this last case, since using $k\theta \leq a+1$, we can choose $c_{\alpha,\beta}$ as

$$0 < c_{\alpha,\beta} \leq (a+1)^{-2}\left((1-\gamma)^{-\frac{4}{4\alpha+1}} - 1\right),$$

which completes the proof of the lemma. □

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